Rend. Mat. Appl. (7).
Volume 42 (2021), 61 – 162

Rendiconti di Matematica
e delle sue Applicazioni

A Hitchhiker's Guide to Endomorphisms and Automorphisms of Cuntz Algebras

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Abstract. *We present a broad selection of results on endomorphisms and automorphisms of the Cuntz algebras $\mathcal{O}_n$ that have been obtained in the last decades. A wide variety of open problems is also included.*

Contents

2020 Mathematics Subject Classification: 46L05, 46L55, 46L85, 46L40.
Keywords: Cuntz algebras, C*-algebras, 2-adic ring C*-Algebra, p-adic ring C*-algebras, representations, endomorphisms, automorphisms, entropy, index, noncommutative geometry, Thompson groups, knots.
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1. Introduction

The aim of the present work is twofold: it is undoubtedly intended to quickly acquaint the readers with the basics on Cuntz algebras, but it also pursues the more ambitious goal to introduce them to current research themes. The first task was admittedly easy to accomplish, for exhaustive treatises on the Cuntz algebras are already available. To take a few examples, rather extensive accounts of the basics of the Cuntz algebras are provided in a survey by Rørdam [161], or in the notes by Skoufranis [165]; the monograph by Davidson [83], too, contains a self-contained chapter entirely devoted to the Cuntz algebras. This is ultimately why we feel it is no unreasonable sacrifice to keep the proofs of the results stated and discussed in the first chapters to a bare minimum. The second task is obviously more difficult to accomplish, not least because the literature devoted to the Cuntz algebras is incredibly rich and diverse. This abundance required a choice on what to include and what not to include. This is where the present survey can boast a certain degree of novelty. Indeed, although the Cuntz algebras are very much a vital part of the literature on C^*-algebras, there is no place where an interested reader could find a wide coverage of their endomorphisms and automorphisms, which nowadays can be considered an area in full swing on its own. If a shred of originality was to be achieved, a natural choice to make was then to give more prominence to endomorphisms and automorphisms, on which much of the material presented in the survey is in fact focused. In so doing, we wish to fill the gap alluded to above and provide a friendly yet thorough, concrete and reasonably updated exposition of the topic, which highlights the coexistence of several interesting mathematical structures in close interaction with each other. Not surprisingly, it is also a rich source of more or less challenging problems which, by the authors' decision, will constitute a non-negligible part of the manuscript. Most of the presented material is not new, but here and there results that can hardly be found in the literature are mentioned with precise statements and often fully proved. As a matter of fact, not only do we provide detailed arguments to prove results whose proofs are either scattered in the existing literature or simply missing, but we also include novel results, such as Theorem 2.20, Theorem 2.25, Theorem 2.30, Theorem 7.4, Theorem 7.8, Proposition 10.1, and Theorem 10.5.

As a disclaimer, some topics are certainly oversimplified for the sake of brevity, but we hope we can nevertheless convey at least some bit of the overall flavour of the matter. The level of the exposition will be quite down-to-earth, with the admitted purpose of reaching a broader audience, but still with the pretense of offering a quite comprehensive account, exploring several distinct directions. Likewise, due to space limitations this treatment will by no means be intended to be complete, and we certainly apologize for the inevitable omissions, but the long list of references should help the keenest readers to delve further into the various facets of the subject and possibly meet the highest levels of sophistication.

The survey is organized in eight chapters of different length. This of course reflects the authors' taste, but it is also a result of their working actively more on one area rather than on another.

Chapter 2 provides all the main definitions and results necessary to work with Cuntz algebras including the so-called Cuntz-Takesaki correspondence, by means of which any unitary u in a Cuntz algebra $\mathcal{O}_n$ is associated with an endomorphism $\lambda_u \in \mathrm{End}(\mathcal{O}_n)$ and *vice versa*. The covered material is standard and can certainly be found elsewhere, apart from the results discussed in Sections 2.4, 2.5 and 2.6. In particular, in Theorem 2.20 we investigate the intertwining operators of the generating isometries; in Theorem 2.25 we provide a self-contained proof that the diagonal subalgebras $\mathcal{D}_n$ are maximal abelian in $\mathcal{O}_n$, a result which is known but whose proof is not easily found in the literature of the field; in Theorem 2.30 we devote ourselves to the pairwise intersections of the C*-subalgebras generated by the canonical isometries. In Chapter 3 endomorphisms and automorphisms of the Cuntz algebra are looked at more closely. In particular, we present the long-standing problem of deciding when an endomorphism λ_u is an automorphism. More precisely, one would like to translate the surjectivity of λ_u in terms of working (non-tautological) properties of the corresponding u. Despite its simple formulation, this is a very difficult problem to tackle in its generality and is still far from being resolved. Nevertheless, a satisfactory solution does exist for so-called localized unitaries and is discussed at length is Section 3.2. One conceptual reason for the hurdles one finds along the path might well have to do with the astonishing largeness of the group $\mathrm{Aut}(\mathcal{O}_2)$ of all automorphisms of $\mathcal{O}_2$, the Cuntz algebra generated by two isometries. In Section 3.1 this naive idea is given more substance showing that both $\mathrm{Aut}(\mathcal{O}_2)$ and $\mathrm{Out}(\mathcal{O}_2)$ contain every second countable locally compact group.

Chapter 4 collects a series of problems and questions focused on $\mathrm{Aut}(\mathcal{O}_n)$ and $\mathrm{Out}(\mathcal{O}_n)$ either understood as mere groups or topological groups. To take but two examples, we mention here the problem of studying the simplicity of $\mathrm{Out}(\mathcal{O}_2)$ to illustrate purely algebraic natural questions, and the problem of deciding if $\mathrm{Out}(\mathcal{Q}_2)$ contains all Polish groups as an epitome of the topological problems. The following two chapters are different in character from the first three, in that they are no longer devoted to the Cuntz algebras in themselves, but rather to the interplay between them and other, sometimes seemingly unrelated, areas of research. In Chapter 5 it is shown how so-called multiresolution analysis gives rise to representations of the Cuntz algebras as well as being a key instrument to obtain wavelets and thus orthonormal bases in $L^2(\mathbb{R})$ which are particular suited to applied harmonic analysis. Chapter 6 is an outline of how endomorphisms λ_u of the Cuntz algebras can be employed to exhibit inclusions of subfactors of all types. The definition of Voiculescu's entropy is then recalled to point out a few natural problems concerning the computation of the entropy of endomorphisms of the Cuntz algebras. Chapter 7 keeps the focus on endomorphisms and automorphisms, but the attention is now lavished on their fixed points, which is yet another theme central to much of non-commutative ergodic theory. It also features a couple of new results: Theorem 7.4, which addresses crossed products of Cuntz algebras under the action of finite abelian groups, and Theorem 7.8, which describes the fixed-point subalgebras under the action of a natural class of endomorphisms. Chapter 8 is only a quick taste of the way Cuntz algebras made their appearance in algebraic quantum field theory in the early 1970s. Section 8.2 provides a brief but not undetailed account of recent works where spectral triples in the sense of Connes are introduced on Cuntz algebras to turn them into non-commutative manifolds worthy of attention.

The last two chapters pursue a different direction. Chapter 9 presents an in-depth description of the so-called 2-adic ring C^*-algebra $\mathcal{Q}_2$, which contains $\mathcal{O}_2$ in a natural fashion and can be conceived as a version of the latter where the generating isometries are intertwined by an inner automorphism. Although much work has been done to study the properties of the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$, which is rigid in many respects, little is still known. For instance, at present endomorphisms of $\mathcal{O}_2$ that can be extended to $\mathcal{Q}_2$ are far from being characterized. Representation theory of $\mathcal{Q}_2$, too, appears too wild to be classified. However, so-called permutative representations turn out to be much more amenable to treat, and the emerging picture is presented in Section 9.4.

Chapter 10 hints at possible generalizations to the p-adic ring C^*-algebras and even wider classes of C^*-algebras of the work done for $\mathcal{Q}_2$. Some generalizations are fully developed, such as Theorem 10.5, where the problem of extending a representations of the Cuntz algebra $\mathcal{O}_p$ to the p-adic ring C*-algebras $\mathcal{Q}_p$ is settled, and the answer is the same as the one for $p = 2$.

2. Cuntz algebras

2.1. The definition and some fundamental properties

Let $\mathcal{H}$ be an infinite-dimensional Hilbert space. For any given integer $n \geq 2$, we can consider the C*-algebra $\mathcal{O}_n$ generated by n fixed isometries $S_1, \ldots, S_n \in \mathcal{B}(\mathcal{H})$ such that

$$\sum_{i=1}^{n} S_i S_i^* = 1$$

If $n = \infty$, the equality $\sum_{i=1}^{\infty} S_i S_i^* = 1$ cannot hold true normwise; it is therefore replaced by the inequalities $\sum_{i=1}^{n} S_i S_i^* \leq 1$ for any n, which make perfect sense instead. No wonder, the corresponding C*-algebra is usually denoted by $\mathcal{O}_\infty$. We are thus considering a countable family of concrete C*-algebras generated by n proper isometries, whose ranges decompose the Hilbert space $\mathcal{H}$ into their direct sum. In particular, we also have the relations $S_i^* S_j = \delta_{i,j} I$ for each $i, j = 1, \dots, n$, which most often turn out to be very useful when computing in $\mathcal{O}_n$. Though the present survey we shall not be dealing with the case of a C*-algebra generated by a single isometry. However, we shall rather briefly recall those results that may possibly shed light on the more general case. To begin with, it is a well-known result by Coburn [57] that the C*-algebra generated by a single isometry S does not depend on S as long as S is a proper isometry, *i.e.* $SS^* < 1$. Phrased differently, every unital representation of the universal C*-algebra generated by a proper isometry is in fact faithful provided that it sends the generator into a proper isometry as well, which means there is no need to really worry about the maximal C*-norm. Moreover, the result goes further giving a concrete description of $C^*(S)$: it is an extension of $C(\mathbb{T})$, the algebra of the continuous functions on the one-dimensional torus, through the C* algebra $\mathcal{K}(\mathcal{H})$ of the compact operators on a separable Hilbert space $\mathcal{H}$, i.e. we have a short exact sequence

$$0 \to \mathcal{K}(\mathcal{H}) \to C^*(S) \to C(\mathbb{T}) \to 0\,.$$

Similarly, the cases where $n \geq 2$ enjoy utterly analogous properties of universality, as proved by Cuntz in his most cited paper [78], where the following important fact was first pointed out, along with some consequences we are going to discuss presently:

Theorem 2.1. *For any $n \geq 2$, the universal*[1] *C^*-algebra $C^*(s_1, \dots, s_n)$ generated by n isometries $s_1, \dots, s_n$ such that $\sum_{i=1}^n s_i s_i^* = 1, s_i^* s_i = 1$ is isomorphic to $\mathcal{O}_n$.*

Afterwards, the C*-algebras $\mathcal{O}_n$ were named after Cuntz himself, and have since been referred to as the Cuntz algebras, and this is what they will always be called here as well henceforth. While being a breakthrough result, the theorem will not be proved in the present expository paper, inasmuch as providing all the proofs of the results herein needed is definitely not what this survey aims at. Admittedly, it would also take too long time to prove. However, the reader who wishes to delve further into the details of the proof is suggested to peruse Cuntz's original paper [78], or the book by Davidson [83] for a simpler treatment.

We now proceed to list some properties enjoyed by the Cuntz algebras, which should always be borne in mind when dealing with them. First, the $\mathcal{O}_n$'s are separable by definition, since they are finitely generated as C*-algebras. Among the most important non-trivial properties featured by the Cuntz algebras is the fact that they are simple C*-algebras, namely every closed two-sided ideal $I \subset \mathcal{O}_n$ is trivial. This is in fact a straightforward consequence of a far stronger result, which was also proved by Cuntz in his seminal paper [78] and can be stated as follows:

[1] This is the abstract C^*-algebra $C^*(s_1, s_2, \dots, s_n)$ generated by n isometries s_i satisfying the Cuntz relations such that any other C^*-algebra generated by n isometries S_i still satisfying the Cuntz relations is a quotient of $C^*(s_1, s_2, \dots, s_n)$ through the surjective *-homomorphism that sends each s_i to S_i, for every $i = 1, 2, \dots, n$. In other terms, $C^*(s_1, s_2, \dots, s_n)$ is the completion of the universal *-algebra generated by n isometries s_i with $\sum_{i=1}^n s_i s_i^* = I$, say $\mathcal{A}$, under the C^*-maximal norm, to wit $\|x\|_{\max} := \sup\{p(x) : p \text{ is a } C^*\text{-seminorm on } \mathcal{A}\}$, $x \in \mathcal{A}$.

Theorem 2.2. *Let $X \neq 0$ be an element of $\mathcal{O}_n$. There are $A, B \in \mathcal{O}_n$ such that $AXB = 1$.*

Notably, the last statement betters an old remark by Dixmier, see [84], according to which there exist separable infinite simple C*-algebras, providing a more explicit class of nice examples. In fact, Dixmier was only able to point out abstract reasons why such C*-algebras should exist. His argument ran as follows. The unital C*-algebra $C^*(S_1, S_2)$ generated by two isometries such that $S_1S_1^* + S_2S_2^* = 1$ is clearly separable and infinite, that is to say there exist proper isometries in $C^*(S_1, S_2)$, *e.g.* S_1 and S_2. If $I \subset C^*(S_1, S_2)$ is a maximal two-sided ideal, then the quotient algebra $C^*(S_1, S_2)/I$ has all the desired properties. In hindsight, what the argument lacks to really yield concrete examples is to realize that $C^*(S_1, S_2)$, being already simple, need not be divided by any ideal. Finally, the $\mathcal{O}_n$'s are infinite to a greater extent. Indeed, they are even purely infinite: not only is $\mathcal{O}_n$ infinite for every n, but any hereditary subalgebra of $\mathcal{O}_n$ is infinite as well.

At this stage, the reader may be asking whether the C*-algebras $\mathcal{O}_n$ corresponding to different integers are distinguished. In other words, it is a natural question to pose if $\mathcal{O}_n \cong \mathcal{O}_m$ implies $n = m$. Indeed, this is the case, as first proved in [79] by making intensive use of K-theory. More precisely, it turns out that $K_0(\mathcal{O}_n) = \mathbb{Z}_{n-1}$ and $K_1(\mathcal{O}_n) = 0$ for every $2 \leq n < \infty$, $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$. In addition, the K-theoretical data of the Cuntz algebras provide a useful constraint for a unital inclusion $\mathcal{O}_m \subset \mathcal{O}_n$ to hold: given two integers $m > n \geq 2$, then $\mathcal{O}_m \subset \mathcal{O}_n$ if and only if $n-1$ divides $m-1$. As a particular case we have that each $\mathcal{O}_n$ is contained in $\mathcal{O}_2$, which at first sight might seem counterintuitive.[2] However, it is not any longer as soon as one realizes that it quite similar to what happens with the free groups $\mathbb{F}_n$ on n generators, which are actually all contained in $\mathbb{F}_2$. Nevertheless, $\mathcal{O}_2$ does feature astounding properties when it comes to saying how big it actually is. We are referring to a couple of results by Kirchberg and Phillips [136], who proved that any separable exact simple unital C*-algebra embeds into $\mathcal{O}_2$ and that every separable nuclear simple unital C*-algebra $\mathcal{A}$ is absorbed by $\mathcal{O}_2$, *i.e.* $\mathcal{A} \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. In particular $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, which is a far from obvious property first proved by Elliott. As a consequence, even more can be said: in fact, one has $\bigotimes_{i=1}^\infty \mathcal{O}_2 \cong \mathcal{O}_2$. Moreover, we also have the isomorphism $\mathcal{O}_\infty \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ thanks to the Kirchberg-Phillips theorem, as $\mathcal{O}_\infty$ clearly fulfills its hypotheses. The latter isomorphism plays a vital role when one is faced with the problem of embedding increasingly broad classes of topological groups into $\mathrm{Out}(\mathcal{O}_2)$, as we shall see in the next sections. Finally, for each n, m there is also a canonical unital injection of $\mathcal{O}_{nm}$ into $\mathcal{O}_n \otimes \mathcal{O}_m$ that is worth mentioning here. Indeed, if $\mathcal{O}_n = C^*(S_1, S_2, \ldots, S_n)$ and $\mathcal{O}_m = C^*(T_1, T_2, \ldots, T_m)$, then the nm isometries $S_i \otimes T_j \in \mathcal{O}_n \otimes \mathcal{O}_m$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$ clearly satisfy the relation $\sum_{i,j}(S_i \otimes T_j)(S_i \otimes T_j)^* = 1$, and thus by universality

[2]For instance, one can embed $\mathcal{O}_3 = C^*(T_1, T_2, T_3)$ into $\mathcal{O}_2 = C^*(S_1, S_2)$ by setting $T_1 = S_1$, $T_2 = S_2S_1$ and $T_3 = (S_2)^2$; more generally, can embed $\mathcal{O}_{2n-1}$ into $\mathcal{O}_n$ by setting $T_1 = S_1, T_2 = S_2, \ldots, T_{n-1} = S_{n-1}, T_n = S_nS_1, T_{n+1} = S_nS_2, \ldots, T_{2n-1} = (S_n)^2$.

the C*-subalgebra they generate is a copy of $\mathcal{O}_{nm} \subset \mathcal{O}_n \otimes \mathcal{O}_m$. This C*-algebra has recently been described by A. Morgan [145] in more intrinsic terms as the so-called $\mathbb{T}$-balanced tensor product of $\mathcal{O}_n$ and $\mathcal{O}_m$, namely the fixed-point subalgebra under the action of $\mathbb{T}$ upon $\mathcal{O}_n \otimes \mathcal{O}_m$ given by $\gamma = \alpha \otimes \beta^{-1}$, where α and β denote the gauge actions on $\mathcal{O}_n$ and $\mathcal{O}_m$ respectively (the gauge action is thoroughly discussed in Section 3 of the present survey).

As we have recalled, the K_1-groups of the Cuntz algebras are all trivial. Nevertheless, it is a remarkable result that $\mathcal{O}_n \cong \mathcal{O}_m$ if and only if $\mathcal{U}(\mathcal{O}_n) \cong \mathcal{U}(\mathcal{O}_m)$ as abstract groups, as proved in [21]. We also recall that $\mathcal{U}(\mathcal{O}_n)$ is a connected topological group for any n. As we shall see in the next section, the unitaries of the Cuntz algebras play a major role because there is a bijective correspondence between them and the endomorphisms of the algebras themselves. Therefore, it is also natural to ask the following question, where the involved automorphism groups are thought as both abstract[3] and topological groups. However, fuller information about the topological properties of these groups will only be given afterwards.

Question 2.3. Does $\mathrm{Aut}(\mathcal{O}_n) \cong \mathrm{Aut}(\mathcal{O}_m)$ imply $\mathcal{O}_n \cong \mathcal{O}_m$?

Furthermore, as we shall see, the Cuntz algebras have plenty of outer automorphism. Notably, $\mathrm{Out}(\mathcal{O}_2)$ is an impressively large group. Therefore, the same question as above may be asked for $\mathrm{Out}(\mathcal{O}_n)$ as well although it will always be regarded as an abstract group only. Later on, we shall given a sound reason for that.

Question 2.4. Does $\mathrm{Out}(\mathcal{O}_n) \cong \mathrm{Out}(\mathcal{O}_m)$ imply $\mathcal{O}_n \cong \mathcal{O}_m$?

To the best of our knowledge, nobody seems to have attacked this couple of problems so far.

Now that we have had a quick look at the core properties of the Cuntz algebras, we would like to set some notation to further discuss their vast theory, as it has been accumulating in the overwhelming literature ever since their debut in [78]. Closely following the original notation in [78], we set $W_n^0 := \{0\}$, and for $k \geq 1$

$$W_n^k := \{1, \ldots, n\}^k, \quad W_n := \bigcup_{k=0}^{\infty} W_n^k.$$

Elements $\alpha \in W_n^k$ are called *words* and elements of W_n are called *multi-indices*. For every $\alpha \in W_n^k$ we denote by $l(\alpha)$ the length of the word α with respect to the alphabet $\{1, \cdots, n\}$. For $\alpha = (\alpha_1, \cdots, \alpha_k) \in W_n^k$ we set $S_0 = 1$ and $S_\alpha := S_{\alpha_1} \cdots S_{\alpha_k}$. Let $\alpha = (\alpha_1, \cdots \alpha_k)$, $\beta = (\beta_1, \cdots, \beta_m) \in W_n$. Since $S_i^* S_i = 1$ and $S_j S_j = 0$ if $i \neq j$ it is easy to see that $S_\beta^* S_\alpha \neq 0$ implies that $\beta_s = \alpha_s$ for $1 \leq s \leq \min\{l(\alpha) = k, l(\beta) = m\}$. As a consequence, we also have that if $S_\beta^* S_\alpha \neq 0$ then

[3]The existence of isomorphisms between infinite groups at times may be somewhat unexpected, e.g. it is well known that $\mathbb{R}$ and $\hat{\mathbb{Q}}$ are isomorphic as abstract groups, where $\hat{\mathbb{Q}}$ is the Pontryagin dual of the additive group of the rational numbers thought of as a discrete group. Nevertheless, they are not isomorphic as topological groups, for $\hat{\mathbb{Q}}$ is compact whilst $\mathbb{R}$ is not.

1. $l(\alpha) = l(\beta)$ implies $\alpha = \beta$ and $S_\beta^* S_\alpha = 1$;

2. $l(\beta) < l(\alpha)$ implies there is a $\gamma \in W_n^{k-m}$ such that $S_\beta^* S_\alpha = S_\gamma$;

3. $l(\alpha) < l(\beta)$ implies there is $\gamma \in W_n^{m-k}$ such that $S_\beta^* S_\alpha = S_\gamma^*$.

This means that every non-zero expression in the set $\{S_i, S_i^* : 1 \leq i \leq n\}$ has a unique reduced expression of the form $S_\alpha S_\beta^*$, therefore we can conclude that $\mathcal{O}_n$ is the closed linear span of $S_\alpha S_\beta^*$, for $\alpha, \beta \in W_n$.

Being simple, the C*-algebras $\mathcal{O}_n$ have no proper closed two-sided ideal. Yet they do have interesting subalgebras, two of which we are going to single out to better appreciate the richness of their inner structure. Notably, there is an inclusion $\mathcal{F}_n \subset \mathcal{O}_n$ for each n, where

$$\mathcal{F}_n := \overline{\bigcup_{k\geq 1} \mathcal{F}_n^k},$$

with

$$\mathcal{F}_n^k := \mathrm{span}\{S_\alpha S_\beta^* : \alpha, \beta \in W_n^k\}.$$

Suppose that n is finite. Because $\mathcal{F}_n^k \cong M_{n^k}$ thanks to the identities

$$(S_\alpha S_\beta^*)(S_{\alpha'} S_{\beta'}^*) = \delta_{\alpha,\beta} S_\alpha S_{\beta'},$$

which say that $\{S_\alpha S_\beta^* : \alpha, \beta \in W_n^k\}$ is a set of matrix units for $M_{n^k} \cong M_n \otimes \cdots \otimes M_n$, every $\mathcal{F}_n$ is immediately seen to be a UHF-algebra of type n^∞. In particular $\mathcal{F}_2$ is the CAR algebra. If n is infinite, then $\mathcal{F}_\infty^k$ is isomorphic to $\mathcal{K}(\mathcal{H})$ for any k and $\mathcal{F}_\infty$ is not a UHF-algebra, but an AF-algebra.

The second important subalgebra of $\mathcal{O}_n$ is the so-called diagonal subalgebra $\mathcal{D}_n$. This is the commutative C*-subalgebra of $\mathcal{O}_n$ generated by the projections of the form $P_\alpha := S_\alpha S_\alpha^*$, $\alpha \in W_n$. The subalgebra $\mathcal{D}_n$ is known to be a maximal abelian selfadjoint subalgebra (or MASA for short) of both $\mathcal{O}_n$ and $\mathcal{F}_n$. As for its Gelfand spectrum, it is not difficult to prove that $\mathcal{D}_n \cong C(X_n)$, where

$$X_n := \prod^{\mathbb{N}} \{1, \ldots, n\}$$

is the set of all infinite words in the alphabet $\{1, \ldots, n\}$ equipped with the product topology. It may be worth noting that each X_n is a Cantor space. Like $\mathcal{F}_n$, the C*-algebra $\mathcal{D}_n$ is itself approximately finite dimensional. Indeed, by its definition we have:

$$\mathcal{D}_n = \overline{\bigcup}_{k\geq 1} \mathcal{D}_n^k.$$

with $\mathcal{D}_n^k \doteq \mathcal{D}_n \cap \mathcal{F}_n^k \cong \mathbb{C}^{n^k}$.

We take the opportunity to mention that the UHF algebra $\mathcal{F}_n$ can be seen as the crossed product $\mathcal{D}_n \rtimes \left(\bigoplus \mathbb{Z}_n\right)$ (see e.g. [152, 7.10.8, p. 282], [101], [170]).

Finally, we point out the following isomorphisms: $\mathcal{D}_n \otimes \mathcal{D}_n \cong \mathcal{D}_n$, $\mathcal{F}_n \otimes \mathcal{F}_n \cong \mathcal{F}_n$. Having defined the algebras $\mathcal{D}_n$ and $\mathcal{F}_n$, it is also worth mentioning that

$\mathcal{F}_n \ncong \mathcal{F}_m$ for $m \neq n$ provided that n and m are both prime. This is indeed an easy consequence of Glimm's classification theorem for UHF algebras, a very perspicuous account of which is given in Davidson's book [83]. Quite the opposite, it is always true that $\mathcal{D}_n \cong \mathcal{D}_m$ merely because $\mathcal{D}_n \cong C(K)$ for any n, where $K \subset [0,1]$ is the Cantor ternary set.

Before ending the present section, we should at least point out to the reader that many generalizations of the Cuntz algebras have cropped out since their first appearance; among them, and with increasing generality, Cuntz-Krieger algebras, Pimsner algebras, graph and k-graph C^*-algebras, and DR-algebras must be mentioned. Although we will hardly enter into these topics, we warn the reader that some of the issues discussed in the next sections would also make perfect sense in some of these more general frameworks.

2.2. Endomorphisms and unitaries of $\mathcal{O}_n$

This section is entirely focused on the set of the endomorphisms of the Cuntz algebras, which is denoted by $\mathrm{End}(\mathcal{O}_n)$. Unless otherwise stated, an endomorphism will always be understood as a unital $*$-preserving morphism of $\mathcal{O}_n$ into itself. The first thing to note is that, being the algebras $\mathcal{O}_n$ simple, all their endomorphisms are automatically injective. Then it turns out that they can be thoroughly described in terms of unitaries of the algebras themselves. That is the content of the following easily proved but nonetheless fundamental proposition, which is credited to Takesaki by Cuntz himself, see [77]. We also include its short proof, for it takes next to no time to read and yet it does highlight many of the features peculiar to the Cuntz algebras, which is a very good reason indeed not to miss it.

Proposition 2.5. *For any given $U \in \mathcal{U}(\mathcal{O}_n)$ there is an associated endomorphism $\lambda_U \in End(\mathcal{O}_n)$, which acts on the generating isometries by*

$$\lambda_U(S_i) := US_i, \quad \forall i = 1, \dots, n.$$

Conversely, every endomorphism $\lambda \in \mathrm{End}(\mathcal{O}_n)$ is of this kind, namely there is a unitary $U = U_\lambda$ such that $\lambda = \lambda_U$. Furthermore, the correspondence between $\mathcal{U}(\mathcal{O}_n)$ and $\mathrm{End}(\mathcal{O}_n)$ thus described is one-to-one.

Proof. Thanks to the Cuntz theorem, the map defined on the generating isometries extends to an endomorphism of the algebra $\mathcal{O}_n$. Indeed, if we set $T_i := US_i$, the relations $T_i^*T_i = 1$ for all $i = 1, \dots, n$ and $\sum_i T_iT_i^* = 1$ are immediately seen to hold. Therefore, the T_i's generate an algebra isomorphic to $\mathcal{O}_n$ through the isomorphism sending every S_i into T_i.

Note that we can get a formula for the unitary U in terms of the corresponding λ_U by multiplying $\lambda_U(S_i)$ by S_i^* on the right and then summing over i:

$$\sum_i \lambda_U(S_i)S_i^* = U \sum_i S_iS_i^* = U$$

Now, for any given endomorphism $\lambda \in \mathrm{End}(\mathcal{O}_n)$, we can use the formula above to define $U \doteq \sum_i \lambda(S_i)S_i^*$. It is easy to see that U is a unitary, since

$U^*U = \sum_{i,j} S_i\lambda(S_i^*S_j)S_j^* = \sum_i S_iS_i^* = 1$. $UU^* = 1$ is similarly dealt with. In order to prove that $\lambda = \lambda_U$, take a generator and multiply it by U:

$$US_j = \sum_i \lambda(S_i)S_i^*S_j = \lambda(S_j).$$

This shows that the map $\mathcal{U}(\mathcal{O}_n) \ni U \to \lambda_U \in \mathrm{End}(\mathcal{O}_n)$ is surjective. As claimed, it is injective too: for if $\lambda_U = \lambda_V$, then by definition $US_i = VS_i$ for each i, hence $\sum_i US_iS_i^* = \sum_i VS_iS_i^*$, *i.e.* $U = V$. □

For any finite value of n, the corresponding $\mathcal{O}_n$ carries a distinguished endomorphism, which is known as the canonical endomorphism and is often denoted by φ regardless of the integer n. It is defined by

$$\varphi(x) \doteq \sum_{i=1}^n S_ixS_i^* \qquad \text{for each } x \in \mathcal{O}_n$$

A trivial computation shows it is actually an endomorphism; as such, it also comes from a unitary θ, which is simply given by $\theta = \sum_j \varphi(S_j)S_j^* = \sum_{i,j} S_iS_jS_i^*S_j^*$.

Because $\theta S_hS_k = S_kS_h$ for every integer h and k, as immediately checked, θ is usually referred to as the flip unitary. As the name itself suggests, the canonical endomorphism is a proper endomorphism, *i.e.* its range is not all $\mathcal{O}_n$. This can be seen in many a way. For instance, it is enough to observe that the projection $S_1S_1^*$ is not in its range.[4] Accordingly, $\mathrm{End}(\mathcal{O}_n)$ endowed with the product given by the composition between endomorphisms is only a semigroup. One thing more deserves to be noted: $\varphi\restriction_{\mathcal{F}_n}$ is the one-sided unilateral shift, i.e. $\varphi(x) = 1 \otimes x$ for every $x \in \mathcal{F}_n$. Since $\mathcal{D}_n \cong C(X_n)$, the restriction $\varphi\restriction_{\mathcal{D}_n}$ is the dual map of the surjective map $\phi\colon X_n \to X_n$ defined by $\phi(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$, which plays a major role in much of classical ergodic theory, especially in symbolic dynamics.

At this stage, some remarks are in order:

Remark 2.6. Unfortunately, the correspondence $\mathcal{U}(\mathcal{O}_n) \ni U \to \lambda_U \in \mathrm{End}(\mathcal{O}_n)$ does not preserve the semigroup structure: for $\mathcal{U}(\mathcal{O}_n)$ is a group, whilst $\mathrm{End}(\mathcal{O}_n)$ is not. In fact, the actual composition rule is given by

$$\lambda_U\lambda_V = \lambda_{\lambda_U(V)U}$$

because $\lambda_U(\lambda_V(S_i)) = \lambda_U(VS_i) = \lambda_U(V)US_i = \lambda_{\lambda_U(V)U}(S_i)$ for each $i = 1, \ldots, n$.

Nevertheless, the correspondence is unital, namely $\lambda_I = \mathrm{id}_{\mathcal{O}_n}$. As a consequence, λ_U is an automorphism if and only if $U \in \lambda_U(\mathcal{O}_n)$. To conclude, it might be useful to remark that the equalities $\lambda_U\lambda_V = \lambda_{\lambda_U(V)U}$ for every $U, V \in \mathrm{End}(\mathcal{O}_n)$ are occasionally referred to as the so-called fusion rules of the semigroup $\mathrm{End}(\mathcal{O}_n)$.

[4] First, note that the C*-subalgebra $\varphi(\mathcal{O}_n)$ has a trivial center, since it is simple being isomorphic to $\mathcal{O}_n$. Second, the projection $S_1S_1^*$ commutes with $\varphi(\mathcal{O}_n)$. Therefore, if it also belonged to $\varphi(\mathcal{O}_n)$, it should be a trivial projection, which is not. Of course, all the projections $S_iS_i^*$ will do just the same.

Remark 2.7. Being endomorphisms as well, inner automorphisms $\mathrm{Ad}(U)$, which act on $\mathcal{O}_n$ as $\mathrm{Ad}(U)(A) \doteq UAU^*$ for each $A \in \mathcal{O}_n$, can also be rewritten as a λ_V for a suitable $V \in \mathcal{U}(\mathcal{O}_n)$. This is given by

$$V = \sum_i \mathrm{Ad}(U)(S_i)S_i^* = \sum_i US_iU^*S_i^* = U\sum_i S_iU^*S_i^* = U\varphi(U^*),$$

where φ is the canonical endomorphism. In short, $\mathrm{Ad}(U) = \lambda_{U\varphi(U^*)}$.

Remark 2.8. If $U \in \mathcal{F}_n$, then the inclusion $\lambda_U(\mathcal{F}_n) \subset \mathcal{F}_n$ is easily checked. The reverse implication is not true instead. In other words, it is possible to show that there exists a $U \in \mathcal{U}(\mathcal{O}_n) \setminus \mathcal{F}_n$ such that $\lambda_U(\mathcal{F}_n) \subset \mathcal{F}_n$. Yet $\lambda_U(\mathcal{F}_n) \subset \mathcal{F}_n$ does imply $U \in \mathcal{F}_n$ whenever λ_U is an automorphism, see [70]. Moreover, if $\lambda_U(\mathcal{F}_n) = \mathcal{F}_n$, then $\lambda_U \in \mathrm{Aut}(\mathcal{O}_n)$. The proofs of these interesting facts are all given in [70, Corollary 4.8]. To conclude, it is also worth mentioning that $\lambda_U(\mathcal{D}_n) = \mathcal{D}_n$ does not imply $\lambda_U \in \mathrm{Aut}(\mathcal{O}_n)$.

The last remarks led to the following question, which actually first appeared in [70, p. 606]. Let $U \in \mathcal{U}(\mathcal{O}_n) \setminus \mathcal{F}_n$ such that $\lambda_U(\mathcal{F}_n) \subset \mathcal{F}_n$. Is it always true that there exists some unitary $V \in \mathcal{F}_n$ such that $\lambda_U \restriction_{\mathcal{F}_n} = \lambda_V \restriction_{\mathcal{F}_n}$? This question has been answered in the negative in [111]. Actually a concrete example of such a unitary U is provided for $n = 2$ such that in addition U belongs to $\mathcal{S}_2$[5] and $\dim(\lambda_U(\mathcal{F}_2)' \cap \mathcal{O}_2) < +\infty$. Anyway, all in all, not only one has

$$\mathcal{U}_{\mathcal{F}_n} := \{u \in \mathcal{U}(\mathcal{O}_n) \mid \lambda_u(\mathcal{F}_n) \subset \mathcal{F}_n\} \supsetneq \mathcal{U}(\mathcal{F}_n)$$

but also the strict inclusion of subsemigroups of $\mathrm{End}(\mathcal{F}_n)$

$$\{\lambda_u|_{\mathcal{F}_n} \mid u \in \mathcal{U}_{\mathcal{F}_n}\} \supsetneq \{\lambda_u|_{\mathcal{F}_n} \mid u \in \mathcal{U}(\mathcal{F}_n)\}\ .$$

To better discuss the last two questions as well as some examples we shall take into account later, we now need to recall some very basic facts about the strict tensor structure carried by the set of endomorphisms of a C*-algebra. In fact, for any given unital C*-algebra, $\mathrm{End}(A)$ is not only a semigroup. It is also a strict C*-tensor category, whose objects are unital endomorphisms of A. The set of arrows between two objects $\rho, \sigma \in \mathrm{End}(A)$ is given by

$$(\rho, \sigma) \doteq \{a \in A : a\rho(b) = \sigma(b)a \text{ for any } b \in A\}$$

In particular, $(\rho, \rho) = \rho(A)' \cap A$ for any object. Furthermore, the composition product between arrows is simply given by the product between elements of the C*-algebra. We will only consider C*-algebras with trivial center $Z(A) :=\doteq A \cap A' = \mathbb{C}$. Note that this is certainly the case of the Cuntz algebras, in that they are even simple. Finally, an object ρ is said to be irreducible if $(\rho, \rho) = \mathbb{C}1$. Under this assumption, automorphisms are all examples of irreducible objects. In particular, the identical automorphism of A, which in this context is often denoted by ι, being the monoidal unit of the tensor category, is also irreducible.

[5] The definition of $\mathcal{S}_2$ is given in the next page.

Later on, it will also be useful to have an explicit description of the C*-algebra of the intertwiners of a given endomorphism $\lambda_U \in \mathrm{End}(\mathcal{O}_n)$. Actually, its elements enjoy a fixed-point property. More precisely, one has the equality

$$(\lambda_U, \lambda_U) = \{x \in \mathcal{O}_n : x = (\mathrm{Ad}(U) \circ \varphi)(x)\} = \mathcal{O}_n^{\mathrm{Ad}(U)\circ\varphi},$$

where φ is the canonical endomorphism. In fact, an element $x \in \mathcal{O}_n$ belongs to (λ_U, λ_U) if and only if $x\lambda_U(S_i) = \lambda_U(S_i)x$ for every $i = 1, 2, \ldots, n$. We can rewrite the last equality as $xUS_i = US_ix$ for every i. If we multiply on the right by S_i^*, we get $xUS_iS_I^* = US_ixS_i^*$ for every i. Summing over i, we finally find $xU = U\varphi(x)$, as claimed. In particular, $U \in (\lambda_U, \lambda_U)$ implies $U = \lambda 1$, since there are no non-trivial solutions of the equation $U = \varphi(U)$. Aside, it can also be shown that $\lambda_U(\mathcal{F}_n)' \cap \mathcal{O}_n = \bigcap_{k\geq 1} \left(\mathrm{Ad}(U) \circ \varphi\right)^k (\mathcal{O}_n)$ [70, Proposition 2.3] and, if $U \in \mathcal{F}_n$, $\lambda_U(\mathcal{F}_n)' \cap \mathcal{F}_n = \bigcap_{k\geq 1} \left(\mathrm{Ad}(U) \circ \varphi\right)^k (\mathcal{F}_n)$.

That said, we can at last move back to Remark 2.4. In particular, we would like to point out that an explicit example of a unitary $W \in \mathcal{U}(\mathcal{O}_n) \setminus \mathcal{F}_n$ such that $\lambda_W(\mathcal{F}_n) \subset \mathcal{F}_n$ has been given by Conti, Rørdam and Szymański in [70], for more details see Example 2.5 in their paper. This is obtained by taking $W = U\varphi(V)$, where (U, V) is a pair of unitaries with $U \in \mathcal{P}_2^4$ and $V \in \mathcal{S}_2 \setminus \mathcal{P}_2$ such that $V \in (\lambda_U, \lambda_U)$. However brilliant, their example does remain an isolated manifestation of an interesting and yet not very well-understood phenomenon that the authors would like to fathom themselves. In an attempt to fill this gap, the following question was therefore posed in [65], cf. Problem 5.1. In fact, the situation we are in vaguely resembles so-called modular invariants, with the commutant of a (finite-dimensional) representation of $\mathrm{SL}(2, \mathbb{Z})$ being replaced by $(\lambda_U, \lambda_U) = \lambda_U(\mathcal{O}_n)' \cap \mathcal{O}_n$ and the matrix Z by V, cf. [134, Section 5].

Question 2.9. Let U be a given unitary in $\mathcal{P}_n \subset \mathcal{U}(\mathcal{O}_n)$. Is it possible to find and classify in a suitable sense all unitaries $V \in \mathcal{S}_n \setminus \mathcal{P}_n$ such that $V \in (\lambda_U, \lambda_U)$?

It is now high time we mentioned two noteworthy classes of unitaries, which will show up here and there throughout the following sections. The first one forms a set $\mathcal{S}_n$ given by:

$$\mathcal{S}_n \doteq \{U \in \mathcal{U}(\mathcal{O}_n) \mid U = \sum_{k=1}^{h} S_{\alpha_k} S_{\beta_k}^*\}.$$

Notably, $\mathcal{S}_n$ has been shown to be isomorphic to the Higman-Thompson group $G_{n,1}$, see [147] for the proof. The second one is:

$$\begin{aligned}\mathcal{P}_n &\doteq \mathcal{S}_n \cap \mathcal{F}_n \\ &= \left\{ U \in \mathcal{U}(\mathcal{O}_n) \;\middle|\; U = \sum_{k=1}^{h} S_{\alpha_k} S_{\beta_k}^*, l(\alpha_k) = l(\beta_k)\ k \in \{1, 2, \ldots, h\}, h \in \mathbb{N} \right\}.\end{aligned}$$

$\mathcal{P}_n$ is easily recognized to be an inductive limit of permutation groups $\mathbb{P}_{n^k}$, and for this reason its elements are also known as permutation unitaries.

We cannot end this section without enhancing our knowledge about the correspondence between unitaries and endomorphisms, whose topological aspects have been neglected so far. In this regard, the first thing to be noted is that $\mathcal{U}(\mathcal{O}_n)$ is a Polish[6] space with respect to the topology induced by the norm, merely because it is a closed subspace of a separable complete metric space as $\mathcal{O}_n$ is. If $\mathrm{End}(\mathcal{O}_n)$ is endowed with the topology of pointwise norm convergence, then a moment's reflection[7] shows that the correspondence $\mathcal{U}(\mathcal{O}_n) \ni U \to \lambda_U \in \mathrm{End}(\mathcal{O}_n)$ is indeed a homeomorphism between topological spaces. As a result, $\mathrm{End}(\mathcal{O}_n)$ is a Polish space. We are now ready to state the result we will need later on.

Theorem 2.10. $\mathrm{Aut}(\mathcal{O}_n)$ *is a Polish group for each* n.

Proof. Since the inclusion $\mathrm{Aut}(\mathcal{O}_n) \subset \mathrm{End}(\mathcal{O}_n)$ holds and $\mathrm{End}(\mathcal{O}_n)$ is a Polish space, it is enough to check that $\mathrm{Aut}(\mathcal{O}_n)$ is a G_δ-subset of $\mathrm{End}(\mathcal{O}_n)$. To this aim, we only have to prove that the set $\{U \in \mathcal{U}(\mathcal{O}_n) : \lambda_U \in \mathrm{Aut}(\mathcal{O}_n)\} \subset \mathcal{U}(\mathcal{O}_n)$ is a G_δ because of the correspondence $U \to \lambda_U$ being a homeomorphism. As λ_U is surjective if and only if $U \in \lambda_U(\mathcal{O}_n)$ and the image of an endomorphism is closed thanks to general results, we can rewrite our set as an intersection $\cap_k A_k$, where $A_k \doteq \{U \in \mathcal{U}(\mathcal{O}_n) : d(U, \lambda_U(\mathcal{O}_n)) < \frac{1}{k}\}$. By a standard $\frac{\varepsilon}{3}$-argument each A_k is immediately recognised as being an open set. That ends the proof. $\square$

In particular, $\mathrm{Aut}(\mathcal{O}_2)$ is a Polish group. Among all the groups $\mathrm{Aut}(\mathcal{O}_n)$, it plays a prominent role, insofar as it is the largest of them. Indeed, as a simple consequence of the isomorphisms $\mathcal{O}_n \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, we get the continuous embeddings[8] $\mathrm{Aut}(\mathcal{O}_n) \subset \mathrm{Aut}(\mathcal{O}_2)$.

Remark 2.11. The alert reader will have undoubtedly noticed that a slightly different argument can be used to prove that, more generally, $\mathrm{Aut}(A)$ is a Polish group whenever A is any separable C*-algebra.

We feel that this is the right place in the survey where to point out the next problem, since it is concerned with the issue of whether quite a remarkable group obtained out of a Cuntz algebra is or not a Polish group.

Question 2.12. Is $\mathrm{Aut}(\mathcal{U}(\mathcal{O}_n))$ a Polish group? If so, is it also a universal Polish group?

[6] A topological space is said to be a Polish space if it is homeomorphic to a separable complete metric space. A Polish group is a topological group that is also a Polish space with respect to the topology which makes it a topological group. Finally, a group is called a universal Polish group if every Polish group is isomorphic to a closed subgroup of it. For instance, the group of the homeomorphisms of the Hilbert cube $[0,1]^{\mathbb{N}}$ is the textbook example of a universal Polish group. That is nonetheless a non-trivial fact, which was first proved by V. V. Uspenskii in [169].

[7] In fact, the map is continuous: if $\|U_k - U\| \to 0$, then $\|\lambda_{U_k}(S_i) - \lambda_U(S_i)\| \to 0$ for each $i = 1, 2, \dots, n$, hence $\|\lambda_{U_k}(a) - \lambda_U(a)\| \to 0$ for every $a \in \mathcal{O}_n$. Furthermore, its inverse is continuous as well. Indeed, if λ_{U_k} tends to λ_U pointwise in norm, then $U_k S_i \to U S_i$ in norm, so $U_k S_i S_i^* \to U S_i S_i^*$ for every $i = 1, 2, \dots, n$. Summing over i, we finally get that $U_k \to U$ in norm.

[8] They are simply given by $\mathrm{Aut}(\mathcal{O}_n) \ni \varphi \to \varphi \otimes \mathrm{id}_{\mathcal{O}_2} \in \mathrm{Aut}(\mathcal{O}_n \otimes \mathcal{O}_2) \cong \mathrm{Aut}(\mathcal{O}_2)$.

We now recast our question more precisely. First, whenever G is a topological group, we shall denote by $\mathrm{Aut}(G)$ the group of the continuous automorphisms of G. Second, we shall always think of $\mathrm{Aut}(G)$ as being given the topology of pointwise convergence with respect to the topology of G itself. As the reader may have already sensed, the main issue to be dealt with to work the problem out is to further investigate the algebraic structure of $\mathrm{Aut}(\mathcal{U}(\mathcal{O}_n))$ as a group. In fact, we still do not know much about the general structure of a continuous homomorphism of G into itself. In this regard, later on we shall raise the question whether all automorphisms of $\mathcal{U}(\mathcal{O}_n)$ can or cannot be extended (in a unique way if they can) to automorphisms of $\mathcal{O}_n$ as a C*-algebra.

As for $\mathrm{Aut}(\mathcal{O}_2)$, there is still some information we would like to add to give the reader but a taste of its complexity. In fact, in a very recent work Gardella and Lupini [100] have shown that the problem of classifying automorphisms of $\mathcal{O}_2$ up to suitable notions of equivalence is quite unapproachable, at least in its full generality. More precisely, the relations of conjugacy and cocycle conjugacy are there both proved to be analytic sets of $\mathrm{Aut}(\mathcal{O}_2) \times \mathrm{Aut}(\mathcal{O}_2)$; as such, they are not Borel. What is still more relevant to the discussions we will make afterwards is their further result that from the point of view of Borel complexity theory the problem of classifying automorphisms of $\mathcal{O}_2$ up to conjugacy or cocycle conjugacy is strictly more difficult than classifying up to isomorphism any class of countable structures with Borel isomorphism relation. Therefore, as a drawback of these results, there appear to exist serious obstructions coming from Logic which might prevent classification problems for automorphisms of $\mathcal{O}_2$ up to such and such equivalence notion from ever coming to a satisfactory conclusion.

2.3. Gauge automorphisms

Given any $z \in \mathbb{T}$, we can define an endomorphism λ_{z1} of $\mathcal{O}_n$ if we set $\lambda_{z1}(S_i) \doteq zS_i$ for every $i = 1, 2, \ldots, n$. In other words, λ_{z1} is the endomorphism associated with the unitary $z1 \in \mathcal{O}_n$. It actually turns out to be an automorphism merely because $z1$ is in its range, given that one trivially has $\lambda_{z1}(z1) = z1$. More importantly, as a straightforward consequence of a theorem of K. Matsmuto and J. Tomiyama proved in [143], each λ_{z1} is an outer automorphism if $z \neq 1$. Furthermore, the composition law $\lambda_{z1}\lambda_{w1} = \lambda_{zw1}$ is immediately checked. This means that each $\mathcal{O}_n$ is acted upon by the one-dimensional torus $\mathbb{T}$. The action is easily proved to be continuous, to wit the function $\mathbb{T} \ni z \to \lambda_{z1}(a) \in \mathcal{O}_n$ is norm-continuous for every $a \in \mathcal{O}_n$. Following the well-established terminology in the existing literature, we shall refer to it as to the gauge action of $\mathbb{T}$ on $\mathcal{O}_n$. Sometimes for brevity we will write α_z instead of λ_{z1}. It is also common to refer to the 2π-periodic gauge action α_t of $\mathbb{R}$ given by $\lambda_{e^{it}1}$. It follows from general results on C*-dynamical systems, a standard reference of which is definitely the classic textbook by Pedersen [152], that the gauge action allows us to think of $\mathcal{O}_n$ as a $\mathbb{Z}$-graded C*-algebra. This is a short way to say that $\mathcal{O}_n$ can be decomposed as a topological direct sum of spectral subspaces $\mathcal{O}_n^k$ with $\mathcal{O}_n^k\mathcal{O}_n^l \subset \mathcal{O}_n^{k+l}$ and $(\mathcal{O}_n^k)^* = \mathcal{O}_n^{-k}$ for any $k, l \in \mathbb{Z}$, where $\mathcal{O}_n^k \doteq \{a \in \mathcal{O}_n : \lambda_{z1}(a) = z^k a \ \forall z \in \mathbb{T}\}$. Note that this provides quite a natural context where the problem of norm-convergence of Fourier-like series for elements in $\mathcal{O}_n$ can be framed; for instance, an analogue of the Fejér theorem can be found in [83] (which we will state for $\mathcal{O}_2$ in the next section). In particular, $\mathcal{O}_n^0$ is the fixed-point subalgebra with respect the gauge action, which may be computed leisurely: it is precisely the UHF subalgebra $\mathcal{F}_n \subset \mathcal{O}_n$. Accordingly, there is a faithful conditional expectation E of $\mathcal{O}_n$ onto $\mathcal{F}_n$, which is obtained by averaging the gauge action with respect to the invariant measure of $\mathbb{T}$. Among other things, this provides a simple proof that $\mathcal{O}_n$ is a nuclear C*-algebra. Indeed, a C*-algebra $\mathcal{A}$ acted upon by a compact group G is nuclear if and only if the fixed-point subalgebra $\mathcal{A}^G$ is nuclear, see [90], and $\mathcal{F}_n = \mathcal{O}_n^{\mathbb{T}}$ is definitely such an algebra, for it is even approximately finite dimensional. Moreover, the composition $E \circ \tau$, with τ being the unique tracial state on $\mathcal{F}_n$, is a KMS state on $\mathcal{O}_n$ with respect to the $\mathbb{T}$ gauge action. The corresponding GNS representation yields a $\mathrm{III}_{1/n}$ type factor $\pi_{E\circ\tau}(\mathcal{O}_n)''$, see [78].

2.4. Intertwining operators between the generating isometries

In this section we discuss the existence of suitable elements intertwining two generating isometries S_i and S_j for some $i, j \in \{1, \ldots, n\}$ of $\mathcal{O}_n$. For simplicity we stick to the case $n = 2$ and $i = 1$, $j = 2$, but the argument works in greater generality. For unitary elements, the case $i = j$ was already settled in [143].

Let us consider the closed subspace

$$C \doteq \{x \in \mathcal{O}_2 \mid xS_1 = S_2 x\} .$$

For any $i \in \mathbb{N}_0$ we define the linear maps $F_i : \mathcal{O}_2 \to \mathcal{O}_2^{\mathbb{T}}$ given by

$$F_i(x) \doteq \int_{\mathbb{T}} \alpha_z[x(S_2^*)^i]dz$$

$$F_{-i}(x) \doteq \int_{\mathbb{T}} \alpha_z[S_2^i x]dz$$

where dz is the normalized Haar measure on the one-dimensional torus and we adopt the convention $S_2^0 = 1$. For every $i \in \mathbb{N}$, it holds $F_i(\mathcal{O}_2) = \mathcal{O}_2^i S_2^i$ and $F_{-i}(\mathcal{O}_2) = (S_2^*)^i \mathcal{O}_2^i$. We observe that, for any $i \in \mathbb{N}$, it holds

$$F_i(x) = F_i(x) S_2^i (S_2^*)^i \tag{2.1}$$

$$F_{-i}(x) = S_2^i (S_2^*)^i F_{-i}(x) \tag{2.2}$$

see e.g. [143].

Given x in $\mathcal{O}_2$, the terms $F_i(x)$ and $F_{-i}(x)$, with $i \in \mathbb{N}_0$, should be regarded as its Fourier coefficients.

We recall the following important facts (see e.g. [83] and [77, Section 1.10]).

Theorem 2.13. *Every x in $\mathcal{O}_2$ can be expressed as the limit*

$$x = \lim_n F_1(x) + \sum_{i=1}^n \left(1 - \frac{|i|}{2n+1}\right)(S_2^*)^i F_{-i}(x) + \left(1 - \frac{|i|}{2n+1}\right) F_i(x) S_2^i .$$

Corollary 2.14. *Let $x \in \mathcal{O}_2$. If $F_i(x) = 0$ for all $i \in \mathbb{Z}$, then $x = 0$.*

The goal of this section is to provide a quick proof of the fact that there are no intertwining operators in C of some special form.

In the following series of lemmas we analyse the positive Fourier coefficients of the elements in C.

Lemma 2.15. *For all $i \geq 0$ and for any $x \in C$ we have $F_i(x^*) = (S_1^*)^k F_i(x^*) S_2^k$ for all $k \in \mathbb{N}$.*

Proof. We have that

$$
\begin{aligned}
(S_1^*)^k F_i(x^*) S_2^k &= \int_{\mathbb{T}} \alpha_z[(S_1^*)^k x^* (S_2^*)^i S_2^k] dz \\
&= \int_{\mathbb{T}} \alpha_z[x^* (S_2^*)^k (S_2^*)^i S_2^k] dz = F_i(x^*)
\end{aligned}
$$

where we used that $(S_1^*)^k x^* = x^* (S_2^*)^k$, $k \in \mathbb{N}$. □

Lemma 2.16. *For any $x \in \mathcal{F}_2^k$ we have that $(S_1^*)^k x S_2^k \in \mathbb{C}$.*

Proof. Suppose that $x = S_\alpha S_\beta^*$, where $|\alpha| = |\beta| = k$. We have that

$$
(S_1^*)^k x S_2^k = (S_1^*)^k S_\alpha S_\beta^* S_2^k = \delta_{\alpha,\underline{2}} \delta_{\beta,\underline{1}}.
$$

□

Lemma 2.17. *Let $z \in \mathcal{F}_2$ be such that $(S_1^*)^k z S_2^k = z$ for some $k \in \mathbb{N}$, then $z \in \mathbb{C}$.*

Proof. Clearly, it holds $(S_1^*)^{hk} z S_2^{hk} = z$ for all $h \in \mathbb{N}$. Now just approximate each z with a sequence $\{y_m\}$ with $y_m \in \mathcal{F}_2^{f(m)}$ for some increasing function f and then apply Lemma 2.16. □

Lemma 2.18. *For all $x \in C$ we have that $F_i(x^*) \in \mathbb{C}$ for all $i \geq 0$.*

Proof. The claim follows from Lemmas 2.15 and 2.17. □

Lemma 2.19. *For all $x \in C$ we have that $F_i(x^*) = 0$ for all $i \geq 1$.*

Proof. Since $F_i(x) \in \mathbb{C}$, then the claim follows from the formula (2.1). □

We consider some notable subsets of elements in C:

$$
C^u := C \cap \mathcal{U}(\mathcal{O}_n) \qquad \text{and} \qquad C^{s.a.} := C \cap \mathcal{O}_n^{s.a.},
$$

where $\mathcal{O}_n^{s.a.} := \{x \in \mathcal{O}_n \ \mid x = x^*\}$.

Theorem 2.20. *The sets C^u and $C^{s.a.}$ are empty.*

Proof. We first deal with the unitary case. Let i be in $\mathbb{N}$. For all $k > i$, we have

$$\begin{aligned}(S_1^*)^{k-i}(S_2^*)^i F_{-i}(x^*)S_2^k &= (S_1^*)^{k-i}(S_2^*)^i \left(\int_{\mathbb{T}} \alpha_z(S_2^i x^*)dz\right) S_2^k \\ &= \int_{\mathbb{T}} \alpha_z((S_1^*)^{k-i} x^* S_2^k)dz \\ &= \int_{\mathbb{T}} \alpha_z(x^*(S_2^*)^{k-i} S_2^k)dz = \int_{\mathbb{T}} \alpha_z(x^* S_2^i)dz \\ &= \int_{\mathbb{T}} \alpha_z(S_1^i x^*)dz = S_1^i (S_2^*)^i F_{-i}(x^*)\end{aligned} \tag{2.3}$$

where in the third line we used that $x^*S_2 = S_1 x^*$ by unitarity. It follows from an argument similar to the proof of Lemma 2.17 that $S_1^i(S_2^*)^i F_{-i}(x^*) =: \lambda_i \in \mathbb{C}$. Since $F_{-i}(x^*) = S_2^i(S_2^*)^i F_{-i}(x^*) = S_2^i(S_1^*)^i(S_1^i(S_2^*)^i F_{-i}(x^*))$, we have $F_{-i}(x^*) = \lambda_i S_2^i(S_1^*)^i$ for all $i \geq 1$. By formula (2.3) this implies that $\lambda_i = 0$ and thus $F_{-i}(x*) = 0$ for all $i \geq 1$. Combined with Lemma 2.19, Lemma 2.18, and Proposition 2.14, we deduce that $x^* = F_0(x^*) = F_0(x)^* \in \mathbb{C}$. Clearly no scalar intertwines S_1 and S_2.

Now we move on to the self-adjoint case. For all $i \geq 1$, it holds

$$\begin{aligned}F_{-i}(x^*)^* = \left(\int_{\mathbb{T}} \alpha_z(S_2^i x^*)dz\right)^* &= \int_{\mathbb{T}} \alpha_z(x(S_2^*)^i)dz \\ &= \int_{\mathbb{T}} \alpha_z(x^*(S_2^*)^i)dz = F_i(x^*) = 0\end{aligned}$$

where we used Lemma 2.19. Again this means that $x = x^* = F_0(x^*) \in \mathbb{R}$ and we are done. ☐

Corollary 2.21. *There are no inner automorphisms of $\mathcal{O}_n$ mapping one isometry into another one.*

We have not discussed if C is 0 altogether. We leave it as a question.

Question 2.22. Is C trivial?

2.5. The diagonal subalgebra is a MASA

It is well known that $\mathcal{D}_n$ is a maximal abelian selfadjoint subalgebra of $\mathcal{O}_n$, but to the knowledge of the authors there is no elementary proof available in the literature. The aim of this short section is to give a simple and self-contained proof of this fact. The strategy of the proof is similar to that in [10]. We introduce the following representation of the Cuntz algebra

$$\begin{aligned}\pi &: \mathcal{O}_n \to \mathcal{B}(\ell_2(\mathbb{Z})) \\ \pi(S_i)e_k &\doteq e_{i-1+kn}\ .\end{aligned}$$

Throughout we omit the symbol π for simplicity.

Lemma 2.23. *In the former representation we have $\mathcal{D}_n' = l_\infty(\mathbb{Z})$.*

Proof. It is easy to see that $\mathcal{D}_n$ is contained in $\ell_\infty(\mathbb{Z})$. As $\ell_\infty(\mathbb{Z})$ is a MASA of $B(\ell_2(\mathbb{Z}))$, it is enough to prove that $\mathcal{D}_2'' = \ell_\infty(\mathbb{Z})$. First of all we observe that the sequence $\{S_1^k(S_1^*)^k\} \subset \mathcal{D}_2$ strongly converges to E_0, the projection onto $\mathbb{C}e_0$ (similarly E_k denotes the projection onto $\mathbb{C}e_k$). The vector e_0 is a cyclic vector. Moreover, for any $k \in \mathbb{Z}$, there exists an $\alpha \in W_n$ such that $S_\alpha e_0 = e_k$. This implies that the sequence $\{S_\alpha S_1^n (S_1^*)^n S_\alpha^*\} \subset \mathcal{D}_2$ strongly converges to E_k. Since $\{E_k\}_{k\in\mathbb{Z}}'' = \ell_\infty(\mathbb{Z})$ we are done. □

For the next preliminary result we need to set some notation. We denote by E the unique (faithful) conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\ell_\infty(\mathbb{Z})$ (for more detail see [132]). As known, this is simply given by $(E[T]e_i, e_j) = (Te_i, e_j)\delta_{i,j}$.

Lemma 2.24. *The following relations hold:*

- $E[S_\alpha S_\alpha^*] = S_\alpha S_\alpha^*$,
- *if* $|\alpha| \neq |\beta|$, $E[S_\alpha S_\beta^*]$ *is either* 0 *or a* E_{-1}, E_0,
- *if* $|\alpha| = |\beta|$, $E[S_\alpha S_\beta^*]$ *is either* 0 *or a* $S_\alpha S_\beta^*$.

Proof. The first equality needs no proof. For the second it is not restrictive to suppose that $|\beta| > |\alpha|$. Now, we have that $(E[S_\alpha S_\beta^*]e_h, e_h) = (S_\alpha S_\beta^* e_h, e_h)$ is non-zero only if $e_h = S_\beta e_i$, in which case

$$\begin{aligned}
(E[S_\alpha S_\beta^*]e_h, e_h) &= (S_\alpha S_\beta^* e_h, e_h) \neq 0 \\
&= (S_\alpha S_\beta^* S_\beta e_i, e_h) \\
&= (S_\alpha e_i, e_h) \\
&= (S_\alpha e_i, S_\beta e_i) \\
&= (e_i, S_\alpha^* S_\beta e_i).
\end{aligned}$$

The former coefficient is non-trivial unless $\beta = (\alpha, \nu)$, and in this case we get

$$\begin{aligned}
(E[S_\alpha S_\beta^*]e_h, e_h) &= (e_i, S_\alpha^* S_\beta e_i) \\
&= (e_i, S_\nu e_i)
\end{aligned}$$

and the former element is non-trivial only if $S_\nu e_i = e_i$. Since the operators S_ν have eigenvectors only when $\nu = (2, \ldots, 2)$ or when $\nu = (1, \ldots, 1)$, we get that the claim.

For the third, we have that $(E[S_\alpha S_\beta^*]e_h, e_h) = (S_\alpha S_\beta^* e_h, e_h)$ is non-zero only if

$e_h = S_\beta e_i$. In this case

$$\begin{aligned}(E[S_\alpha S_\beta^*]e_h, e_h) &= (S_\alpha S_\beta^* e_h, e_h) \neq 0 \\ &= (S_\alpha S_\beta^* S_\beta e_i, e_h) \\ &= (S_\alpha e_i, e_h) \\ &= (S_\alpha e_i, S_\beta e_i) \\ &= (e_i, S_\alpha^* S_\beta e_i) \\ &= \delta_{\alpha,\beta}(e_i, e_i)\end{aligned}$$

and we are done. □

Being $\mathcal{D}_n \subset \mathcal{F}_n$ =UHF(n^∞), we have a canonical conditional expectation $\Theta\colon \mathcal{O}_n \to \mathcal{D}_n$ obtained by the composition of $F_0\colon \mathcal{O}_n \to \mathcal{F}_n$ and the canonical conditional expectation $\mathcal{F}_n \to \mathcal{D}_n$.

Theorem 2.25. *The diagonal subalgebra $\mathcal{D}_n \subset \mathcal{O}_n$ is a maximal abelian selfadjoint subalgebra.*

Proof. As usual, all we have to do is make sure that the relative commutant $\mathcal{D}_n' \cap \mathcal{O}_n = \ell_\infty(\mathbb{Z}) \cap \mathcal{O}_n$ reduces to $\mathcal{D}_n$. Let $x \in \ell_\infty(\mathbb{Z}) \cap \mathcal{O}_n$, then there exists a sequence $\{x_k\}$ converging normwise to x with each of the x_k of the form $\sum c_{\alpha,\beta} S_\alpha S_\beta^*$. As above, $x = E(x) = \lim_k E(x_k)$. Thanks to the former lemmas, we can rewrite $E(x_k)$ as $d_k + f_k$, where $d_k \in \mathcal{D}_n$ and f_k are finite rank operators of the form $a_0(k)E_0 + a_{-1}(k)E_{-1}$. Now, being $d_k = \Theta(x_k)$, we see that d_k must converge to some $d \in \mathcal{D}_n$. But then f_k converge normwise to a diagonal finite rank operator (in particular compact operator), say f, which means $f = x - d$ is in $\mathcal{O}_n$, hence $f = 0$, and $x = d \in \mathcal{D}_n$. □

2.6. On the intersection $C^*(S_i) \cap C^*(S_j)$

In this subsection we collect a few (possibly unknown) results on subalgebras of $\mathcal{O}_n$ generated by proper subsets of the defining isometries, of which we give complete proofs for want of a reference. We start with $\mathcal{O}_2$. As there are only two isometries to deal with, we have to focus only on $C^*(S_1)$ and $C^*(S_2)$. For brevity we set $P_1 := S_1 S_1^*$ and $P_2 := S_2 S_2^*$. More generally, we denote by $P_{11\ldots 1}$ (1 is repeated n times) the projection $(S_1)^n (S_1^*)^n$ and analogously $P_{22\ldots 2}$ (2 is repeated n times) is the projection $(S_2)^n (S_2^*)^n$.

Since $P_1 + P_2 = 1$, both P_1 and P_2 sit in the intersection $C^*(S_1) \cap C^*(S_2)$. Actually, one of the goals of the section is to prove that the intersection $C^*(S_1) \cap C^*(S_2)$ is just the unital C^*-algebra generated by P_1 and P_2, that is $\mathbb{C}P_1 + \mathbb{C}P_2$. To achieve this result, we first need to do some preliminary work, much of which has an interest in its own.

Lemma 2.26. *If $C^*(S_1)^{\mathbb{T}}$ is the gauge-invariant subalgebra of $C^*(S_1)$, we have*

$$C^*(S_1)^{\mathbb{T}} = C^*(1, P_1, P_{11}, \ldots) = \overline{\operatorname{span}(1, P_1, P_{11}, \ldots)} \subset \mathcal{D}_2.$$

Proof. We only have to prove the inclusion $C^*(S_1)^{\mathbb{T}} \subset C^*(1, P_1, P_{11}, \ldots)$. In fact, the others are immediately recognized to hold true. Let then $x \in C^*(S_1)$ be a $\mathbb{T}$-invariant element. By definition there is a sequence $\{x_k\} \in C^*(S_1)_{\mathrm{alg}}$ such that x_k is norm convergent to x. It is not restrictive to assume that the sequence is $\mathbb{T}$-invariant itself, for $E(x_k)$ converges to x as well. Since the equality $\left(C^*(S_1)_{\mathrm{alg}}\right)^{\mathbb{T}} \subset C^*(1, P_1, P_{11}, \ldots)_{\mathrm{alg}}$ is easily verified, the conclusion is as easy to get to. $\square$

Obviously, the analogous statement is true for $C^*(S_2)^{\mathbb{T}}$ as well, hence the following result.

Corollary 2.27. *The following equality holds:*

$$P_1 C^*(S_2)^{\mathbb{T}} = \mathbb{C}P_1.$$

Proposition 2.28. *We have*

$$C^*(1, P_1, P_{11}, P_{111}, \ldots) \cap C^*(1, P_2, P_{22}, P_{222}, \ldots) = \mathbb{C}P_1 + \mathbb{C}P_2\,.$$

Proof. Let x be an element of the intersection above. As $x = P_1 x + P_2 x$ and both $P_1 x$ and $P_2 x$ are still in the intersection, the former corollary applies. $\square$

We are now in a position to prove the announced result.

Theorem 2.29. *If S_1, S_2 are the generating isometries of $\mathcal{O}_2$, then*

$$C^*(S_1) \cap C^*(S_2) = \mathbb{C}P_1 + \mathbb{C}P_2\,.$$

Proof. This will at once be achieved as soon as we have proved that $C^*(S_1) \cap C^*(S_2)$ is nothing but $C^*(1, P_1, P_{11}, P_{111}, \ldots) \cap C^*(1, P_2, P_{22}, P_{222}, \ldots)$. To this aim, let $x \in C^*(S_1) \cap C^*(S_2)$ be a self-adjoint element. As an element of $C^*(S_1)$, it can be approximated by a sequence of self-adjoint elements of the form

$$x_k = d_1^{(k)}(0) + \sum_{h>0} d_1^{(k)}(h) S_1^h + (S_1^*)^h d_1^{(k)*}(h),$$

where all the $d_1^{(k)}(h)$'s belong to $\mathrm{span}(1, P_1, P_{11}, \ldots)$. As an element of $C^*(S_2)$, it can likewise be approximated by a sequence

$$y_k = d_2^{(k)}(0) + \sum_{h>0} d_2^{(k)}(h) S_2^h + (S_2^*)^h d_2^{(k)*}(h),$$

with all the $d_2^{(k)}$'s being in $\mathrm{span}(1, P_2, P_{22}, \ldots)$. We now need to recall that the h^{th} spectral component of an element $z \in \mathcal{O}_2$ can be defined as

$$z^{(h)} = \frac{1}{2\pi} \int_0^{2\pi} \alpha_{e^{it}}(z) e^{-iht} \mathrm{d}t, \quad h \in \mathbb{Z}.$$

As easily verified, the sequences $x_k^{(h)}$ and $y_k^{(h)}$ both converge to $x^{(h)}$ in norm for each h. Furthermore, $x_k^{(h)} = d_1^{(k)}(h)S_1^h$ for every $h > 0$. Therefore, we have

$$(x^{(h)})^* x^{(h)} = \lim(d_1^{(k)}(h)S_1^h)^* d_2^{(k)}(h)S_2^h = 0$$

for $h > 0$. It follows at once that $x^{(h)} = 0$ for all $h \neq 0$. Since x is uniquely determined by its spectral components $x^{(h)}$, we must then have $x = E(x) \in C^*(S_1)^{\mathbb{T}} \cap C^*(S_2)^{\mathbb{T}} = \mathbb{R}P_1 + \mathbb{R}P_2$. Our task is thus accomplished. □

Now it might come as a little surprise to the reader that the intersection $C^*(S_1) \cap C^*(S_2)$ is no longer $\mathbb{C}P_1 + \mathbb{C}P_2$ when one focuses on a Cuntz algebra $\mathcal{O}_n$ with $n \geq 3$. In fact, in this case that intersection is always trivial.

Theorem 2.30. *Let n be an integer greater than 2. If $S_1, S_2, \ldots S_n$ are the generating isometries of $\mathcal{O}_n$, then $C^*(S_i) \cap C^*(S_j) = \mathbb{C}1$ if $i \neq j$.*

Proof. There is no loss of generality if we take $i = 1$ and $j = 2$. The proofs we gave above continue to work. Accordingly, we still have:

1. $C^*(S_1)^{\mathbb{T}} = C^*(1, P_1, P_{11}, \ldots)$,
2. $P_1 C^*(S_2) = \mathbb{C}P_1$,
3. $C^*(S_1) \cap C^*(S_2) = C^*(S_1)^{\mathbb{T}} \cap C^*(S_2)^{\mathbb{T}} = C^*(1, P_1, P_{11}, \ldots) \cap C^*(1, P_2, P_{22}, \ldots)$.

Now let x be an element of $\in C^*(S_1) \cap C^*(S_2)$. Then $P_1 x \in \mathbb{C}P_1$, while $\sum_{k=2}^n P_k x \in \mathbb{C}(\sum_{k=2}^n P_k)$. This enables us to rewrite x as the sum $x = P_1 x + (\sum_{k=2}^n P_k)x = \lambda_1 P_1 + \lambda_2(\sum_{k=2}^n P_k)$. By symmetry, we can also rewrite x as $x = \tilde{\lambda}_1 P_2 + \tilde{\lambda}_2(\sum_{k\neq 2} P_k)$. If we equate these two expressions, we get the following relation:

$$\lambda_1 P_1 + \lambda_2 \Big(\sum_{k=2}^n P_k \Big) = \tilde{\lambda}_1 P_2 + \tilde{\lambda}_2 \Big(\sum_{k\neq 2} P_k \Big)$$

By looking at the coefficients of P_1, P_2, P_n, the three equalities $\lambda_1 = \tilde{\lambda}_2$, $\lambda_2 = \tilde{\lambda}_1$, and $\lambda_2 = \tilde{\lambda}_2$ are obtained respectively, which implies that x is a multiple of 1. □

3. Automorphism groups

3.1. Embedding topological groups into automorphism groups of the Cuntz algebras

Once one is given a C*-algebra, studying the structure of its automorphism group is not an idle problem as well as being a somewhat difficult issue to tackle even in commutative cases, where it reduces to dealing with the properties of the homeomorphism group of an assigned compact space. In fact, an in-depth understanding of the several automorphism groups of an assigned C*-algebra can often shed new light on many aspects of its theory unravelling the unexpected interplay between them. In this respect, the Cuntz algebras are no exception. The present section starts by discussing those automorphism groups of the Cuntz algebras that seem to us to be the most relevant to the current themes of research. As the title of the section itself suggests, the problem of embedding wider and wider classes of groups is then addressed.

In Section 2.3 we have introduced the gauge automorphisms of $\mathcal{O}_n$ of the form λ_{z1} for $z \in \mathbb{T}$. Since $\mathbb{T}$ is the unitary group of the one-dimensional Hilbert space $\mathbb{C}$, one might ask whether the higher dimensional unitary groups act on $\mathcal{O}_n$ as well. This is easily achieved by making use of the identification $\mathcal{F}_n^k \cong M_{n^k}$ explained above. In the case $k = 1$ the endomorphisms obtained in this way are known as Bogolubov automorphisms or quasi-free automorphsms, see [93]. Let $g \in \mathcal{U}(\mathcal{F}_n^1)$, we denote the associated Bogolubov automorphisms by α_g They are indeed automorphisms since it is straightforward to verify that the fusion rules in this case reduce to $\lambda_U\lambda_V = \lambda_{UV}$ for each $U, V \in \mathcal{F}_n^1$. These Bogolubov automorphisms provide a broad class of examples of outer automorphisms as was first proved by Enomoto, Takehana and Watatani [92]. Let G be a compact group. It is a consequence of the Peter-Weyl Theorem that G is a Lie group if and only if it has a faithful finite-dimensional representation (see [97, Theorem 5.13]). Thus, every Lie group can be seen as a closed subgroup of a unitary group. Since the correspondence $U \to \lambda_U$ is one-to-one, our swift overview on the Bogolubov automorphisms shows that in particular every compact Lie group embeds into $\mathrm{Aut}(\mathcal{O}_n)$ for some n. In particular, after the composition with canonical map from $\mathrm{Aut}(\mathcal{O}_n) \to \mathrm{Out}(\mathcal{O}_n)$ we also have an embedding in $\mathrm{Out}(\mathcal{O}_n)$. We observe that it is also possible to embed all compact Lie groups in $\mathrm{Aut}(\mathcal{O}_2)$. In fact, if $\psi : \mathcal{O}_n \otimes \mathcal{O}_2 \to \mathcal{O}_2$ is any isomorphism, we can consider the following embedding

$$G \to \mathrm{Aut}(\mathcal{O}_n) \to \mathrm{Aut}(\mathcal{O}_n \otimes \mathcal{O}_2) \to \mathrm{Aut}(\mathcal{O}_2)$$
$$g \to \alpha_g \to \alpha_g \otimes \mathrm{id}_{\mathcal{O}_2} \to \psi(\alpha_g \otimes \mathrm{id}_{\mathcal{O}_2})\psi^{-1}$$

That should already be enough to give but a rough idea how big $\mathrm{Aut}(\mathcal{O}_n)$ is.

Actions of finite groups, too, are worth studying. Notably, $\mathbb{Z}_2$ acts on $\mathcal{O}_2$ through the idempotent automorphism $\sigma \in \mathrm{Aut}(\mathcal{O}_2)$ that switches the two generators, that is to say $\sigma(S_1) = S_2$ and $\sigma(S_2) = S_1$. The automorphism σ is well known to be outer; a three-line proof of this is found in [55]. Following Archbold, [26], the map σ is customarily called the *flip-flop* automorphism and it can be written as λ_f with $f = S_1S_2^* + S_2S_1^*$. This action features unexpected nontrivial properties, which are the main concern of a paper by Choi and Latrémolière [55], where the fixed-point algebra $\mathcal{O}_2^\sigma$ and the crossed product $\mathcal{O}_2 \rtimes_\sigma \mathbb{Z}_2$ are both proved to be $*$-isomorphic to $\mathcal{O}_2$. Actually, the former result had been known for about ten years. In fact, Pinzari and Izumi (see [116, Example 3.7], [153]) proved independently that any finite group yields an isomorphism $\mathcal{O}_{|G|}^{\alpha_G} \cong \mathcal{O}_{|G|}$, where α_G is the regular action by Bogolubov automorphisms induced by the left regular representation.[9] When $G = \mathbb{Z}_2$ we can immediatey recognise the result stated

[9] See [120, Lemma 5.1] for a generalisation to arbitrary finite-dimensional unitary representations

above. Besides, the action itself had already been known for quite a long time before the above-named authors achieved their results. For instance, as far back as 1979 in the aforementioned paper [26] Archbold exhibited σ as an example of a remarkable outer automorphism of $\mathcal{O}_2$, but it was not until 2004 that Izumi [120] proved that it enjoys a form of Rohlin property.[10]

This enabled him to show that the two $\mathbb{Z}_2$ actions $x \otimes y \to y \otimes x$ and λ_u on $\mathcal{O}_2 \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$ and on $\mathcal{O}_2$, respectively, where $u(S_1) = S_1$ and $u(S_2) = -S_2$ are actually conjugate, and thus also conjugate to the regular quasi-free action.

We can now conclude with three theorems dealing with the possibility of embedding general classes of groups into $\mathrm{Aut}(\mathcal{O}_2)$ or even into $\mathrm{Out}(\mathcal{O}_2)$. Their proofs are also included, since we do not know any reference for them, and, more importantly, they should be new, albeit possibly already implicit in the vast literature, results.

The first one focuses on discrete countable groups.

Proposition 3.1. *Every countable group G embeds into* $\mathrm{Aut}(\mathcal{O}_2)$.

Proof. We shall distinguish two cases: finite groups and countable groups. The proof for the latter case also works in the former. However, we do believe that having around two different embeddings might turn out to be useful. In the first case we consider the left regular representation. Any element is thus associated with a unitary matrix. To any unitary matrix in $M_{|G|}(\mathbb{C})$ we can then associate a Bogolubov automorphism, and by doing so we get a map $G \to \mathrm{Aut}(\mathcal{O}_{|G|})$ given by $g \to \alpha_g$. Finally, considering the isomorphism $\phi : \mathcal{O}_{|G|} \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ we have the following map $G \to \mathrm{Aut}(\mathcal{O}_2)$ defined by $g \to \phi(\alpha_g \otimes \mathrm{id}_{\mathcal{O}_2})\phi^{-1}$.

Let us now consider the second case. As recalled in the first section, we have an isomorphism $\psi : \bigotimes_{g\in G} \mathcal{O}_2^{(g)} \to \mathcal{O}_2$, where $\mathcal{O}_2^{(g)} \doteq \mathcal{O}_2$. To each $h \in G$ we can associate the Bernoulli shift λ_h which sends $\mathcal{O}_2^{(g)}$ to $\mathcal{O}_2^{(hg)}$. The composition with ψ and ψ^{-1} as in the previous case gives the sought map. $\square$

The second result has to do with embedding locally compact groups instead.

Proposition 3.2. *Any 2nd countable, locally compact group embeds into* $\mathrm{Aut}(\mathcal{O}_2)$.

Proof. Let G be any group as in the statement and let $u\colon G \to \mathcal{U}(L^2(G,\mu))$ be the left regular representation, where μ is the left Haar measure on G. As G is second countable, the Hilbert space $L^2(G,\mu)$ is isomorphic to $\mathcal{H} \doteq \overline{\mathrm{span}\{S_i\}_{i=1}^{\infty}} \subset \mathcal{O}_\infty$. Each $u(g)$ is a unitary operator on $\mathcal{H}$. By the Cuntz uniqueness theorem it induces

of G.

[10] More precisely, he proved that the action of $\mathbb{Z}_2$ implemented by the flip-flop satisfies the Rohlin property for actions α of finite groups G on a C^*-algebra $\mathcal{A}$. This asks that there exists a partition of unity $\{e_g : g \in G\} \subset \mathcal{A}_\infty$ made of self-adjoint projections such that $(\alpha_g)_\infty(e_h) = e_{gh}$, for every $h \in G$. Here $\mathcal{A}_\infty$ is by definition the intersection $\mathcal{A}^\infty \cap A'$, where $\mathcal{A}^\infty$ is the quotient C^*-algebra $\ell^\infty(\mathcal{A})/c_0(\mathcal{A})$: $\ell^\infty(\mathcal{A})$ is the C^*-algebra of all bounded sequences in $\mathcal{A}$, whereas $c_0(\mathcal{A})$ is the closed two-sided ideal of vanishing sequences. Note that A can be thought of as a subalgebra of $\mathcal{A}^\infty$ if we associate to any $a \in \mathcal{A}$ the equivalence class of th constant sequence always equal to a. Finally, if α is an automorphism of $\mathcal{A}$, α_∞ is the restriction to $\mathcal{A}_\infty$ of the natural extension of α to $\mathcal{A}^\infty$.

an automorphism of $\mathcal{O}_\infty$. We denote this automorphism by α_g. Finally, the morphism $\alpha_g \otimes 1_{\mathcal{O}_2} \in \mathrm{Aut}(\mathcal{O}_\infty \otimes \mathcal{O}_2) \cong \mathrm{Aut}(\mathcal{O}_2)$ gives the embedding whose existence is claimed in the statement. $\square$

Though the last theorem we have proved can actually be thought as a generalization of the first, we do prefer to give two different independent proofs, sure as we are they are both interesting, insofar as they emphasize different aspects of the theory. Moreover, the latter may be further improved. To this aim, we need a technical lemma first.

Lemma 3.3. *Let A be a C^*-algebra with trivial centre. If $\alpha \in \mathrm{Out}(A)$, then $\alpha \otimes \mathrm{id} \in \mathrm{Out}(A \otimes A)$.*

Proof. Suppose that $\alpha \otimes \mathrm{id}$ is inner, i.e there exists a $u \in \mathcal{U}(A \otimes A)$ such that $\alpha \otimes \mathrm{id}(\cdot) = u \cdot u^*$. In particular

$$(\alpha \otimes \mathrm{id})(1 \otimes b) = u(1 \otimes b)u^* = 1 \otimes b \quad \forall b \in A,$$

which implies that

$$(1 \otimes b)u = u(1 \otimes b).$$

Thus $u \in (1 \otimes A)'$. By virtue of [114, Theorem 1], we have that $(1 \otimes A)' = A \otimes 1$. This means that $u = v \otimes 1$ for some $v \in \mathcal{U}(A)$. Therefore

$$\begin{aligned}(\alpha \otimes \mathrm{id})(a \otimes 1) &= (v \otimes 1)(a \otimes 1)(v^* \otimes 1) = \\ &= (vav^* \otimes 1).\end{aligned}$$

Thus α is inner, which is not. $\square$

Clearly, this lemma applies to automorphisms of $A \otimes B$ as well, mutatis mutandis, and thus it can be applied to $\mathcal{O}_\infty \otimes \mathcal{O}_2 \simeq \mathcal{O}_2$.

Here is the promised result.

Theorem 3.4. *Every second countable, locally compact group embeds into* $\mathrm{Out}(\mathcal{O}_2)$.

Proof. If the group is finite, the thesis follows from *Proposition* 3.1 and the fact that Bogolubov automorphisms are outer. If the group is not finite, instead, the result follows from *Proposition* 3.2 and [25, Theorem 4.5]. $\square$

By the same reasoning, it is also clear that the unitary group $\mathcal{U}(\mathcal{H})$ of a separable Hilbert space $\mathcal{H}$ embeds into both $\mathrm{Aut}(\mathcal{O}_2)$ and $\mathrm{Out}(\mathcal{O}_2)$. This is a rather interesting fact since $\mathcal{U}(\mathcal{H})$ is a Polish space when thought as being endowed with the strong operator topology. In this regard, we would like to to warn the reader against overly facile conclusions. The unitary group, in fact, is not strongly closed, as one might naively expect, for a sequence of unitaries may well converge to a proper isometry, see [167] for an enlightening example. However, it is closed under the strong* topology, as immediately checked. Since on the unitary group the two topologies do coincide as well as being the same as the weak topology, we have that

$\mathcal{U}(\mathcal{H})$ is a Polish group with respect to the strong topology too. We also provide the reader with yet another proof. It is actually a bit more involved, but it has the undeniable advantage of giving more information. To state the theorem, we first need to recall that $\mathrm{Iso}(\mathcal{H})$ denotes the semigroup of the isometries of $\mathcal{H}$ into itself.

Theorem 3.5. *$\mathcal{U}(\mathcal{H})$ is a G_δ-subset of $\mathrm{Iso}(\mathcal{H})$ with respect to the strong topology. In particular, $\mathcal{U}(\mathcal{H})$ is a Polish group.*

Proof. First, note that $\mathrm{Iso}(\mathcal{H})$, being a strongly closed subset of the unit ball of $\mathcal{B}(\mathcal{H})$, is strongly complete. In addition, it is also strongly separable because $\mathcal{H}$ is norm separable by assumption. In order to prove that $\mathcal{U}(\mathcal{H})$ is a G_δ-subset of $\mathrm{Iso}(\mathcal{H})$, we observe that for an isometry T to be surjective is enough to have dense range. Now let $\{x_i : i \in \mathbb{N}\} \subset \mathcal{H}$ be a dense sequence in $\mathcal{H}$. We can define $A_{i,k} \doteq \{T \in \mathrm{Iso}(\mathcal{H}) : d(x_i, \mathrm{Ran}T) < \frac{1}{k}\}$. By a straightforward application of the triangle inequality, each $A_{i,k}$ is easily seen to be a strongly open set. To conclude, it is sufficient to note that $\mathcal{U}(\mathcal{H}) = \bigcap_{i,k} A_{i,k}$, in which writing we immediately recognize $\mathcal{U}(\mathcal{H})$ as a G_δ. □

However large it may appear at first glance, $\mathcal{U}(\mathcal{H})$ is not a universal Polish group. In fact, $\mathrm{Homeo}([0,1]^{\mathbb{N}})$ does not embed into it, cf. [169].

3.2. Seeking Automorphisms

One of the most demanding tasks to face when dealing with the endomorphisms of the Cuntz algebras is to find manageable necessary and sufficient conditions for a unitary $U \in \mathcal{O}_n$ in order that the associated endomorphism λ_U be an automorphism, *i.e.* $\lambda_U(\mathcal{O}_n) = \mathcal{O}_n$. As we saw in the first section, for a λ_U to be surjective is enough to prove that U belongs to $\lambda_U(\mathcal{O}_n)$. Yet the condition is admittedly too self-referential to be really useful in practice. This is arguably why the problem, in its full generality, has hitherto remained open. Notwithstanding this inherent difficulty, important progress has been made in a work by Conti and Szymański [75], where the authors achieved conclusive results for unitaries of special kinds. Indeed, they gave quite a satisfactory answer when $u \in \mathcal{F}_n^k$, which is what we are going to discuss now.

Following Longo, when $U \in \mathcal{U}(\mathcal{F}_n^k)$ (for some $k \in \mathbb{N}$) the corresponding endomorphism λ_U is called a localized endomorphism, for the inclusions $\lambda_U(\mathcal{F}_n^h) \subset \mathcal{F}_n^{k+h-1}$ hold for each h, k. Set $\mathrm{End}_{loc}(\mathcal{O}_n) \doteq \{\lambda_U \text{ s.t. } U \in \mathcal{F}_n^k\}$ and $\mathrm{Aut}_{loc}(\mathcal{O}_n) \doteq \mathrm{Aut}(\mathcal{O}_n) \cap \mathrm{End}_{loc}(\mathcal{O}_n)$. Out of a single unitary $U \in \mathcal{U}(\mathcal{O}_n)$ we can get a whole sequence of unitaries $\{U_n : n \in \mathbb{N}\} \subset \mathcal{U}(\mathcal{O}_n)$ if we set $U_n \doteq U\varphi(U)\varphi(U)^2 \ldots \varphi(U)^{n-1}$, where φ is the canonical endomorphism of $\mathcal{O}_n$. We are now ready to state the announced result, which is singled out below for the reader's convenience:

Theorem 3.6. *([75], Theorem 3.2, Corollary 3.3) Let $U \in \mathcal{U}(\mathcal{F}_n^k)$. The localized endomorphism λ_U is an automorphism with localized inverse λ_U^{-1} if and only if the sequence $\{\mathrm{Ad}(U_r^*)(U^*) : r \in \mathbb{N}\}$ eventually stabilizes. Moreover, $\lambda_U^{-1} = \lambda_V$, where $V \in \mathcal{U}(\mathcal{F}_n^h)$ is the limit of the earlier sequence and h is at most $n^{2(k-1)}$.*

Besides, is $n^{2(k-1)}$ the optimal bound? Likely it may be improved.

Lest the reader get confused, we should immediately remark that in their characterization of localizable automorphisms the authors require *a priori* that the inverse should be localized too, which is an assumption one would like to dispense with. If it cannot be dropped, instead, one would like to be able to show explicit counterexamples.

As is well known, real numbers can be described by continued fractions. Rational numbers are precisely those numbers whose continued fraction representation is finite, whereas irrational numbers are those with infinite continued fraction representation. This vague analogy suggests that the unitaries satisfying the conditions in the above theorem should be called rational unitaries, and this is exactly what they will be called in the present survey as of now.

Question 3.7. Is it possible to find more substantial reasons to justify this name?

In [75, 67] a particular family of unitary elements, along with the associated automorphisms, was studied. To be more precise, it has been possible to compute explicitly how many different automorphisms of $\mathcal{O}_n$ arise from the permutation matrices in $\mathcal{P}_n^k$ when $n+k \leq 6$ (and partially for $n+k=7$, [20]). However, an insightful interpretation for the numbers thus obtained is still missing. In a more mathematical jargon, one can therefore ask the following, which no doubt deserves attention:

Question 3.8. Is it possible to exhibit an explicit closed formula for the generating function of these (and related) numbers?

In a series of recent papers with Brenti and Nenashev the case of cycles has been examined in great detail and many enumerative formulas have been obtained for $k=2$, but arbitrary n [45, 46].

Another question related to these automorphisms is the following:

Question 3.9. What properties do the restriction of these automorphisms to the algebra $\mathcal{F}_n$ and $\mathcal{D}_n$ have?

More concretely, at least for the remarkable case $n=2$, all the outer automorphisms of $\mathcal{O}_2$ discovered in [75] restrict to outer automorphisms of the CAR algebra $\mathcal{F}_2$. Therefore, one might wonder what their explicit description in terms of the natural generators of $\mathcal{F}_2$ is. That description may already exist if these automorphism have formerly appeared in the vast literature devoted to the CAR algebra. However, we do not know of any result toward this problem.

The following question, too, is natural to ask being closely related to the previous.

Question 3.10. Are there natural procedures that yield families of unitaries giving rise to outer automorphisms?

In order to have a better understanding of the automorphism groups of the Cuntz algebras it might be useful to fully understand as many cases as possible.

For example, one might consider permutation matrices where 1 is replaced by complex numbers of modulus one, e.g. suitable roots of unity, or also the so-called Hadamard unitaries, cf. [65, Problem 3.1 (a)]. Besides, it is easy to see that the product unitaries, i.e. unitaries of the form $\prod_{r=1}^{k}\varphi^{r-1}(u_r)$ with $u_r \in \mathcal{F}_n^1$ for all $r = 1, \ldots, k$ are rational, thus in a sense the rational unitaries are those for which there is a kind of weak interaction/correlation between the tensor factors.

Example 3.11. As we saw in the second section, the requirement that a unitary $V \in \mathcal{O}_n$ intertwines an endomorphism λ_U is encoded by an equation satisfied by V. Fixed a natural number k, the condition $U \in (\lambda_U^k, \lambda_U^k)$ is accordingly a (non-linear) equation as well, whose solutions provide interesting examples of non-rational unitaries. In the second section, too, the case corresponding to $k = 1$ was seen not to be of much interest since the associated equation has only the trivial solutions, i.e. scalar multiples of the identity. Higher values of the natural parameter k do yield interesting equations, though. The very first thing to note is that if there exists a non-trivial unitary U such that $U \in (\lambda_U^k, \lambda_U^k)$ for some $k \geq 2$, then λ_U must be a proper endomorphism: this is a general remark indeed that only depends on $\mathcal{O}_n$ being a simple C*-algebra. However, we are still working in too much generality to be able to write down manageable equations that may be treated more explicitly. The most convenient thing to do, therefore, is to specialize our equation to cover particular but still remarkable cases. To this aim, we shall not only focus on particular values of k, but we shall also need to limit ourselves to particular classes of unitaries. More precisely, we shall consider those unitaries belonging to the finite-dimensional subalgebras of the type $\mathcal{F}_n^h$, which will be useful to think of as identified with the tensor product $M_n^{\otimes h}$. For a unitary $U \in \mathcal{F}_n^h$ the corresponding $\lambda_U(U)$ can more suitably re-expressed as

$$\lambda_U(U) = U\varphi(U)\varphi^2(U)\ldots\varphi^{h-1}(U)U\varphi^{h-1}(U^*)\ldots\varphi(U^*)U^* \in \mathcal{F}_n^{2h-1} \quad (3.1)$$

The most elementary non-trivial case is obtained by taking $h = k = 2$. The equation this choice leads to is not less than the Yang-Baxter equation on $\mathbb{C}^n$, as the following computation shows:

$$U\lambda_U^2(S_i) = \lambda_U^2(S_i)U$$
$$U\lambda_U(US_i) = \lambda_U(US_i)U$$

by multiplying by S_i^* on the right and then summing over i, we come to the equation

$$U\lambda_U(U)U = \lambda_U(U)U\varphi(U)$$

substituting the corresponding expression for $\lambda_U(U)$ as recalled in 3.1, we finally obtain

$$U\varphi(U)U = \varphi(U)U\varphi(U)$$

as claimed. After the first re-interpretation of the Yang-Baxter equation in the context of Cuntz algebras [80], many more aspects were examined [69, 61, 68].

Taking a step further with $h = 3$ and $k = 2$, we can also get to a more involved equation. In this case λ_U^2 assumes a more complicated form:

$$\lambda_U^2 = \lambda_U \circ \lambda_U = \lambda_{\lambda_U(U)U} = \lambda_{U\varphi(U)\varphi^2(U)U\varphi^2(U^*)\varphi(U^*)}$$

By means of analogous computations we can write down the following equation (in $\mathcal{F}_n^5$)

$$\varphi(U)\varphi^2(U)U\varphi^2(U^*)\varphi(U)\varphi^2(U) = U\varphi(U)\varphi^2(U)U \tag{3.2}$$

It is worthwhile to note that any solution $U \in \mathcal{F}_n^2 \subset \mathcal{F}_n^3$ of the Yang-Baxter equation also solves the equation obtained above, which enables us to regard it as a sort of a generalization of the Yang-Baxter equation itself.

As for the case $h = 2$ and $k = 3$, the following formula is also nedeed for the computations involved:

$$\lambda_U(\varphi(U)) = U\varphi(U)\varphi^2(U)\varphi^2(U^*)\varphi(U^*)$$

In fact,

$$\begin{aligned}
\lambda_U(\varphi(U)) &= \lambda_U(\sum_i S_i U S_i^*) \\
&= \sum_i U S_i \varphi(U) U \varphi(U^*) U^* S_i^* U^* \\
&= U\varphi(U)\varphi^2(U)\varphi^2(U^*)\varphi(U^*).
\end{aligned}$$

The next step to take is to compute λ_U^3

$$\begin{aligned}
\lambda_U^3 &= \lambda_U \circ \lambda_U^2 \\
&= \lambda_U \circ \lambda_{\lambda_U(U)U} \\
&= \lambda_{\lambda_U(\lambda_U(U)U)U}
\end{aligned}$$

we are now left with the task of computing $\lambda_U(\lambda_U(U)U)$:

$$\begin{aligned}
\lambda_U(\lambda_U(U)U)U &= \lambda_U(U\varphi(U)U\varphi(U^*))U \\
&= U\varphi(U)U\varphi^2(U)\varphi(U)\varphi^2(U^*)U\varphi^2(U)\varphi(U^*)\varphi^2(U^*)\varphi(U^*)
\end{aligned}$$

Since $U \in (\lambda_V, \lambda_V)$ if and only if $UV = V\varphi(U)$, the equation $U \in (\lambda_U^3, \lambda_U^3)$ becomes

$$\begin{aligned}
&U\varphi(U)U\varphi^2(U)\varphi(U)\varphi^2(U^*)U\varphi^2(U)\varphi(U^*)\varphi^2(U^*)\varphi(U^*) \\
&\quad = \varphi(U)U\varphi^2(U)\varphi(U)\varphi^2(U^*)U\varphi^2(U)\varphi(U^*)\varphi^2(U^*)
\end{aligned} \tag{3.3}$$

(in $\mathcal{F}_n^4$) which is again a generalization of the Yang-Baxter equation.

Unfortunately, the cases corresponding to greater values of the parameters h and k entailed so time-consuming computations that we decided not to perform them fully. Nonetheless they might be a good start for the interested and sedulous reader to discover new equations possibly linked to already known equations, like those arising in the theory of integrable systems or in the representation theory of remarkable groups. Notably, we should like to raise the following issue.

Question 3.12. Are there any values of h and k such that the corresponding equation reproduces (up to a suitable equivalence), or at least resembles the so-called tetrahedron equation? Consider e.g. $R_{abc}R_{ade}R_{bdf}R_{cef} = R_{cef}R_{bdf}R_{ade}R_{abc}$ on $V_a \otimes V_b \otimes V_d \otimes F_c \otimes F_e \otimes F_f$, [31, eq. (3)].

Going back to the main theorem, it is worth emphasizing that not only does it give an elegant characterization of those unitaries yielding automorphisms, but it also says that they form a real algebraic variety in $\mathcal{U}(\mathcal{F}_n^k)$ (for any fixed k), which would be interesting to investigate with methods of algebraic geometry. For instance, one may wonder whether this variety is connected (or even irreducible) or whether it has singular points, just to take but two examples.

As $\lambda_{U(\mathcal{F}_n^k)}(\mathcal{F}_n^h) \subset \mathcal{F}_n^{k+h-1}$ and $\lambda_U\lambda_V = \lambda_{\lambda_U(V)U}$, it follows that $\mathrm{Aut}_{loc}(\mathcal{O}_n)$ is a semigroup. At this stage, a problem that is easily formulated but is difficult to work out is the following:

Question 3.13. Is $\mathrm{Aut}_{loc}(\mathcal{O}_n)$ a group? Namely, if $U \in \mathcal{F}_n^k$ and $\lambda_U \in \mathrm{Aut}(\mathcal{O}_n)$, is true that $\lambda_U^{-1} = \lambda_V$ for some $V \in \mathcal{F}_n^h$?

As for the general formulation of the problem of whether an endomorphism is actually an automorphism, there is an interesting result by Conti that should be mentioned here: any unital endomorphism of $\mathcal{O}_n$ which fixes the diagonal subalgebra $\mathcal{D}_n$ pointwise is automatically surjective. This fact is explicitly pointed out in [71], where, as well as being proved, it is also said to be already implicit in [77]. However, the interplay between $\mathcal{O}_n$ and its diagonal subalgebra $\mathcal{D}_n$ is more complicated than one might expect based on our discussion. For instance, in [71] it is shown that there exist product-type automorphisms of $\mathcal{D}_n$ that do not extend to endomorphisms of $\mathcal{O}_n$ (yet they do extend to automorphisms of $\mathcal{F}_n$). Besides, a longstanding problem posed by Cuntz himself illustrates further how subtle and little understood this interplay is. It asks if for any $\lambda_U \in \mathrm{Aut}(\mathcal{O}_n)$ there is a unitary $V \in \mathcal{O}_n$ such that $\lambda_U(\mathcal{D}_n) = \mathrm{Ad}(V)(\mathcal{D}_n)$, i.e. the two conjugate MASAs $\mathcal{D}_n$ and $\lambda_U(\mathcal{D}_n)$ are actually inner conjugate. Remarkably, this is not always the case, most of quasi-free automorphisms providing counterexamples [66] (see also [112, 113] for a recent generalization of this result in the setting of graph algebras). Finally, the subalgebra $\mathcal{D}_n$ plays a vital role when meaningful decompositions of $\mathcal{O}_n$ as crossed products are sought. In this spirit, Spielberg [166] proved that $\mathcal{O}_{2m} = \mathcal{D}_{2m} \rtimes \Gamma_{2m}$, where Γ_{2m} is the (non-amenable) group $\mathbb{Z}_m \times (\mathbb{Z}_2 * \mathbb{Z}_{2m+1})$ thought of as a subgroup of $\mathcal{S}_{2m}$, which acts on $\mathcal{O}_{2m}$ by inner automorphisms, under which the diagonal subalgebra is globally invariant. The odd case, though, is out of the reach of his results. As far as we know, no-one has proved analogous results for the odd cases either, which is an issue we would like to raise here:

Question 3.14. Does there exist a decomposition of $\mathcal{O}_{2n+1}$ as a crossed product of the diagonal subalgebra $\mathcal{D}_{2n+1}$ by a (possibly non-amenable) countable group $\Gamma \subset \mathcal{S}_{2n+1}$?

B. Blackadar discovered an automorphism α of the UHF subalgebra such that $\alpha^2 = \mathrm{id}$ and the fixed-point algebra is not an AF-algebra, [37]. This immediately leads to the following problem:

Question 3.15. Does there exist an automorphism $\phi : \mathcal{O}_n \to \mathcal{O}_n$ such that $\phi \restriction_{\mathcal{F}_n} = \alpha$? More generally, are there any necessary and sufficient conditions for an endomorphism of $\mathcal{F}_n$ to extend to $\mathcal{O}_n$? For instance, if $\alpha \in \mathrm{Aut}(\mathcal{F}_n)$ eventually commutes with the shift, does it extend to an automorphism of $\mathcal{O}_n$? Does the Blackadar automorphism "eventually commute with the shift"[11]?

A question that is somehow related to the former group of problems is whether a τ-invariant automomorphism $\alpha \in \mathrm{Aut}(\mathcal{D}_n)$ automatically extends to an endomorphism of $\mathcal{F}_n$ or even to an automorphism.

4. Notable groups associated with the Cuntz algebras

4.1. Constructive and non-constructive aspects

Now that we have seen some of the principal themes concerning the Cuntz algebras, we are ready to collect a number of open problems of different nature. While some of them mostly concern their representation theory, states and automorphisms, others are more linked to actions of suitable groups, on which we will focus presently, as well as presenting their interplay with the huge research field of the classification of C*-algebras. Some might well have already been pointed out elsewhere, some definitely have been in a survey by Conti and Szymański, [73], to which the reader is referred for more information. Finally, some problems should have been raised for the first time instead. Whenever we are aware of a problem having already been posed somewhere, we shall give suitable references. The list we are about to make is a bit lengthy by our own admission, and at first glance the problems herein collected might appear unrelated to one another. However inevitably farraginous the exposition of the problems may be, it should help the reader grasp the many connections between the several different issues being raised all the same – at least this is our hope. This said, we can now proceed without further ado.

As we have seen, there exists an exotic isomorphism $\psi\colon \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$, which was actually discovered by Elliott before Kirchberg proved his celebrated theorem, from which its existence can be immediately inferred as we did in the first section. Thereafter, the proof was simplified by Rørdam in [160]. Let $\mathcal{L}_2 \subset \mathcal{O}_2$ be the $*$-algebra generated by S_1, S_2. In spite of $\mathcal{O}_2 \otimes \mathcal{O}_2$ being isomorphic to $\mathcal{O}_2$, it is possible to prove that $\mathcal{L}_2 \otimes_{alg} \mathcal{L}_2$ is not isomorphic to $\mathcal{L}_2$. As far as we know, P. Ara and G. Cortinas were the first who explicitly proved this result in their paper

[11] Let α be an automorphism of $\mathcal{F}_n$. We recall that α eventually commutes with φ if there exists some non-negative integer m such that $\alpha\varphi^m$ commutes with φ.

[24]. Since we fear that the reader may be dismayed by this result, we also include a rather quick review of the main ingredients used to concoct its proof. As recalled at the beginning of section 4 of the above-mentioned paper, to which the reader is obviously referred for a fuller coverage, it is possible to define the so-called Leavitt path algebra associated with a graph. It then turns out that it can be identified with the *-algebra:

$$L(E) = \mathrm{span}\{\alpha\beta^* | \alpha, \beta \in E^*, r(\alpha) = r(\beta)\}$$

where E^* is the set of all finite paths, r is the map from edges to vertices sending an edge to the range vertex, and the adjoint map sends an edge to its reverse. In light of this last remark, $\mathcal{L}_2$ is easily seen to be the Leavitt path algebra associated with $L(E_2)$, where E_2 is the graph with one vertex and two arrows. By using *Theorem* 5.1 in [24], the sought result is finally obtained. For the sake of completeness we should also mention that in the same work the authors take the opportunity to recall that an alternative proof, due to Bell and Bergman, is available as well. Unlike the proof provided by Ara and Cortinas, theirs uses classic theoretic-module constructions, see [34] for many more details. If we now say that an exotic isomorphism $\psi\colon \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ is constructive when it sends $\mathcal{L}_2$ into $\mathcal{L}_2 \otimes \mathcal{L}_2$ and non-constructive otherwise, the discussion outlined above raises the following problem, which is formulated in terms of a couple of mutually exclusive questions.

Question 4.1. Is any exotic isomorphism automatically non-constructive? Can an example of constructive ψ be exhibited instead?

We end this discussion by remarking that in [70, Cor. 3.8], a *-homomorphism $\sigma : \mathcal{O}_2 \otimes \mathcal{O}_2 \longrightarrow \mathcal{O}_2$ such that $\sigma(\mathcal{F}_2 \otimes \mathcal{F}_2) \subset \mathcal{F}_2$ has been shown to exist, which poses some natural questions:

Question 4.2. Is it possible to define a *-homomorphism $\phi : \mathcal{O}_2 \otimes \mathcal{O}_2 \longrightarrow \mathcal{O}_2$ such that one has $\phi(\mathcal{F}_2 \otimes \mathcal{F}_2) = \mathcal{F}_2$? Furthermore, one might ask whether there exists an isomorphism ϕ as above such that $\phi(\mathcal{F}_2 \otimes \mathcal{F}_2) \subset \mathcal{F}_2$ or even $\phi(\mathcal{F}_2 \otimes \mathcal{F}_2) = \mathcal{F}_2$. It goes without saying that it is natural to ask all the above questions for $\mathcal{D}_2 \subset \mathcal{O}_2$ as well.

We would like to note that associated with any isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ there is a corresponding continuous embedding $\mathrm{Aut}(\mathcal{O}_2) \times \mathrm{Aut}(\mathcal{O}_2) \subset \mathrm{Aut}(\mathcal{O}_2)$. This fact may be regarded as a kind of self-similarity property enjoyed by the group $\mathrm{Aut}(\mathcal{O}_2)$ itself. Among the several questions in this direction, one might wonder whether some of the relevant groups in this survey fit within the framework depicted in [35] or similar approaches taken elsewhere. To our knowledge no suitable notion of self-similarity/fractality seems to exist for non-commutative C*-algebras although such notions have been introduced for certain groups, see for instance [148]. In this respect, it would no doubt be interesting to develop such a theory for non-commutative C*-algebras as well. As far as only commutative cases are dealt with, two approaches are already available: one has to do with self-similarity of the

Gelfand spectrum, while the other applies similar ideas to the algebra itself, thus leading to a notion one would rather be attempted to regard as a co-self-similarity property. We should finally mention that a categorical approach to self-similarity has also been proposed in [141]. We end this discussion by recalling that in the literature devoted to the classification of nuclear C*-algebras a great many examples are known of algebras A such that $A \cong A \otimes A$, where there is no need to specify what completion is meant by the symbol $\otimes$, as the algebras being dealt with are nuclear by assumption. These are often referred to as self-absorbing C*-algebras. To begin with, in the classical setting a commutative C*-algebra $C(X)$ is self-absorbing exactly when its Gelfand spectrum X is homeomorphic to the product $X \times X$, as easily checked. Examples of such compact spaces abound. Notably, Cantor spaces are definitely of this type. To the best of our knowledge, the converse is not known yet. More precisely, one can ask whether any compact space X of topological dimension zero such that $X \cong X \times X$ is necessarily a Cantor space. The condition on the topological dimension must needs be included to avoid coming across trivial counterexamples, such as $X \doteq \mathbb{T}^{\mathbb{N}}$, which is obviously homeomorphic to $X \times X$ without being a Cantor space. As for non-commutative algebras, the compact operator algebra $\mathcal{K}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$, the Cuntz algebras $\mathcal{O}_2$ and $\mathcal{O}_\infty$, UHF algebras of infinite type, the Jiang-Su algebra $\mathcal{Z}$ are all well-known examples of self-absorbing C*-algebras. However, if we are to give a more accurate and up-to-date list, we must also mention a comparatively recent and perhaps lesser-known object, the so-called Jacelon algebra $\mathcal{R}$. First appeared in [123], $\mathcal{R}$ has there been proved to have noticeable properties, especially to those people interested in the classification program initiated by G. Elliott, and has since received much attention. Even so, $\mathcal{R}$ has not been proved to be a self-absorbing C*-algebra though it is widely believed to be such an algebra. Going back to the general discussion, there is no a priori reason to focus on C*-algebras only. Although the material we have been expounding so far is a bit biased towards C*-algebra theory, we do think it might be worth considering von Neumann algebras as well. The tensor product between any two of them is now understood in the spatial sense: if $\mathfrak{R}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathfrak{R}_2 \subset \mathcal{B}(\mathcal{H}_2)$ are von Neumann algebras, then $\mathfrak{R}_1 \otimes \mathfrak{R}_2$ is the von Neumann algebra acting on the Hilbert tensor product space $\mathcal{H}_1 \otimes \mathcal{H}_2$ generated by the operators $T_1 \otimes T_2$ with $T_i \in \mathfrak{R}_i$. A von Neumann algebra $\mathfrak{R}$ is then said to be self-absorbing if $\mathfrak{R} \cong \mathfrak{R} \otimes \mathfrak{R}$. Examples are $\mathcal{B}(\mathcal{H})$ for any infinite-dimensional Hilbert space $\mathcal{H}$, and R, the hyperfinite II_1 factor.

Recall that one has $\mathcal{F}_n = (\mathcal{O}_n)^{\mathbb{T}}$, $\mathrm{Aut}_{\mathcal{F}_n}(\mathcal{O}_n) = \{\lambda_{z1} \mid z \in \mathbb{T}\}$, $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) = \{\lambda_u \in \mathrm{Aut}(\mathcal{O}_n) \mid u \in \mathcal{U}(\mathcal{F}_n)\} =: \lambda(\mathcal{U}(\mathcal{F}_n))^{-1}$ and $N_{\mathcal{F}_n}(\mathcal{O}_n) = \mathcal{U}(\mathcal{F}_n)$; indeed, if $u \in \mathcal{U}(\mathcal{O}_n)$ satisfies $u\mathcal{F}_n u^* = \mathcal{F}_n$ one has, for every $x \in \mathcal{F}_n$ and $z \in \mathbb{T}$, $\lambda_{z1}(uxu^*) = \lambda_{z1}(u)x\lambda_{z1}(u^*) = uxu^*$, that is $u^*\lambda_{z1}(u) \in \mathcal{F}_n' \cap \mathcal{O}_n = \mathbb{C}$, and it readily follows that $u \in \mathcal{U}(\mathcal{F}_n)$. Moreover, $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) = \lambda(\mathcal{U}(\mathcal{D}_n))^{-1} = \{\lambda_d \mid d \in \mathcal{U}(\mathcal{D}_n)\} \simeq \mathcal{U}(\mathcal{D}_n)$, $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) = \lambda(N_{\mathcal{D}_n}(\mathcal{O}_n))^{-1}$ and $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) = \lambda(N_{\mathcal{D}_n}(\mathcal{F}_n))^{-1}$, where $N_{\mathcal{D}_n}(\mathcal{O}_n) = \mathcal{U}(\mathcal{D}_n) \cdot \mathcal{S}_n$ and $N_{\mathcal{D}_n}(\mathcal{F}_n) = \mathcal{U}(\mathcal{D}_n) \cdot \mathcal{P}_n$ [75].

Denote by π the canonical projection of $\mathrm{Aut}(\mathcal{O}_n)$ on $\mathrm{Out}(\mathcal{O}_n)$, where $\mathrm{Out}(\mathcal{O}_n)$ is the quotient group $\mathrm{Aut}(\mathcal{O}_n)/\mathrm{Inn}(O_n)$. We can now consider the following groups:

$$\big(\mathrm{Aut}(\mathcal{O}_n,\mathcal{D}_n)\cap\mathrm{Aut}(\mathcal{O}_n,\mathcal{F}_n)\big)/\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)\ ;\ \mathrm{Aut}(\mathcal{O}_n,\mathcal{D}_n)/\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$$

They are often referred to as the *restricted Weyl group* and the *Weyl group* of $\mathrm{Aut}(\mathcal{O}_n)$, respectively, whereas their images through π are known as the *outer restricted Weyl group* and the *outer Weyl group*. Here is a couple of natural questions one may possibly start with:

Question 4.3. Do the various Weyl groups depend on n, as abstract groups?

It turns out that the restricted Weyl is isomorphic (under restriction to $\mathcal{D}_n$) with the group of those homeomorphisms of the full n-shift space X_n that eventually commute, along with their inverses, with the shift map. Notice that the group of homeomorphisms of X_n commuting with the shift is much smaller, for instance it is isomorphic with $\mathbb{Z}_2$ for $n = 2$. Before proceeding with further observations about the Weyl groups, we need to recall the following definition. A group G is said to be residually finite if for every element $g \in G$ with $g \neq e$ there exist a finite group F and a surjective homomorphism $\psi\colon G \to F$ such that $\Psi(g) \neq e$. The condition for a group G to be residually finite can be re-expressed in a variety of ways, for instance G is residually finite if and only if it embeds into the direct product of a family of finite groups. From this the class of residually finite groups is seen at once to be closed under both inclusions and direct products, whereas quotients of residually finite groups may fail to be residually finite. In fact, every finitely generated group is a quotient of a free group, and free groups are known to be residually finite. Yet not every finitely generated group is residually finite, as is known. The outer restricted Weyl group has been shown to be residually finite in [62, Prop. 4.1.7]. In addition, in [64, Cor. 5.3] the outer Weyl group is shown to be strictly bigger than the outer restricted Weyl group. Although we have no precise tool to say how marked this difference is, we do expect the outer Weyl group not to be residually finite instead. Also, it is shown in [62] that, for n prime, the restricted outer Weyl group is (abstractly) isomorphic to a group that has been extensively studied in the theory of dynamical systems/symbolic dynamics, namely the automorphism group of the two-sided shift (the group of homeomorphisms of the bilateral n-shift space $\{1, 2, \ldots, n\}^{\mathbb{Z}}$ that commute with the shift map) divided by its center.

Question 4.4. Is there some dynamical characterization of the outer Weyl group as well, at least for some values of n?

In the context of graph algebras, some partial generalizations of the above results dealing with the analogous notions of restricted Weyl groups are discussed in [63].

Question 4.5. Is there a characterization of those homeomorphisms of the shift space X_n arising from unitaries $w \in \mathcal{S}_n$ with $\lambda_w(\mathcal{D}_n) = \mathcal{D}_n$ (but with the λ_w not necessarily automorphisms of $\mathcal{O}_n$)?

Results of this type for $w \in \mathcal{P}_n$ are discussed in [62, Section 3], especially Theorem 3.6 therein.

Question 4.6. Is it possible to develop a Weyl chamber theory for these Weyl groups?

Contrary to what might be hoped, it is shown in [66, Theorem 3.7] that given $\alpha \in \mathrm{Aut}(\mathcal{O}_n)$ it is not necessarily true that one can find $U \in \mathcal{U}(\mathcal{O}_n)$ such that $\alpha(\mathcal{D}_n) = \mathrm{Ad}(U)(\mathcal{D}_n)$. Even so, one might wonder all the same whether there exist $\beta \in \mathrm{Aut}(\mathcal{O}_n)$ and $U \in \mathcal{U}(\mathcal{O}_n)$ such that $\beta\alpha\beta^{-1}(\mathcal{D}_n) = \mathrm{Ad}(U)(\mathcal{D}_n)$. The set of all automorphisms $\alpha \in \mathrm{Aut}(\mathcal{O}_n)$ such that $\alpha(\mathcal{D}_n) = \mathrm{Ad}(U)(\mathcal{D}_n)$ for a suitable $U \in \mathcal{U}(\mathcal{O}_n)$ is easily checked to be a subgroup of $\mathrm{Aut}(\mathcal{O}_n)$ containing both $\mathrm{Inn}(\mathcal{O}_n)$ and $\mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)$, which we shall denote by $\mathrm{AutInn}(\mathcal{O}_n, \mathcal{D}_n)$. In terms of this group, the above question may be recast in a far more intelligible way, namely whether every automorphism $\alpha \in \mathrm{Aut}(\mathcal{O}_n)$ is conjugate to an element of $\mathrm{AutInn}(\mathcal{O}_n, \mathcal{D}_n)$.

One can also give a closer look to the conjugacy properties of the canonical UHF subalgebra. In [113] it is shown that if $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$ with $u \in \mathcal{S}_n$ and there is a sequence of projections P_k in $\mathcal{D}_n$ such that

$$\lim_{k\to\infty} \frac{\tau(\lambda_u(P_k))}{\tau(P_k)} \to 0 \text{ or } +\infty$$

then one has $\lambda_u \neq \mathrm{Ad}w\lambda_v$, for all $v \in \mathcal{U}(\mathcal{F}_n)$ and $w \in \mathcal{U}(\mathcal{O}_n)$. In particular, this implies that $\mathcal{F}_n$ and $\lambda_u(\mathcal{F}_n)$ are not inner conjugate. An example of this sort is provided by $u = S_{22}S_{212}^* + S_{212}S_{22}^* + P_{211} + P_1 \in \mathcal{S}_2$ w.r.t. to the sequence $P_k = P_{\underbrace{22\cdots 2}_{k}}$.

There are a great many natural questions still awaiting an answer. For instance, the following issue, where a remarkable abelian subgroup is investigated.

Question 4.7. Is $\pi(\lambda_{\mathcal{U}(\mathcal{D}_n)})$ a maximal abelian subgroup of $\mathrm{Out}(\mathcal{O}_n)$?

If $\mathcal{U}(\mathcal{O}_n)$ denotes the unitary group of $\mathcal{O}_n$, we have the following the chain of inclusions $\mathcal{P}_n \subset \mathcal{S}_n \subset \mathcal{U}(\mathcal{O}_n)$. Both $\mathcal{P}_n$ and $\mathcal{S}_n$ enjoy the ICC property (see [76]), namely every nontrivial conjugacy class contains infinitely many elements. As already observed, $\mathcal{S}_2$ is isomorphic to the Thompson group $V = G_{2,1}$ and thus is infinite and simple. As for $\lambda(\mathcal{P}_n)^{-1}$ and $\lambda(\mathcal{S}_n)^{-1}$, they are easily seen to be the associated automorphisms through the map $\lambda\colon \mathcal{U}(\mathcal{O}_n) \to \mathrm{Aut}(\mathcal{O}_n)$. A great many questions arise naturally when these groups are dealt with. Among others, the following really seems worth asking:

Question 4.8. Is $\pi(\lambda(\mathcal{S}_n)^{-1})$ ICC? Is it residually finite? Is it a Coxeter group?

We can take the subgroups $\left\langle \lambda(\mathcal{S}_n)^{-1}_{fin} \right\rangle$ and $\langle \lambda(\mathcal{S}_n)^{-1}_{\infty} \rangle$, where $\lambda(\mathcal{S}_n)^{-1}_{fin}$ are the elements of finite order and $\lambda(\mathcal{S}_n)^{-1}_{\infty}$ those of infinite order.

Question 4.9. What can be said about $\left\langle \lambda(\mathcal{S}_n)^{-1}_{fin} \right\rangle$ and $\langle \lambda(\mathcal{S}_n)^{-1}_{\infty} \rangle$?

Going back to the outer automorphisms, a more ambitious task to accomplish would be answering the following:

Question 4.10. Is $\pi(\lambda(\mathcal{S}_n)^{-1})$ contained in the commutator $[\mathrm{Out}(\mathcal{O}_n), \mathrm{Out}(\mathcal{O}_n)]$?

The next question, if a bit too technical at first glance, is nevertheless a necessary preliminary step to take in order to tackle the last problem we have pointed out. For any given $U \in \lambda(\mathcal{S}_n)^{-1}$, we can consider λ_U and λ_t, where $t \in \mathbb{T}$ is not a root of unity.

Question 4.11. What is the order of $\pi(\lambda_U \circ \lambda_t)$? More precisely, is it infinite as one might expect?

For $U \in \mathcal{P}_n$, trivially one has $\lambda_{zI}\lambda_U = \lambda_U\lambda_{zI}$ for any $z \in \mathbb{T}$.

Question 4.12. Given $U \in \lambda(\mathcal{S}_n)^{-1}$, let us consider the subgroup of $\mathbb{T}$ given by

$$G_U = \{z \in \mathbb{T} \mid \lambda_{zI}\lambda_U\lambda_{zI}^{-1}\lambda_U^{-1} \in \mathrm{Inn}(\mathcal{O}_n)\} \ .$$

How does $G_U \subset \mathbb{T}$ depend on the choice of U? Can G_U be both finite and infinite depending on U?

Being able to answer the above question would undoubtedly be of great usefulness in that both $\mathrm{Out}(\mathcal{O}_n)$ and his commutator are generated by their elements of finite order [66].

It is an acknowledged fact that results on injectivity of suitable restriction maps are eagerly sought in many contexts. The Cuntz algebras also provide a framework where this kind of questions arise quite naturally. In particular, we feel the following couple of problems are worth asking because they address a notable subset of unitaries that we have come across many a time throughout our exposition.

Question 4.13. Given $W \in \mathcal{S}_n$ such that $\lambda_W \restriction_{\mathcal{S}_n} = \mathrm{id}$, does it then follow that $W = 1$?

For the sake of completeness, we should like to point out that an affirmative answer has already been given to the same question for $\mathcal{P}_n$ in [73, Sect. 3.3].

Question 4.14. Does the C*-algebra generated by $\mathcal{S}_n$ coincide with $\mathcal{O}_n$? (This is true for $n = 2$, see the paragraph following Remark 4.40.) If that is not the case, we would like to say more about the C*-subalgebra obtained in this way. Especially, can it be described more explicitly?

The two questions outlined above are obviously related to one another. More precisely, an affirmative answer to the former would clearly imply an affirmative answer to the latter as well.

As it is possible to restrict any automorphism of $\mathcal{O}_n$ to an automorphism of $\mathcal{U}(\mathcal{O}_n)$, it is quite natural to wonder whether every group automorphism of $\mathcal{U}(\mathcal{O}_n)$ can be extended to an algebra automorphism of $\mathcal{O}_n$. In a few words, we have the following issue:

Question 4.15. Is the restriction map from $\mathrm{Out}(\mathcal{O}_n)$ to $\mathrm{Out}(\mathcal{U}(\mathcal{O}_n))$ surjective?

The same question may of course be asked for $\mathrm{Aut}(\mathcal{O}_n)$ too, and, more generally, for any given C*-algebra. In this regard, at least two comments are in order:

Remark 4.16. That the answer to the above question may in general be negative is already seen by looking at commutative C^*-algebras. Indeed, for any compact Hausdorff space X, the unitary group $\mathcal{U}(C(X))$ of $C(X)$ is simply given by all continuous functions on X taking values on $\mathbb{T}$. Obviously, $\eta(u)(x) = \overline{u(x)}$ defines a group automorphism of $\mathcal{U}(C(X))$, which does not come from any *-automorphism of $C(X)$. For otherwise, there should exist a homeomorphism $\Phi \in \mathrm{Homeo}(X)$ such that $u(\Phi(x)) = \overline{u(x)}$, $x \in X$, for any unitary function u, which becomes an absurd equality as soon as we compute it on a constant function $u(x) = \lambda$ for any $x \in X$ with $\lambda \neq \bar{\lambda}$.

Other group endomorphisms of $\mathcal{U}(C(X))$ can also be exhibited that do not arise by restriction of *-endomorphisms of $C(X)$. Indeed, with any integer k it is possible to associate a group endomorphism φ_k of $\mathcal{U}(C(X))$ which is given by $\varphi_k(u) = u^k$ pointwise. Obviously, $k = 1$ corresponds to $\mathrm{id}_{\mathcal{U}(C(X))}$ and $k = -1$ corresponds to the automorphism discussed above. When $k \neq \pm 1$, φ_k is an endomorphism which will fail to be injective. Now it is not hard to see that φ_k cannot be obtained by restricting to $\mathcal{U}(\mathcal{C}(X))$ a *-endomorphism of $C(X)$: if there existed a $\phi_k \in \mathrm{Aut}(C(X))$ such that $\Phi_k \restriction_{U(C(X))} = \varphi_k$, then, for any complex number $\lambda \in \mathbb{T}$ we should have $\lambda = \Phi_k(\lambda) = \varphi_k(\lambda) = \lambda^k$, which is absurd.

Remark 4.17. The answer is easily seen to be negative for $\mathcal{B}(\mathcal{H})$ as well. It is a well-known fact that every automorphism of $\mathcal{B}(\mathcal{H})$ is inner. However, if $U : \mathcal{H} \to \mathcal{H}$ is any antiunitary map, then $\mathcal{U}(\mathcal{H}) \ni V \to UVU^* \in \mathcal{U}(\mathcal{H})$ does define a group automorphism of $\mathcal{U}(\mathcal{H})$ although it cannot possibly be extended to a linear automorphism of $\mathcal{B}(\mathcal{H})$. As a matter of fact, this is the worst that can happen: as a consequence of an old result by Sakai, cf. [163], every uniformly continuous group automorphism of $\mathcal{U}(\mathcal{H})$ is indeed of the form $\mathcal{U}(\mathcal{H}) \ni V \to UVU^* \in \mathcal{U}(\mathcal{H})$, where U is either a unitary or anti-unitary operator of $\mathcal{H}$. For a thorough discussion of the bijections of $\mathcal{U}(\mathcal{H})$ preserving its various algebraic structures, the interested reader can also see a comparatively recent work by Molnár ans Šemrl, [144].

Remark 4.18. We doubt whether there may ever exist a C*-algebra A for which the restriction map $\mathrm{Aut}(A) \ni \varphi \to \varphi \restriction_{\mathcal{U}(A)} \in \mathrm{Aut}(\mathcal{U}(A))$ is surjective. However, we have not really examined as many examples as to provide evidence for this statement, nor are we aware of an example where the map is surjective instead.

Question 4.19. Is it possible to define and study Weyl groups for $\mathrm{Aut}(\mathcal{U}(\mathcal{O}_n))$?

From now on, we shall only consider problems concerning some aspects of group theory.

Question 4.20. Given a unitary $U \in \mathcal{O}_n$, in general it is not true that if λ_U is an automorphism, then λ_{U^*} is also an automorphism. Is it possible to define an index that in a sense measures their difference? In particular, we can consider inner autormorphisms. As we have already seen, the following identity holds $\mathrm{Ad}(U) = \lambda_{U\varphi(U^*)}$, where $\varphi : \mathcal{O}_n \to \mathcal{O}_n$ is the canonical endomorphism defined by $\varphi(x) = \sum S_i x S_i^*$. In this case we would consider $\lambda_{\varphi(U)U^*}$. When is it still an automorphism? This is clearly the case when U and $\varphi(U)$ commute, e.g. when $U \in \mathcal{U}(\mathcal{F}_n^1)$.

It has been recently shown in [46] that if $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$, where $u \in \mathcal{P}_n^2$, then $\lambda_{Fu^*F} \in \mathrm{Aut}(\mathcal{O}_n)$, thus showing that the set of rational permutations of the square $\{1,\dots,n\}^{\times 2}$ admits a natural involutive symmetry. It is likely that one can extend this result to all permutations in $\mathcal{P}_n^k$, for all k.

Question 4.21. Let U be in $\mathcal{U}(\mathcal{O}_n)$. In particular we may suppose that it is in $\mathcal{U}(\mathcal{D}_n)$ or $\mathcal{U}(\mathcal{F}_n)$. When does there exist $V \in \mathcal{U}(\mathcal{O}_n)$ (or $\mathcal{U}(\mathcal{D}_n)$ and $\mathcal{U}(\mathcal{F}_n)$, respectively) such that $U = V\varphi(V^*)$? Moreover, if this is the case, how can one exhibit such V explicitely? It follows from the above discussion that an algorithm answering this question can be effectively used to recognize inner automorphisms.

As we have proved, every second countable locally compact group embeds into both $\mathrm{Aut}(\mathcal{O}_2)$ and $\mathrm{Out}(\mathcal{O}_2)$. This seems to suggest that the two groups are likely to enjoy some universal property. In fact, in his paper [162] Sabok had already raised the question whether $\mathrm{Aut}(\mathcal{O}_2)$ is a universal Polish group. In this regard, it is really worth noting that so far most of the classes of topological groups we know of to embed into $\mathrm{Aut}(\mathcal{O}_2)$ are also embeddable into $\mathrm{Out}(\mathcal{O}_2)$. Actually, we do not know either if $\mathrm{Out}(\mathcal{O}_2)$ embeds into $\mathrm{Aut}(\mathcal{O}_2)$ or if $\mathrm{Aut}(\mathcal{O}_2)$ embeds into $\mathrm{Out}(\mathcal{O}_2)$. We should observe, however, that if $\mathrm{Aut}(\mathcal{O}_2)$ does embed into $\mathrm{Out}(\mathcal{O}_2)$, then every $\mathrm{Aut}(\mathcal{O}_n)$ do as well. In a sense, the following question is a conceivable generalization of that posed by Sabok.

Question 4.22. Does any Polish group embed into $\mathrm{Out}(\mathcal{O}_2)$?

Remark 4.23. The embedding in the above question is merely understood as a map between abstract groups irrespective of the topology carried by the Polish group to embed into $\mathrm{Out}(\mathcal{O}_2)$. In fact, we shall refrain from regarding $\mathrm{Out}(\mathcal{O}_2)$ as a topological group. There are many a reason to do so. The chief reason is that the natural topology carried by $\mathrm{Out}(\mathcal{O}_2)$ is not Hausdorff because the group is obtained as a quotient $\mathrm{Aut}(\mathcal{O}_2)/\mathrm{Inn}(\mathcal{O}_2)$ where the normal subgroup $\mathrm{Inn}(\mathcal{O}_2)$ is not closed, being even dense. This property is implicit in a paper by Rørdam, where an ostensibly stronger result is proved: given any $\varphi,\ \eta \in \mathrm{Aut}(\mathcal{O}_2)$, then there exists a sequence of unitaries $u_n \in \mathcal{O}_2$ such that $u_n\varphi(a)u_n^* \to \eta(a)$ in norm for each $a \in \mathcal{O}_2$. Taking $\eta = \mathrm{id}_{\mathcal{O}_2}$, we immediately find the density of $\mathrm{Inn}(\mathcal{O}_2)$ in $\mathrm{Aut}(\mathcal{O}_2)$. For an easy-to-follow but comprehensive account of all the material needed to prove Rørdam's theorem, the reader can consult the useful Operator Theory Notes written by P. Skoufranis (see [165, p. 56]), which are available for free on his webpage. To conclude, we should also acknowledge the possibility that $\mathrm{Out}(\mathcal{O}_2)$ might well be endowed with interesting if unusual topologies. However, none is yet known to us.

Now we should like to take this opportunity to comment further on the content of the previous remark. In particular, there is yet another result to point up. In fact, $\mathrm{Inn}(\mathcal{O}_2)$ is not only dense in $\mathrm{Aut}(\mathcal{O}_2)$, but it is even dense in $\mathrm{End}(\mathcal{O}_2)$, cf. [165]. The case $n = 2$ has nothing special. In fact, it is known that $\overline{\mathrm{Inn}(\mathcal{O}_n)} = \mathrm{Aut}(\mathcal{O}_n)$ for any finite n as well, where the closures are obviously understood with respect to the topology of pointwise norm-convergence. The reader interested in the proof can consult [120, p. 272]. As a matter of fact, the inner automorphisms are even dense in the whole semigroup of endomorphisms. This was first shown in [161] (for the even Cuntz algebras $\mathcal{O}_{2n}$) and later by Bratteli and Kishimoto ([40]). See also Farah et al. ([95]) for another recent proof that all endomorphisms of $\mathcal{O}_3$ are approximately unitarily equivalent.

More generally, we can also ask the following.

Question 4.24. What groups embed into the following: $\mathcal{U}(\mathcal{O}_n)$, $\mathrm{Aut}(\mathcal{O}_n)$, $\mathrm{Out}(\mathcal{O}_n)$, $\mathrm{Aut}(\mathcal{U}(\mathcal{O}_n))$, $\mathrm{Out}(\mathcal{U}(\mathcal{O}_n))$?

Remark 4.25. Elliott and Rørdam proved in [91] that $\mathcal{U}(\mathcal{O}_2)/\mathbb{T}$ is topologically simple, viz. the only closed normal subgroups are the trivial ones.

Another problem we would like to briefly discuss here has to do with the canonical decomposition of automorphisms. As is known [66, Corollary 3.9], not every automorphism can be decomposed as a product $\mathrm{Ad}(U) \circ \lambda_d \circ \lambda_V$, with $U \in \mathcal{U}(\mathcal{O}_n)$, $d \in \mathcal{U}(\mathcal{D}_n)$ and $V \in \mathcal{S}_n$ with $\lambda_V \in \mathrm{Aut}(\mathcal{O}_n)$. Even if $d \in \mathcal{D}_n$ were replaced with $f \in \mathcal{F}_n$, the corresponding decomposition might fail to work for every automorphism as well, although this last issue has not hitherto been investigated to the best of our knowledge. Nonetheless we are all the more motivated to point out the following problem.

Question 4.26. Is it possible to decompose any $\alpha \in \mathrm{Aut}(\mathcal{O}_n)$ in a canonical way, possibly resembling the Iwasawa KAN decomposition of a semisimple Lie group?

Question 4.27. Given $U \in \mathcal{U}(\mathcal{O}_n)$, does there exist $\alpha \in \mathrm{Aut}(\mathcal{O}_n)$ such that $\alpha(U) \in \mathcal{F}_n$?

Trivially, $\mathrm{Aut}(\mathcal{O}_n)$ is not simple for every n, for $\mathrm{Inn}(\mathcal{O}_n)$ is a proper normal subgroup. However, it is not closed. In the case $n = 2$ even more can actually be said: $\mathrm{Aut}(\mathcal{O}_2)$ is topologically simple, see [159, Theorems 3.6 and 8.2]. Even so, neither $\mathrm{Aut}(\mathcal{O}_2)$ nor $\mathrm{Out}(\mathcal{O}_2)$ are very well understood as abstract groups. The following questions represent the very bare minimum one would have to know to carry out further research on them.

Question 4.28. Consider $\mathrm{Out}(\mathcal{O}_2)$ and $\mathrm{Aut}(\mathcal{O}_2)$:

1. Is $\mathrm{Out}(\mathcal{O}_2)$ simple?

2. Are they perfect groups?

3. Are they ICC?

4. Are their centers trivial?

Obviously, the four questions gathered above are not indipendent. For example, a positive answer to 1 implies a positive answer to 2, 3, and 4, whereas a positive answer to 3 implies a positive answer to 4. It follows from the computations in [75] that the conjugacy class of the flip-flop automorphism is infinite as $\#\{\lambda_A^n \lambda_f \lambda_A^{-n} \mid n \in \mathbb{Z}\} = \infty$ in $\mathrm{Out}(\mathcal{O}_2)$. The careful reader must have realized that there is nothing particular with $\mathcal{O}_2$. Indeed, the same questions may safely be asked when $\mathcal{O}_2$ is replaced by $\mathcal{O}_n$ as well as by $\mathcal{F}_n$. We should acknowledge, though, that we have not thoroughly examined the relative literature to make sure the answers are already known for $\mathcal{F}_n$. In addition, as far as the first question of the group is concerned, there are definitely some results to point out if a sufficient degree of completeness is to be achieved in our exposition. More importantly, these will also give further material with which to compare our problems if a fuller contextualization is needed. In the first place, since $\mathcal{D}_n \cong C(K)$, as we saw in the first section, we have $\mathrm{Aut}(\mathcal{D}_n) = \mathrm{Out}(\mathcal{D}_n) =$Homeo$(K)$, with K being the ternary Cantor set, which implies that $\mathrm{Aut}(\mathcal{D}_n)$ is a simple group by virtue of a well-known theorem proved by Anderson in [23]. In the second place, we should also mention that $\mathrm{Out}(R)$ was shown by Connes to be simple if R is the hyperfinite type II_1 factor. This striking result is in fact a consequence of his in-depth classification of automorphisms for factors. By contrast, the outer automorphism group of a type III factor can never be simple because its modular group always sits in the center. The same argument does not work for $\mathcal{O}_n$ since the group of gauge automorphisms is not contained in the center of $\mathrm{Out}(\mathcal{O}_n)$ [66, Remark 4.4].

Question 4.29. We have the following questions concerning the Aut and Out groups of $\mathcal{O}_n$, $\mathcal{F}_n$ and $\mathcal{D}_n$:

1. Are they Coxeter groups? Are they generated by involutions?

2. Are they sofic groups?

As a partial but interesting answer, we recall that $\mathrm{Aut}(\mathcal{D}_n)$ is in fact known to be generated by its involutions [23].

The next question has to do with the permutation unitary groups $\mathcal{S}_n$ that we defined in the second section.

Question 4.30. Let u be a unitary in $\mathcal{S}_n$. Is it possible to give an explicit algorithm for determining whether λ_U is an automorphisms or not?

This question was originally posed in [70, Remark 4.14]. An efficient algorithm for deciding when $\lambda_u(\mathcal{D}_n) = \mathcal{D}_n$ for $u \in \mathcal{S}_n$ was finally exhibited in [64]. A similar procedure in the setting of graph algebras has been later described in [124].

Remark 4.31. An interesting research project that might be worth undertaking is to propose meaningful and workable notions of Weyl groups for tensor products of two (possibly different) Cuntz algebras. Any such tensor product yields an example of a 2-graph C*-algebra.

4.2. An embedding of the Thompson groups into $\mathcal{U}(\mathcal{O}_2)$

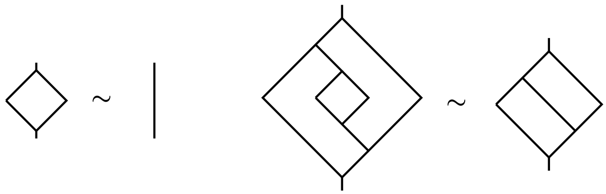
The Thompson groups $F \leq T \leq V$ were introduced in the 1960s by Richard Thompson. The groups T and V were the first known examples of infinite, finitely presented, simple groups. The Thompson group F can be described as the group of the orientation-preserving piecewise-linear homeomorphisms from the unit interval $[0,1]$ into itself, differentiable everywhere but at finitely many points that are all dyadic rationals and with slopes taking values in $2^{\mathbb{Z}}$. The group T is defined similarly but identifying the endpoints of $[0,1]$ (so that T is a subgroup of $\mathrm{Homeo}(S^1)$). The elements of the Thompson group V may be described in terms of pairs of standard dyadic partitions of $[0,1]$. We refer to [50] and [32] for a nice introduction to the Thompson groups.

Many subfactors arise from Conformal Field Theory (CFT). However, there are some subfactors for which no CFT is known to exist, for example those constructed by Haagerup [109, 27]. Recently, the problem of constructing a CFT for the Haagerup's subfactors has attracted great deal of attention. In particular, there is a paper of Jones [127] where, in an attempt to construct such a CFT using some idea inspired by the method of block spin renormalization, he produces some unitary representations of the Thompson groups F and T. The Thompson group T is a subgroup of piecewise-linear homeomorphism of S^1. Jones's idea is that T should play the role of $\mathrm{Diff}^+(S^1)$ in the construction of a CFT and that its elements should be thought of as local scale transformations. One of the main ingredients in his work is the description of elements of F and T as pairs of trees. As already mentioned, the Thompson group V is isomorphic with $\mathcal{S}_2 \subset \mathcal{U}(\mathcal{O}_2)$, thus its subgroups F and T both embed into $\mathcal{U}(\mathcal{O}_2)$. It follows at once that any representation of $\mathcal{O}_2$ gives rise to a unitary representation of all these groups [47, 28]. A question related to the construction of a CFT for the Haagerup subfactor and Jones's attempt is the following:

Question 4.32. Does $\mathrm{Diff}^+(S^1)$ embed into $\mathrm{Out}(\mathcal{O}_2)$? Can one embed the Thompson groups in $\mathrm{Out}(\mathcal{O}_2)$ using their description in terms of pairs of trees? For example, to do so we would like to use the isomorphisms between $\mathcal{O}_2$ and $\mathcal{O}_2 \otimes \mathcal{O}_2$.

Now it is worth recalling a classical result that T can be embedded into $\mathrm{Diff}^+(S^1)$, [102]. In passing, we observe a couple of unrelated problems in the context of AQFT. The first one is whether a pair of DR-isometries (implementing a DR-morphism with statistical dimension 2) may provide a representation of $\mathcal{Q}_2$ (see Chapter 9); in particular, is there any specific interpretation for the unitary U? The second one asks whether the unitary representation of Thompson group F associated to the representation of $\mathcal{O}_2$ arising again from a sector of dimension 2 can be related to the usual construction of the link invariants through the braiding (or give any other insightful interpretation of the role of F in this setting).

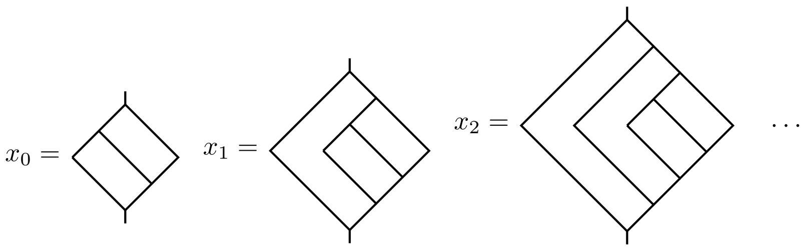
Here we briefly recall another equivalent description of F. More detailed information, however, can be found in the classical references [50] and [32]. Elements g of the Thompson group F can actually be described as pairs (T, T') of trees with the same number of leaves. More precisely, if we denote by $\mathcal{T}_n$ the set of rooted planar binary tress with n leaves, any such g can be represented as $g(T,T')$ with $T,T \in \mathcal{T}_n$, for some n depending on g. In this description the order of the trees is important, for $g(T,T') = g(T',T)^{-1}$ for any $T,T' \in \mathcal{T}_n$. Furthermore, in general there will be several pairs of trees associated with the same element of F. However, different pairs of trees associated with the same element $g \in F$ will only differ by deletion/addition of pairs of opposing "carets", [32]. Here below are a pair of opposing carets and two pairs of trees representing the same element of F.



In terms of generators and relations, the Thompson group F can also be presented as the group generated by the infinite family of elements $\{x_n\}_{n\geq 0}$ satisfying the following relation

$$x_n x_k = x_k x_{n+1} \qquad \text{for all } k < n$$

These elements correspond to the following pair of binary trees:



We would like to outline a construction due to Fiore and Leinster somehow related to Question 4.32. In particular, our aim is to give a sketched proof of the following result.

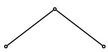
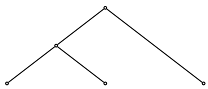
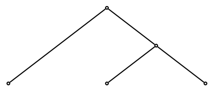
Theorem 4.33 ([96, Theorem 1.1, p. 2]). *Let $\mathcal{A}$ be the strict monoidal category freely generated by an idempotent object (A,α), namely an object A and an isomorphism $\alpha\colon A\otimes A \to A$. Then $\mathrm{Aut}_{\mathcal{A}}(A)$ is isomorphic with Thompson group F.*

Before proceeding with the proof, we recall the definition of a monoidal category freely generated by an idempotent object: it is a monoidal category $\mathcal{A}$ together with an idempotent object (A,α) satisfying the following universal property

- let $\mathcal{C}$ be a monoidal category with an idempotent object (M,μ). Then there exists a unique monoidal functor $F\colon \mathcal{A}\to\mathcal{C}$ such that $F(A) = M$ and $F(\alpha) = \mu$.

We observe that, as usual, the universal property uniquely determines the triple $(\mathcal{A}, A, \alpha)$ (up to isomorphism)

Let $\mathcal{A}$ be a category whose objects are $\mathbb{N}$ and morphisms $\phi\colon m \to n$ are piecewise-linear, bijective maps $f\colon [0,m] \to [0,n]$ with slopes that are integer powers of 2 and all breakpoints have dyadic rational coordinates. The tensor functor $\otimes\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ is defined on objects as $m\otimes n \doteq m+n$ and on morphisms by juxtaposition. The isomorphism $\alpha : 2 := 1\otimes 1 \to 1$ is given by division by 2. By definition $F = \mathrm{Aut}_{\mathcal{A}}(1)$, so we only have to prove that the universal property holds. Take a monoidal tensor category $\mathcal{C}$ with idempotent object (M,μ). As already mentioned the elements of the Thompson group F can be described by pairs of bifurcating trees. First of all we describe the correspondence between trees and morphisms, then we will consider the case of elements of F. Given a tree $T \in \mathcal{T}_m$ we can define a morphism $\mu_T : M^{\otimes m} \to M$. The idea essentially is the following: starting from the root, whenever we meet bifurcation on a vertex we apply $\mu : M\otimes M \to M$. In order to clarify the above procedure, we take some examples. Consider the following trees T_1, T_2, T_3

$T_1 =$  $T_2 =$  $T_3 =$ 

The associated morphisms are

- $\mu_{T_1} = \mu\colon M\otimes M \to M$;
- $\mu_{T_2} = \mu\circ(\mu\otimes\mathrm{id})\colon M\otimes M\otimes M \xrightarrow{\mu\otimes\mathrm{id}} M\otimes M \xrightarrow{\mu} M$;
- $\mu_{T_3} = \mu\circ(\mathrm{id}\otimes\mu)\colon M\otimes M\otimes M \xrightarrow{\mathrm{id}\otimes\mu} M\otimes M \xrightarrow{\mu} M$.

More generally, one can define the morphism μ_f associated with a forest $f = (T_1, T_2, \ldots, T_k)$ as $\mu_f = \mu_{T_1}\otimes\mu_{T_2}\otimes\cdots\otimes\mu_{T_k}$. Now any morphism $\phi\in\mathcal{A}$ between m an n can be shown to factor as $\phi = \alpha_f\circ\alpha_{f'}^{-1}$ for suitable forests f, f'. The monoidal functor $G\colon \mathcal{A}\to\mathcal{C}$ is defined by $G(1) = M$, $G(\alpha) = \mu$ and for $\phi\in Mor(m,n)$ we have that $G(\phi) = \mu_f\circ\mu_{f'}^{-1}$. As examples, the morphisms corresponding to x_0, x_1 and x_2 are, respectively,

$$
\begin{aligned}
&\mu\circ(\mu\otimes\mathrm{id})\circ(\mu\circ(\mathrm{id}\otimes\mu))^{-1},\\
&\mu\circ(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\mu\otimes\mathrm{id})\circ(\mu\circ(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu))^{-1},\\
&\mu\circ(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu\otimes\mathrm{id})\\
&\qquad\circ(\mu\circ(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mu))^{-1}.
\end{aligned}
$$

The morphism $G(\phi)$ can be shown not to depend on the choice of the forests f, f' through which ϕ is factored.

As already mentioned, there exists an isomorphism $\mu\colon \mathcal{O}_2\otimes\mathcal{O}_2\to\mathcal{O}_2$. Therefore, the above procedure can be used to provide an embedding of F into $\mathrm{Aut}(\mathcal{O}_2)$. At this point, one might wonder if it is even possible to faithfully represent F through *outer* automorphism of $\mathcal{O}_2$.

Question 4.34. Does the above procedure produce an embedding of F into $\mathrm{Out}(\mathcal{O}_2)$?

As already mentioned, there are actually many examples of C^*-algebras and von Neumann algebras A such $A \cong A \otimes A$, and the above question makes sense for all of them.

Remark 4.35. We remark that the answer to the former question must depend on structure of $\mathcal{O}_2$. In fact, $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$ (here $\otimes$ denotes the von Neumann tensor product). As $\mathcal{B}(\mathcal{H})$ has only inner automorphisms, the embedding of F into $\mathrm{Aut}(\mathcal{B}(\mathcal{H}))$ cannot induce an embedding into $\mathrm{Out}(\mathcal{B}(\mathcal{H})) = \{1\}$.

We have mentioned may a time that the Thompson group F embeds into $\mathcal{U}(\mathcal{O}_2)$. Now we discuss an explicit description of this embedding also providing a graphical interpretation of the map, which first appeared in [147]. We claim that the elements

$$\begin{aligned} x_0 &\doteq S_1 S_1 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^* S_2^* \\ x_k &\doteq 1 - S_2^k S_2^{*k} + S_2^k x_0 S_2^{*k} \qquad k \geq 1 \end{aligned}$$

generate a subgroup of $\mathcal{U}(\mathcal{O}_2)$ isomorphic with F. First , we check that these elements satisfy the relations

$$x_n x_k = x_k x_{n+1} \qquad \text{for } k < n.$$

We start by dealing with the case $k \neq 0$. On the one hand, we have the chain of equalities

$$\begin{aligned} x_n x_k &= [1 - S_2^n S_2^{*n} + S_2^n x_0 S_2^{*n}][1 - S_2^k S_2^{*k} + S_2^k x_0 S_2^{*k}] \\ &= 1 - S_2^n S_2^{*n} + S_2^n x_0 S_2^{*n} - S_2^k S_2^{*k} + S_2^n S_2^{*n} - S_2^n x_0 S_2^{*n} \\ &\quad + S_2^k x_0 S_2^{*k} - S_2^n S_2^{*n-k} x_0 S_2^{*k} + S_2^n x_0 S_2^{*n-k} x_0 S_2^{*k} \\ &= 1 - S_2^k S_2^{*k} + S_2^k x_0 S_2^{*k} - S_2^n S_2^{*n-k} x_0 S_2^{*k} + S_2^n x_0 S_2^{*n-k} x_0 S_2^{*k} \\ &= x_k - S_2^n S_2^{*n-k} x_0 S_2^{*k} + S_2^n x_0 S_2^{*n-k} x_0 S_2^{*k}. \end{aligned}$$

On the other hand, we also have

$$\begin{aligned} x_k x_{n+1} &= [1 - S_2^k S_2^{*k} + S_2^k x_0 S_2^{*k}][1 - S_2^{n+1} S_2^{*n+1} + S_2^{n+1} x_0 S_2^{*n+1}] \\ &= 1 - S_2^k S_2^{*k} + S_2^k x_0 S_2^{*k} - S_2^{n+1} S_2^{*n+1} + S_2^{n+1} S_2^{*n+1} - S_2^k x_0 S_2^{n+1-k} S_2^{*n+1} \\ &\quad + S_2^{n+1} x_0 S_2^{*n+1} - S_2^{n+1} x_0 S_2^{*n+1} + S_2^k x_0 S_2^{n+1-k} x_0 S_2^{*n+1} \\ &= 1 - S_2^k S_2^{*k} + S_2^k x_0 S_2^{*k} - S_2^k x_0 S_2^{n+1-k} S_2^{*n+1} + S_2^k x_0 S_2^{n+1-k} x_0 S_2^{*n+1} \\ &= x_k - S_2^k x_0 S_2^{n+1-k} S_2^{*n+1} + S_2^k x_0 S_2^{n+1-k} x_0 S_2^{*n+1}. \end{aligned}$$

Therefore, it is enough to prove the following equality

$$-S_2^n S_2^{*n-k} x_0 S_2^{*k} + S_2^n x_0 S_2^{*n-k} x_0 S_2^{*k} = -S_2^k x_0 S_2^{n+1-k} S_2^{*n+1} + S_2^k x_0 S_2^{n+1-k} x_0 S_2^{*n+1}.$$

On the left-hand side we have

$$\begin{aligned}
&-S_2^n S_2^{*n-k} x_0 S_2^{*k} + S_2^n x_0 S_2^{*n-k} x_0 S_2^{*k}\\
&\quad = -S_2^n S_2^{*n-k+1} S_2^{*k} + S_2^n (S_1^2 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^{*2}) S_2^{*n+1}\\
&\quad = -S_2^n S_2^{*n+1} + S_2^n (S_1^2 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^{*2}) S_2^{*n+1}.
\end{aligned}$$

On the right-hand side we have

$$\begin{aligned}
&-S_2^k x_0 S_2^{n+1-k} S_2^{*n+1} + S_2^k x_0 S_2^{n+1-k} x_0 S_2^{*n+1}\\
&= -S_2^k (S_1^2 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^{*2}) S_2^{n+1-k} S_2^{*n+1} +\\
&+ S_2^k (S_1^2 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^{*2}) S_2^{n+1-k} (S_1^2 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^{*2} + S_2 S_2^{*2}) S_2^{*n+1}\\
&= -S_2^{k+1} S_2^{n-1-k} S_2^{*n+1} + S_2^{k+1} S_2^{n-1-k} (S_1^2 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^{*2} + S_2 S_2^{*2}) S_2^{*n+1}\\
&= -S_2^n S_2^{*n+1} + S_2^n (S_1^2 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^{*2} + S_2 S_2^{*2}) S_2^{*n+1}.
\end{aligned}$$

The above computations prove that $x_n x_k = x_k x_{n+1}$ for $0 \neq k < n$. We need to consider the case $k = 0$ aside. In other words, the relation $x_n x_0 = x_0 x_{n+1}$ is yet to be verified. This means that we have to show that

$$(1 - S_2^n S_2^{*n} + S_2^n x_0 S_2^{*n}) x_0 = x_0 (1 - S_2^{n+1} S_2^{*n+1} + S_2^{n+1} x_0 S_2^{*n+1}),$$

thus it is enough to prove that

$$(-S_2^n S_2^{*n} + S_2^n x_0 S_2^{*n}) x_0 = x_0 (-S_2^{n+1} S_2^{*n+1} + S_2^{n+1} x_0 S_2^{*n+1}).$$

On the one hand, we have that

$$\begin{aligned}
&(-S_2^n S_2^{*n} + S_2^n x_0 S_2^{*n}) x_0 = (-S_2^n S_2^{*n} + S_2^n x_0 S_2^{*n}) S_2 S_2^{*2}\\
&\quad = (-S_2^n S_2^{*n} + S_2^n (S_1^2 S_1^* + S_1 S_2 S_1^* S_2^* + S_2 S_2^{*2}) S_2^{*n}) S_2 S_2^{*2}\\
&\quad = -S_2^n S_2^{*n+1} + S_2^n S_1^2 S_1^* S_2^{*n+1} + S_2^n S_1 S_2 S_1^* S_2^{*n+2} + S_2^{n+1} S_2^{*n+3}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&x_0 (-S_2^{n+1} S_2^{*n+1} + S_2^{n+1} x_0 S_2^{*n+1}) = S_2 S_2^{*2} (-S_2^{n+1} S_2^{*n+1} + S_2^{n+1} x_0 S_2^{*n+1})\\
&\quad = S_2 S_2^{*2} (-S_2^{n+1} S_2^{*n+1} + S_2^{n+1} (S_1^2 S_1^* S_2^* + S_2 S_2^* 2) S_2^{*n+1})\\
&\quad = -S_2^n S_2^{*n+1} + S_2^n S_1^2 S_1^* S_2^{*n+1} + S_2^n S_1 S_2 S_1^* S_2^{*n+2} + S_2^{n+1} S_2^{*n+3}.
\end{aligned}$$

This finally ends the proof since every proper quotient of F is abelian, [50, Theorem 4.3].

Remark 4.36. Let $\lambda_f \in \mathrm{Aut}(\mathcal{O}_2)$ be the flip-flop automorphism. Then $\lambda_f(x_0) = x_0^{-1} \in \mathcal{U}(\mathcal{O}_2)$. In fact, we have that

$$\lambda_f(x_0) = S_2S_2S_2^* + S_2S_1S_2^*S_1^* + S_1S_1^*S_1^* = x_0^*$$

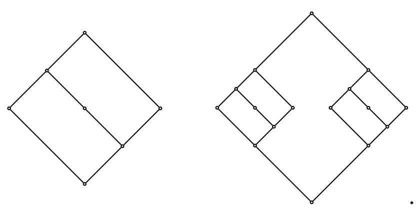
and

$$\begin{aligned}\lambda_f(x_0)x_0 &= S_2(S_2S_2^*)S_2^* + S_2(S_1S_1^*)S_2^* + S_1S_1^* = S_2S_2^* + S_1S_1^* = 1\\ x_0\lambda_f(x_0) &= S_1S_1S_1^*S_1^* + S_1S_2S_2^*S_1^* + S_2S_2^* = S_2S_2^* + S_1S_1^* = 1.\end{aligned}$$

First we observe that $\lambda_f(F) = F$. The restriction of λ_f to $F \subset \mathcal{U}(\mathcal{O}_2)$ admits two nice equivalent interpretations. In terms of tree diagrams, λ_f is just the reflection about a vertical line. When F is described as group of homeomorphisms of $[0, 1]$, the action of λ_f is in fact given by the conjugation by $\sigma(t) := 1 - t$, for $t \in [0, 1]$.

The above embedding has a nice graphical interpretation. It is well known that each element $g \in F$ can be represented by a pair of bifurcating trees $T_+, T_- \in \mathcal{T}_n$ for some $n \in \mathbb{N}$. In the sequel we will denote g also by $g(T_+, T_-)$. If we label all the NE/SW edges by 1 and all the NW/SE edges by 2, then each leaf of the tree can be determined by an array with entries 1 and 2. Let $f \in V(T_\pm)$ be a leaf, we denote by $\alpha_{\pm,f}$ the associated array. Given an element $g = g(T_+, T_-) \in F$, we consider the element of the Cuntz algebra defined as $z_g \doteq \sum_f S_{\alpha_{+,f}} S^*_{\alpha_{-,f}}$. It is easily verified that $z_{x_0} = S_1^2S_1^* + S_1S_2(S_2S_1)^* + S_2(S_2^2)^*$.

The canonical endomorphism of the Cuntz algebra φ can be restricted to an endomorphism of $F \subset \mathcal{U}(\mathcal{O}_2)$ and has an interesting graphical representation which we illustrate with an example. The generator x_0 and $\varphi(x_0) = S_1x_0S_1^* + S_2x_0S_2^*$ are associated to the following graphs, respectively



Therefore, the canonical endomorphisms duplicates the pair of trees given as input. As φ can be restricted to an endomorphism of F, it is clear that some elements of F give rise to automorphisms of $\mathcal{O}_2$. Indeed, we have that $\lambda_{g\varphi(g)^*} = \mathrm{Ad}(g)$ for all $g \in F$.

Question 4.37. It is natural to ask if λ_{x_k} is an automorphism of $\mathcal{O}_2$, $k = 0, 1, 2, 3, \ldots$ In general, which are the elements of F inducing automorphisms of $\mathcal{O}_2$? Is it true that $\lambda_F \cap \mathrm{Aut}(\mathcal{O}_2) \subset \mathrm{Inn}(\mathcal{O}_2)$? As $\mathcal{S}_2 \subset N_{\mathcal{O}_2}(\mathcal{D}_2)$ (unitary normalizer), $Ad(g)$ with $g \in F$ restricts to an automorphism of $\mathcal{D}_2$. More generally, for which $g \in F$ does λ_g restrict to an automorphism of $\mathcal{D}_2$? (This is a weaker requirement than asking that $\lambda_g \in \mathrm{Aut}(\mathcal{O}_2)$.)

The above question can actually be answered, albeit partially. Indeed, the following result holds.

Proposition 4.38. *Each $\{x_k\}_{k\in\mathbb{N}}$ gives rise to a proper endomorphism of $\mathcal{O}_2$.*

The restriction $\lambda_{x_0}|\mathcal{D}_2$ had already been shown not to be an automorphism in [64, Example 3.3] As for the x_k's, we can adopt a similar strategy to that employed in the aforementioned paper. To this aim, it is enough to only make sure that $\lambda_{x_k}|\mathcal{D}_2$ is not surjective, and apply [67, Proposition 1.1, (b)] to conclude that these homomorphisms cannot be automorphisms of $\mathcal{O}_2$.

Lemma 4.39. *With the above notations, $\lambda_{x_k}|\mathcal{D}_2\colon \mathcal{D}_2 \to \mathcal{D}_2$ is not surjective.*

Proof. Set $u = x_k \in \mathcal{S}_2$ and define $u_h = u\varphi(u)\cdots\varphi(u)^{h-1}$, where φ is the canonical endomorphism. By [64, Lemma 3.4, p.7] it is enough to exhibit a projection $P = P_\alpha$, with $|\alpha| = k+2$, such that the sequence $\{u_h^* P u_h\}$ does not eventually stabilize. Let $P = S_2^{k+2} S_2^{*k+2}$. We claim that $u_h^* P u_h = S_2^{k+2+h}(S_2^*)^{k+2+h}$. We give a proof by induction on h. When $h = 1$ we have

$$\begin{aligned}
x_k^* P x_k &= x_k^* S_2^{k+2}[S_2^{*k+2} x_k] \\
&= x_k^* S_2^{k+2}[S_2^{*k+2} - S_2^{*k+2} + S_2^{*2} x_0 S_2^{*k}] \\
&= x_k^* S_2^{k+2}[S_2^{*2} x_0 S_2^{*k}] \\
&= [x_k^* S_2^{k+2}] S_2^{*2} x_0 S_2^{*k} \\
&= [S_2^{k+2} - S_2^{k+2} + S_2^k x_0^* S_2^{*2}] S_2^{*2} x_0 S_2^{*k} \\
&= [S_2^k x_0^* S_2^2] S_2^{*2} x_0 S_2^{*k} \\
&= S_2^{k+3} S_2^{*(k+3)}.
\end{aligned}$$

Suppose that the formula holds for $h \geq 1$, and consider the case $h+1$. Note that, for every $n \in \mathbb{N}$, it holds

$$S_2^n (S_2^*)^n \varphi^n(P) = S_2^{k+2+n}(S_2^*)^{k+2+n}\ .$$

Now the following computations prove the claim

$$\begin{aligned}
u_{h+1}^* P u_{h+1} &= \varphi^h(x_k)^*(u_h^* P u_h)\varphi^h(x_k) \\
&= \varphi^h(x_k)^* S_2^{k+2+h}(S_2^*)^{k+2+h}\varphi^h(x_k) \\
&= \varphi^h(x_k)^* S_2^h (S_2^*)^h \varphi^h(P)\varphi^h(x_k) \\
&= S_2^h (S_2^*)^h \varphi^h(x_k)^* \varphi^h(P)\varphi^h(x_k) \\
&= S_2^h (S_2^*)^h \varphi^h(x_k^* P x_k) \\
&= S_2^h (S_2^*)^h \varphi^h(S_2^{k+3} S_2^{*(k+3)}) \\
&= S_2^{k+2+h+1}(S_2^*)^{k+2+h+1}.
\end{aligned}$$

□

The Thompson group T, too, can be embedded into $\mathcal{U}(\mathcal{O}_2)$. We would like to sketch a proof of this. For more details, the reader is referred to [50]. We first recall that T admits the following finite presentation with generating set $\{x_0, x_1, c\}$ satisfying the relations

1. $[x_0x_1^{-1}, x_0^{-1}x_1x_0] = 1$
2. $[x_0x_1^{-1}, x_0^{-2}x_1x_0^2] = 1$
3. $x_1c_3 = c_2x_2$
4. $cx_0 = c_2^2$
5. $x_1c_2 = c$
6. $c^3 = 1$

where $c_n = x_0^{-n+1}cx_1^{n-1}$ for $n \geq 0$ (so that $c_1 = c$). The embedding of T into $\mathcal{U}(\mathcal{O}_2)$ can be obtained by taking for x_0 and x_1 the same unitaries of the Cuntz algebra we used for F, whereas c is sent to $S_1S_1^*S_2^* + S_2S_1(S_2^*)^2 + S_2^2S_1^*$.

Obviously, Relations (1) and (2) hold by the previous discussion.

We check that Condition (3) holds, namely $x_1x_0^{-2}cx_1^2 = x_0^{-1}cx_1x_2$. We recall that $x_0 = S_1^2S_1^* + S_1S_2S_1^*S_2^* + S_2(S_2^*)^2$ and

$$x_1 = 1 - S_2S_2^* + S_2(S_1^2S_1^* + S_1S_2S_1^*S_2^* + S_2(S_2^*)^2)S_2^*.$$

On the left-hand side we have

$$\begin{aligned}
&x_1x_0^{-2}cx_1^2\\
&\quad = x_1(S_2^3S_2^* + S_2^2S_1S_2^*S_1^* + S_2S_1S_2^*(S_1^*)^2 + S_1(S_1^*)^3)cx_1^2\\
&\quad = x_1(S_2^3S_1(S_2^*)^2 + S_2^4S_1^* + S_2^2S_1S_2^*S_1^*S_2^* + S_2S_1S_2^*(S_1^*)^2S_2^* + S_1(S_1^*)^3S_2^*)x_1^2\\
&\quad = (S_1(S_1^*)^3S_2^* + S_2S_1^2S_2^*(S_1^*)^2S_2^* + S_2S_1S_2S_2^*S_1^*S_2^* + S_2^2S_1(S_2^*)^2 + S_2^3S_1^*)x_1^2\\
&\quad = (S_1(S_1^*)^2S_2^* + S_2S_1^2S_2^*S_1^*S_2^* + S_2S_1S_2S_1^*(S_2^*)^2 + S_2^2S_1(S_2^*)^3 + S_2^3S_1^*)x_1\\
&\quad = S_1S_1^*S_2^* + S_2S_1^2S_1^*(S_2^*)^2 + S_2S_1S_2S_1^*(S_2^*)^3 + S_2^2S_1(S_2^*)^4 + S_2^3S_1^*.
\end{aligned}$$

On the right hand-side we have

$$\begin{aligned}
x_0^{-1}cx_1x_2 &= x_0^{-1}(S_1^2S_1^*S_2^* + S_1S_2S_1^*(S_2^*)^2 + S_2S_1(S_2^*)^3 + S_2^2S_1^*)x_2\\
&= (S_1S_1^*S_2^* + S_2S_1S_1^*(S_2^*)^2 + S_2^2S_1(S_2^*)^3 + S_2^3S_1^*)x_2\\
&= S_1S_1^*S_2^* + S_2S_1^2S_1^*(S_2^*)^2 + S_2S_1S_2S_1^*(S_2^*)^3 + S_2^2S_1(S_2^*)^4 + S_2^3S_1^*.
\end{aligned}$$

As for Relation (4), namely $cx_0 = x_0^{-1}cx_1x_0^{-1}cx_1$, we make the following computations. On the left- hand side we simply have

$$\begin{aligned}
cx_0 &= (S_1S_1^*S_2^* + S_2S_1(S_2^*)^2 + S_2^2S_1^*)(S_1^2S_1^* + S_1S_2S_1^*S_2^* + S_2(S_2^*)^2)\\
&= S_1S_1^*(S_2^*)^2 + S_2S_1(S_2^*)^3 + S_2^2S_1S_1^* + S_2^3S_1^*S_2^*.
\end{aligned}$$

On the right-hand side we start by computing

$$\begin{aligned} x_0^{-1}cx_1 &= (S_1(S_1^*)^2 + S_2S_1S_2^*S_1^* + S_2^2S_2^*)(S_1S_1^*S_2^* + S_2S_1(S_2^*)^2 + S_2^2S_1^*)x_1 \\ &= (S_1(S_1^*)^2S_2^* + S_2S_1S_2^*S_1^*S_2^* + S_2^2S_1(S_2^*)^2 + S_2^3S_1^*) \\ &\qquad \times (S_1S_1^* + S_2S_1^2S_1^*S_2^* + S_2S_1S_2S_1^*(S_2^*)^2 + S_2^2(S_2^*)^3) \\ &= S_1S_1^*S_2^* + S_2S_1S_1^*(S_2^*)^2 + S_2^2S_1(S_2^*)^3 + S_2^3S_1^* \end{aligned}$$

which gives

$$\begin{aligned} x_0^{-1}cx_1x_0^{-1}cx_1 &= (S_1S_1^*S_2^* + S_2S_1S_1^*(S_2^*)^2 + S_2^2S_1(S_2^*)^3 + S_2^3S_1^*) \\ &\qquad \times (S_1S_1^*S_2^* + S_2S_1S_1^*(S_2^*)^2 + S_2^2S_1(S_2^*)^3 + S_2^3S_1^*) \\ &= S_1S_1^*(S_2^*)^2 + S_2S_1(S_2^*)^3 + S_2^2S_1S_1^* + S_2^3S_1^*S_2^*. \end{aligned}$$

Relation (5), namely $x_1x_0^{-1}cx_1 = c$, holds as well. Inded, we have that

$$\begin{aligned} x_1x_0^{-1}cx_1 &= x_1x_0^{-1}(S_1^2S_1^*S_2^* + S_1S_2S_1^*(S_2^*)^2 + S_2S_1(S_2^*)^3 + S_2^2S_1^*) \\ &= x_1(S_1S_1^*S_2^* + S_2S_1S_1^*(S_2^*)^2 + S_2^2S_1(S_2^*)^3 + S_2^3S_1^*) \\ &= S_1S_1^*S_2^* + S_2S_1^2S_1^*(S_2^*)^2 + S_2S_1S_2(S_2^*)^3 + S_2^2S_1^* = c. \end{aligned}$$

Finall, Relation (6) holds too.

$$\begin{aligned} c^3 &= (S_2^2S_1^*S_2^* + S_1(S_2^*)^2 + S_2S_1S_1^*)(S_1S_1^*S_2^* + S_2S_1(S_2^*)^2 + S_2^2S_1^*) \\ &= S_1S_1^* + S_2S_1S_1^*S_2^* + S_2^2(S_2^*)^2 = 1. \end{aligned}$$

Since T is simple [50, Theorem 5.8 and Corollary 5.9], the subgroup of $\mathcal{U}(\mathcal{O}_2)$ generated by x_0, x_1 and c as above is isomorphic to T itself.[12]

Remark 4.40. We observe that $c_0 = f$, where f is the unitary element that gives rise to the flip-flop automorphism λ_f.

Further results about embeddings of $F \subset T \subset V$ into $\mathcal{U}(\mathcal{O}_2)$ are described in the recent paper [110], especially see Proposition 4.3 therein where it is shown that $C^*(V) = \mathcal{O}_2$. This provides a positive answer to Question 4.14 for $n = 2$.

Remark 4.41. As $\mathcal{S}_2$ is isomorphic with $V = V_2$, the group $\mathcal{S}_n$ is isomorphic with the Higman-Thompson group $V_n = G_{n,1}$, [147]. Since $\mathcal{S}_n \subset \mathcal{U}(\mathcal{O}_n)$, it is clear that any (unital) representation of $\mathcal{O}_n$ gives rise to a unitary representation of $\mathcal{S}_n$, cf. [47]. This way, for $n = 2$, one can easily produce several unitary representations of the Thompson groups $F \subset T \subset V$. For instance, thanks to [110], by restricting the GNS representation of the unique KMS state ω of $\mathcal{O}_2$, one gets a unitary representation π of V such that $\pi(V)''$ is a factor of type $\mathrm{III}_{1/2}$. Among other possibilities, it is clear that one can introduce permutative representations of the

[12] In a different way, one might have observed that the copies of F and T inside $\mathcal{S}_2$ are nothing but the images of F and T as subgroups of V under the isomorphism of V with $\mathcal{S}_2$ found in [147].

Thompson groups [43], see also [19] for a very recent step in this direction. A class of unitary representations of V arising from certain representations of $\mathcal{O}_2$ are discussed in some detail in [28], with a characterisation of unitary equivalence and irreducibility.

The potential interplay between endomorphisms and automorphisms of the Cuntz algebras and the various instances of the Thompson groups is another source of inspiration. In particular, one may look for endomorphisms/automorphisms of the Thompson groups F, T and V arising by restriction of endomorphisms/automorphisms of $\mathcal{O}_2$ (but similar issues, of course, make sense for any $\mathcal{O}_n$ and the corresponding Higman-Thompson group $G_{n,1}$ along with some selected subgroups). It is clear that F and V are invariant under φ[13]. Moreover, $\lambda_u(V) \subseteq V$, when u belongs to V. In [29], it is proved that an endomorphisms $\lambda_u \in \mathrm{End}(\mathcal{O}_2)$ preserves V, that is $\lambda_u(V) \subseteq V$, if and only if $u \in V$. In other words, one obtains a huge family of endomorphisms of V simply by restricting to V the endomorphisms of $\mathcal{O}_2$ induced by unitaries in V itself. Moreover, if $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2)$, then $u \in V$ is also equivalent to $\lambda_u(F) \subseteq V$. We have already observed that $\varphi(F) \subset F$. Also note that it may happen that $\lambda_u(F) = F$ for some $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2)$ even though u itself is not in F. A notable example is given by the flip-flop λ_f, which means that f lies in $T \setminus F$. One of the ingredients of the proof of the results mentioned above is the notion of *modestly scaling* endomorphism, taken from [29], which we are going to recall below.[14]

Defintion 4.42. An endomorphism $\lambda \in \mathrm{End}(\mathcal{O}_n)$ is modestly scaling if, for every sequence (μ_k) of non-empty multi-indices in W_n, and $p \in P(\mathcal{O}_n)$ such that $p \leq \lambda(P_{\mu_1\mu_2\cdots\mu_k})$ for all $k \in \mathbb{N}$, one has $p = 0$.

For example, all automorphims of $\mathcal{O}_n$ are modestly scaling, as are endomorphisms $\lambda \in \mathrm{End}(\mathcal{O}_n)$ such that $\lambda(\mathcal{F}_n) \subseteq \mathcal{F}_n$. However, not all endomorphisms are modestly scaling. A cheap example is the following. For $k \geq 1$ and $i \in \{1, \ldots, n\}$ denote by $i(k) \in W_n^k$ the multi-index of length k of only i's. Pick $n = 2$ and let v be a partial isometry given by

$$v = S_2S_1(S_1S_2)^* + S_2S_2S_1(S_2S_1)^* + S_2S_2S_2(S_2S_2)^* \ .$$

Then $v^*v = P_{12} + P_{21} + P_{22} = 1 - P_{11}$ and $vv^* = P_{21} + P_{221} + P_{222} = P_2 = 1 - P_1$. Let $w \in \mathcal{U}(\mathcal{O}_2)$ be given by $w = v + S_1S_1^*S_1^*$. Then it is not difficult to check that $0 \neq P_1 \leq \lambda_w(P_{1(k)})$ for all $k \geq 1$ (here, the above condition is violated for the sequence $\mu_k = 1$ for all k).

We have already seen that all the generators $x_k, k \geq 0$, of F give rise to proper endomorphisms of $\mathcal{O}_2$ (and $\mathcal{D}_2$). Furthermore, all these endomorphisms turn out not to be modestly scaling.

We now examine the endomorphism of $\mathcal{O}_2$ associated with the additional generator of T.

[13]Curiously enough, $\varphi(T)$ is obviously contained in V but not in T. For instance, $\varphi(f) \notin T$.
[14]See Remark 3.2 in loc.cit. for a "topological" interpretation of this property.

Proposition 4.43. *With the notation set above, we have $\lambda_c(\mathcal{D}_2) \subsetneq \mathcal{D}_2$. Moreover, $\lambda_c(F) \subseteq F$, and $\lambda_c(T) \subseteq T$.*

Proof. The first property will follow as an application of [64, Lemma 3.4, p.7] if we exhibit a projection $P \in \mathcal{D}_2$ such that the sequence $\{u_n P u_n^*\}$ does not eventually stabilizes, where $u_k := c\varphi(c)\ldots\varphi^{k-1}(c)$. We next show that the projection P_{11} works. Indeed, the it holds $u_k^* P_{11} u_k = P_{2^k 11} := S_2^k S_1^2 (S_1^*)^2 (S_2^*)^k$. This can be proved by induction on k. Recall that $c = S_1 S_1^* S_2^* + S_2 S_1 (S_2^*)^2 + S_2^2 S_1^*$. The base of the induction can be checked as follows:

$$\begin{aligned} u^* P_{11} u &= (S_2 S_1 S_1^* + S_2^2 S_1^* S_2^* + S_1 (S_2^*)^2) P_{11} (S_1 S_1^* S_2^* + S_2 S_1 (S_2^*)^2 + S_2^2 S_1^*) \\ &= S_2 S_1^2 (S_1^*)^2 S_2^* = P_{211}. \end{aligned}$$

As for the inductive step, we have:

$$\begin{aligned} u_k^* P_{11} u_k &= \varphi^{k-1}(u)^* u_{k-1}^* P_{11} u_{k-1} \varphi^{k-1}(u) \\ &= \varphi^{k-1}(u)^* P_{2^{k-1}11} \varphi^{k-1}(u) \\ &= S_2^{k-1} u^* P_{11} u (S_2^*)^{k-1} = S_2^{k-1} P_{211} (S_2^*)^{k-1} = P_{2^k 11}. \end{aligned}$$

The inclusion $\lambda_c(F) \subset F$ amounts to showing $\lambda_c(x_0), \lambda_c(x_1) \in F$, which can be done by simple computations that are left to the reader's care. Finally, the inclusion $\lambda_c(T) \subset T$ holds because $\lambda_c(c) = S_2^2 S_2^* S_1^* S_2^* + S_2 S_1 S_2 S_1 S_1^* + S_2 S_1 S_2^2 (S_1^*)^2 S_2^* + S_1 S_1^* (S_2^*)^2 + S_2 S_1^2 (S_2^*)^3$ is in T. □

Question 4.44. Characterize the unitaries $u \in \mathcal{U}(\mathcal{O}_2)$ such that $\lambda_u(F) \subseteq F$, resp. $\lambda_u(T) \subseteq T$.

For which unitaries u does it hold $\lambda_u(F) = F$, or $\lambda_u(T) = T$?

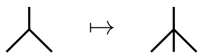
Suppose that $\lambda_u \in \mathrm{End}(\mathcal{O}_2)$ satisfies $\lambda_u(V) = V$. Then $u \in V$ and, by [110], $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2)$ (and actually $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2, \mathcal{D}_2)$). Conversely, if $u \in V$ is such that $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2)$ then $\lambda_u(V) = V$, since $\lambda(V)^{-1} = \{\lambda_v \mid v \in V\} \cap \mathrm{Aut}(\mathcal{O}_2)$ is a group [75, Theorem 2.1]. All in all, we have proved the following result.

Proposition 4.45. *For a unitary $u \in \mathcal{U}(\mathcal{O}_2)$ one has $\lambda_u(V) = V$ if and only if $u \in \{v \in V \mid \lambda_v \in \mathrm{Aut}(\mathcal{O}_2)\}$.*

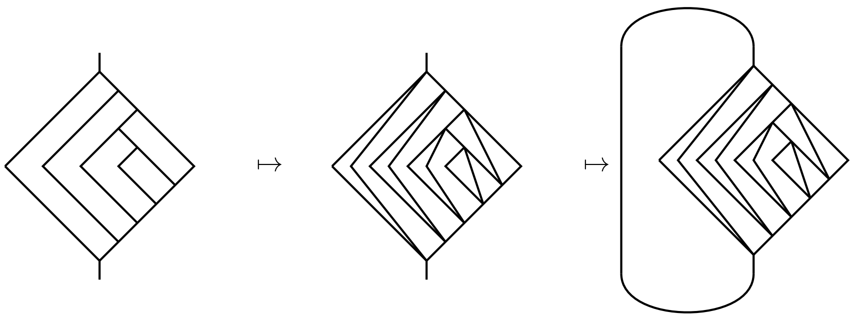
Remark 4.46. We would like to stress that the Weyl group of $\mathcal{O}_2$ embeds into $\mathrm{Aut}(V)$. Indeed, the Weyl group is isomorphic with $\lambda(V)^{-1}$, which is seen at once to embed (under restriction) into $\mathrm{Aut}(V)$ thanks to Proposition 4.45 and [110]. At this point, it would also be interesting to find an intrinsic description of the subgroup of $\mathrm{Aut}(V)$ obtained as the image of the above embedding.

In his work on the Thompson groups, Jones discovered that these groups may be used to produce knots, just like the braid groups [127]. More precisely, the Thompson group F can be used to produce all unoriented knots. Since these knots do not have a canonical orientation, Jones introduced the oriented subgroup $\vec{F} \leq F$, which may be used to construct all oriented knots [127, 4]. The oriented subgroup $\vec{F}$ can also be defined as the subgroup generated by x_0x_1, x_1x_2, x_2x_3, [106] (see also [107] for more information on $\vec{F}$). More recently, the construction of both unoriented and oriented knots was extended in [128] to F_3 and $\vec{F}_3$ (see also [17] for a study of $\vec{F}_3 \leq F_3$). In passing, we mention that thanks to this connection with knots, several unitary representations of both F and $\vec{F}$ related to knot and graph invariants have been defined [127, 7, 8, 9, 156, 6, 16].

Example 4.47. In this example we briefly review Jones' construction of links from elements of the Thompson group and thus from certain unitaries in the Cuntz algebras. Starting from a pair of trees (T_+, T_-) in F, first we turn all the 3-valent vertices into 4-valent ones (according to the rule shown below)

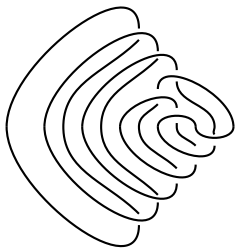


and join the two roots by an edge

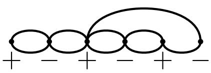


We note that the corresponding element in Cuntz algebra is $S_1S_1^* + S_2S_1(S_2S_1)^* + S_2^2S_1^2(S_2S_1)^* + S_2^2S_1S_2S_1(S_2^2S_1)^* + S_2^2S_1S_2^2(S_2^4S_1)^* + S_2^3(S_2^*)^5$. In order to obtain a link diagram, all we have to do is turn the 4-valent vertices into crossings and for this we follow this rule

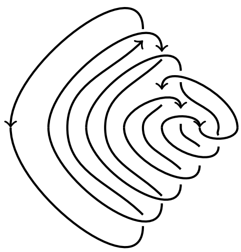
This is the resulting link diagram (actually this is a trivial link with four components)

$\mathcal{L}(T_+, T_-) =$ 

In order to obtain an oriented link diagram, one has to restrict this construction to the elements of Jones' oriented subgroup $\vec{F}$, which consists of the elements of F for which the Tait graph $\Gamma(T_+, T_-)$ of the link constructed above is 2-colourable. We denote by $+$ and $-$ the two colours. Since $\Gamma(T_+, T_-)$ is connected, if it is 2-colourable, there are exactly two colourings and by convention we choose the one in which the left-most vertex has colour $+$. In our example, the Tait diagram is

$\Gamma(T'_+, T'_-) =$ 

We observe that the vertices of the Tait graph sit in the black regions of the checkerboard shading of $\mathcal{L}(T_+, T_-)$. When a region has colour $+$, its boundary is oriented counter-clockwise. Otherwise, the orientation is clockwise. Therefore, in our example we get

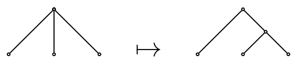
$\vec{\mathcal{L}}(T_+, T_-) =$ 

An element in F is said to be positive if it belongs to the monoid generated by x_0, x_1, ... We denote this monoid by F_+. Motivated by the investigation on the monoid of positve braids, the positive oriented Thompson links (that is the links of the $\vec{\mathcal{L}}(g)$ with g in the monoid of positive oriented elements $\vec{F}_+ := \vec{F} \cap F_+$) have been studied in a recent paper [5], where all such links have been shown to be positive and alternating. As $\vec{F}_+$ may be seen as a subgroup of the unitary group of $\mathcal{O}_2$, it might be interesting to investigate how the knots change under the action of suitable endomorphisms of the Cuntz algebra.

The (Brown-)Thompson groups F_N were introduced by Brown in [48]. These groups admit the following infinite presentation

$$\langle x_0, x_1, x_2, \dots \mid x_n x_k = x_k x_{n+N-1},\ k < n\rangle \ .$$

Clearly for $N = 2$ we get the Thompson group $F = F_2$. Similarly to the case of F, the elements of this group may be described in terms of pairs of N-ary trees with the same number of leaves. Since $\mathcal{U}(\mathcal{O}_n)$ contains a copy of V_n, it also contains (T_n and) F_n. In particular, we may identify $F = F_2$ with its copy in $\mathcal{U}(\mathcal{O}_2)$ and, more generally, F_n with its copy in $\mathcal{U}(\mathcal{O}_n)$. Now, every $\mathcal{O}_n$ can be faithfully embedded into $\mathcal{O}_2$ in some natural way. For each n, any of these embeddings induces an embedding of F_n into F_2. Here we consider the embedding of $\mathcal{O}_3 = C^*(T_1, T_2, T_3)$ into $\mathcal{O}_2 = C^*(S_1, S_2)$ such that $T_1 \mapsto S_1$, $T_2 \mapsto S_2S_1$, $T_3 \mapsto S_2S_2$. The map $F_3 \ni g \mapsto \tilde{g} \in F = F_2$ thus obtained has a natural pictorial interpretation, that is it maps a pair of triadic trees into a pair of dyadic trees by replacing any 4-valent vertex with a trivalent configuration according to the rule



(4.1)

It is surprising that this map is exactly the one defined in [156] to provide a proof of the fact that the oriented subgroup $\vec{F}$ is isomorphic with F_3 (this fact was originally proved in [106]).

In a recent work [38], Pinto et al. describe how to obtain representations of Cuntz algebras starting from a single isometry satisfying some condition. One might ask for conditions ensuring the extendability of these representations to $\mathcal{Q}_n$ and/or a characterisation of their restrictions to the Thompson groups, cf. Chapter 10. Likewise, one might consider extending the results of [28] to $\mathcal{Q}_2$, *mutatis mutandis.*

5. Cuntz Algebras and Wavelets

This section aims to briefly describe a connection between Cuntz algebras and wavelets. A wavelet on $\mathbb{R}$ is a norm-one function in $L^2(\mathbb{R})$ satisfying some particular properties which allow to define the so-called continuous wavelet transform. Wavelets are important, for example, in signal theory because they allow to express a signal in terms of functions that are localized in time-frequency. From a mathematical point of view, wavelets define good orthonormal bases: given a wavelet ψ, the set $\{2^{\frac{j}{2}}\psi(x-k)\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis called orthonormal wavelet basis. For further details the interested reader is referred to [82] for a nice introduction to wavelets and to [130] for the interplay between wavelets and Cuntz algebras (for the latter topic see e.g. also [41, 43, 39, 131]).

5.1. Some Facts About Wavelets

We write $\mathbb{R}^\times$ to denote the set $\{x \in \mathbb{R} : x \neq 0\}$. For a norm-one function $\psi \in L^2(\mathbb{R})$ we define

$$\psi_{a,b}(x) := \frac{1}{\sqrt{|a|}}\psi\left(\frac{x-b}{a}\right),$$

where $b \in \mathbb{R}$ and $a \in \mathbb{R}^\times$.

Defintion 5.1. For every $f \in L^2(\mathbb{R})$, the continuous wavelet transform of f with respect to ψ is

$$W_\psi f(a,b) := \langle f, \psi_{a,b}\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{\psi_{a,b}(x)}dx,$$

$a \in \mathbb{R}^\times$, $b \in \mathbb{R}$.

A wavelet is a function ψ such that the continuous wavelet transform with respect to ψ is an injective bounded operator from $(L^2(\mathbb{R}), dx)$ to $(L^2(\mathbb{R}\rtimes\mathbb{R}^\times), d\mu(a,b))$, where dx is the Haar measure on $\mathbb{R}$ and $d\mu(b,a) = \frac{dadb}{a^2}$ is the left Haar measure on $\mathbb{R} \rtimes \mathbb{R}^\times$. As shown in [82, Section 2.4], this happens when $\psi \in L^2(\mathbb{R})$ satisfies the following admissibility condition

$$\int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}d\xi < \infty,$$

where $\widehat{\psi}$ denotes the Fourier transform of ψ,

$$\hat{\psi}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-ix\xi}\psi(x)dx \; .$$

More precisely, we have the following result.

Theorem 5.2 ([82, Prop. 2.4.1.]). *If ψ satisfies the admissibility condition above, then for every $f, g \in L^2(\mathbb{R})$ the following formula holds:*

$$\int_G W_\psi f(a,b)\overline{W_\psi g(a,b)}d\mu(a,b) = C_\psi \langle f, g\rangle_{L^2},$$

where $d\mu(a,b) := \frac{dadb}{a^2}$ denotes the left Haar measure of $G := \mathbb{R} \rtimes \mathbb{R}^\times$ and

$$C_\psi = 2\pi\int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|}d\xi.$$

An interesting consequence of this theorem is the following reconstruction formula

$$f(x) = \frac{1}{C_\psi}\int_G W_\psi f(a,b)\psi_{a,b}(x)d\mu(a,b),$$

where the convergence of the integral is weak, namely the integral converges after taking its inner product with any function g in $L^2(\mathbb{R})$. Choosing $g = f$, we obtain the formula which expresses the norm of f,

$$\|f\|_{L^2}^2 = \frac{1}{C_\psi} \int_G |W_\psi f(a,b)|^2 d\mu(a,b).$$

Furthermore, it is now clear that the operator $W_\psi : L^2(\mathbb{R}) \longrightarrow L^2(G)$ is bounded (because it has norm $\sqrt{C_\psi}$) and injective. Thus we can use the inner product notation and write

$$C_\psi \langle f, g\rangle = \langle W_\psi f, W_\psi g\rangle_{L^2(G)}.$$

What we have seen above can be expressed using representations of locally compact groups and it is what we will do shortly (see [82, Sec. 2.5] [98]). Recalling that $G = \mathbb{R} \rtimes \mathbb{R}^\times$, it is easy to see that $\psi_{a,b}$ belongs to $L^2(\mathbb{R})$, thus we can define $U\colon G \longrightarrow \mathbb{B}(L^2(\mathbb{R}))$ by setting $(U(b,a))(\psi(x)) = \psi_{a,b}(x)$. A direct computation shows that U is a unitary representation of G on $L^2(\mathbb{R})$. Furthermore, using the Fourier transform, it can be proved that U is irreducible. Indeed, U is irriducible if and only if the representation $\hat{U}$ is, where

$$(\hat{U}(b,a))(\hat{\psi}(\xi)) = |a|^{\frac{1}{2}} \hat{\psi}(a\xi) e^{-ib\xi},$$

$(b,a) \in G$, is the representation U under the Fourier transform. Using this fact it is easy to see directly that every characteristic function, orthogonal to $\{\hat{U}(b,a)\hat{\psi} : (b,a) \in G,\ \ \psi \in L^2(\mathbb{R})\}$, must be the zero function.

Defintion 5.3. We say that U is square integrable if there exists a function $\psi \in L^2(\mathbb{R})$ such that

$$\int_G |\langle U(b,a)\psi, \psi\rangle_{L^2(G)}|^2 \mu(b,a) < \infty.$$

Using $\hat{U}$, the representation U can be shown to be square integrable more easily. Since

$$\int_G \langle \hat{U}(b,a)\hat{\psi}, \hat{\psi}\rangle_{L^2(\hat{\mathbb{R}})} \mu(b,a) = 2\pi \|\psi\|^2 \int_{\mathbb{R}} \frac{|\widehat{\psi}(\xi)|^2}{|\xi|} d\xi,$$

where $\hat{\mathbb{R}}$ is the dual of $\mathbb{R}$, this is equivalent to the admissibility condition of ψ.

In general a representation π of a locally compact group G on a Hilbert space $\mathcal{H}_\pi$ is said to be square integrable if there exists a vector $\psi \in \mathcal{H}_\pi$ such that

$$\int_G |\langle \pi(g)\psi, \psi\rangle_{\mathcal{H}_\pi}|^2 dg < \infty,$$

where dg is the left Haar measure of G. Irreducible square integrable representations on an abstract group G allow to develop a general theory of wavelets, in which a reconstruction theorem can still be proved (see [98]). Moreover, this kind of representations are all equivalent to subrepresentation of the left regular representation of G.

Going back to the concrete theory set on $\mathbb{R}$, in the next subsection we will introduce the "multiresolution" version of the classical theory which is more easily connectable to Cuntz algebras.

5.2. Multiresolution Analysis

One of the most important ways to obtain wavelets comes from the concept of multiresolution analysis. A dyadic multiresolution analysis consists of a family of closed subspaces of $L^2(\mathbb{R})$, $\{V_i\}_{i\in\mathbb{Z}}$, and a function $\varphi \in L^2(\mathbb{R})$ ("scaling function") which satisfies the following properties [82, Sec. 1.3.3]

1. $\cdots \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots$;
2. $\overline{\bigcup_{i\in\mathbb{Z}} V_i} = L^2(\mathbb{R})$ and $\bigcap_{i\in\mathbb{Z}} V_i = \{0\}$;
3. f is in V_j if and only if $f(2^j\cdot)$ is in V_0;
4. if $f \in V_0$, then $f(\cdot - n) \in V_0$ for all $n \in \mathbb{Z}$;
5. The function φ belongs to V_0 and is such that $\{\varphi(x-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_0 (note that the previous condition implies that for a fixed integer j, the set $\{2^{\frac{j}{2}}\varphi(2^j x - n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_j, [115, Sec. 2.1]).

It can be shown that whenever we have a multiresolution analysis there exists an orthonormal wavelet basis induced by a wavelet ψ such that [82, Formula (1.3.5)]

$$P_{j-1}f = P_j f + \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}$$

where $\psi_{j,k}(x) := 2^{-\frac{j}{2}}\psi(2^{-j}x - k)$, $(j,k) \in \mathbb{Z}^2$, and P_j is the orthogonal projection onto V_j. The proof of this fact [82, 43] also shows that there are two functions $m_0, m_1 \in L^\infty(\mathbb{T})$, with $m_1(x) = e^{ix}\overline{m_0(x+\pi)}$, such that

$$\hat{\varphi}(2x) = m_0(x)\hat{\varphi}(x), \quad \hat{\psi}(2x) = m_1(x)\hat{\varphi}(t) \;\; a.e.$$

and

$$\begin{cases} |m_i(x)|^2 + |m_i(x+\pi)|^2 = 1, & i = 0,1 \\ m_1(x)\overline{m_0(x)} + m_1(x+\pi)\overline{m_0(x+\pi)} = 0 & \end{cases} \quad a.e.$$

where $\hat{f}$ denotes the Fourier transform of $f \in L^2(\mathbb{R})$. These two functions will turn out to be useful to define a representation of $\mathcal{O}_2$.

5.3. Representations of Cuntz Algebras, Coding Spaces and Fractals

Let $G := \mathbb{Z}_N = \{0, 1, \ldots, N-1\}$, $N \geq 2$, be the cyclic group of order N and define

$$\Omega := \prod_1^\infty G.$$

Denote by μ the infinite product measure on Ω corresponding to the distribution of probabilities $p_i \in (0,1)$ at the points i of G. With the product topology and coordinatewise operations, Ω is an abelian compact group, and the choice $p_i = \frac{1}{N}$ corresponds to the Haar measure μ on Ω. An element $x = (i_1, i_2, \ldots) \in \Omega$ is usually called an infinite code, and corresponds to the representation of the real numbers $x \in [0,1]$ in base N, as

$$x = \sum_n \frac{i_k}{N^k}.$$

Consider the following maps $\sigma\colon \Omega \to \Omega$ and $\sigma_i\colon \Omega \to \Omega$

$$\begin{aligned} &\sigma(i_1, i_2, \cdots) \doteq (i_2, i_3, \cdots) \\ &\sigma_i(i_1, i_2, \cdots) \doteq (i, i_1, i_2, \cdots) \qquad i = 0, 1, \ldots, N-1. \end{aligned}$$

Clearly, σ is a N-to-1 map and each σ_i is a right section of σ, that is $(\sigma \circ \sigma_i)(x) = x$, $x \in \Omega$, in particular σ is surjective and σ_i is injective for every $i \in G$. Furthermore

$$\int_\Omega f d\mu = \sum_{i \in G} p_i \int_\Omega (f \circ \sigma_i) d\mu$$

for every measurable function f on Ω. Now fix $i \in G$ and let z_i be a complex number such that $|z_i|^2 = p_i$. Define the operators

$$\begin{aligned} &S_i\colon L^2(\Omega, \mu) \longrightarrow L^2(\Omega, \mu), \qquad i = 0, 1, \ldots, N-1 \\ &(S_i f)(i_1, i_2, \ldots) = \frac{1}{z_i} \delta_{i, i_1} f(i_2, i_3, \ldots), \quad f \in L^2(\Omega, \mu). \end{aligned}$$

It can be shown that these operators actually define a representation π of the Cuntz algebra $\mathcal{O}_N$ acting on the Hilbert space $L^2(\Omega, \mu)$, [130]. In this section we shall write $Rep(\mathcal{O}_N, \mathcal{H})$ to denote the representations of $\mathcal{O}_N$ on the Hilbert space $\mathcal{H}$.

Now let $m_0, \ldots, m_{N-1}$ be $L^\infty(\Omega, \mu)$-functions such that the $N \times N$ matrix $(m_i \circ \sigma_j)_{i,j \in G}$ is unitary almost everywhere. This means that the following relations hold

$$\begin{aligned} &\sum_{i \in G} \overline{m_i(h, i_1, \ldots)} m_i(k, i_1, \ldots) = \delta_{h,k} \qquad \text{for all } h,\, k \in \mathbb{Z}_N \\ &\sum_{h \in G} \overline{m_i(h, i_1, \ldots)} m_j(h, i_1, \ldots) = \delta_{i,j} \qquad \text{for all } i,\, j \in \mathbb{Z}_N,\ \mu\text{-a.a. } (i_1, \ldots) \in \Omega \end{aligned}$$

These functions allow us to define the operators

$$T_i\colon L^2(\Omega,\mu) \longrightarrow L^2(\Omega,\mu) \qquad i=0,1,\ldots,N-1$$
$$T_i(f)(i_1,i_2,\ldots) \doteq \sqrt{N} m_i(i_1,i_2,\ldots)f(i_2,i_3,\ldots).$$

As shown in [130, Sec. 2], the operators T_i provide a representation ρ of $\mathcal{O}_N$ on $L^2(\Omega,\mu)$ defined by $\rho(s_i) \doteq T_i$, $i \in G$.

In general, given any two representations $\{S_i\}_{i\in G}$ and $\{T_i\}_{i\in G}$ of the Cuntz algebra $\mathcal{O}_N$ on the same Hilbert space $\mathcal{H}$, they are related by an operator-valued unitary matrix $(U_{i,j})$. More precisely, its entries satisfy the following conditions

$$U_{i,j} \in \mathcal{B}(\mathcal{H}), \;\; U_{i,j} = S_i^* T_j \quad \text{and with} \quad T_i = \sum_{j\in G} S_j U_{j,i}\,.$$

In particular, for the representations π and ρ defined above, the matrix $(U_{i,j})$ is

$$(U_{i,j}f)(i_1,i_2,\ldots) = (m_j \circ \sigma_i)(i_1,i_2,\ldots)f(i_1,i_2,\ldots) = m_j(i,i_1,i_2,\ldots)f(i_1,i_2,\ldots).$$

see [130, Prop. 2.1].

5.4. Representations of $\mathcal{O}_2$ and Wavelets

We have seen above that wavelet theory deals with Hilbert spaces such as $L^2(\mathbb{R})$ or $L^2(\mathbb{T})$, whereas we have exhibited two representations of $\mathcal{O}_2$ on the Hilbert of square integrable functions on the fractal Ω which, in this case, is the infinite product of $\mathbb{Z}_2$, the cyclic group with elements 0, 1. The next result say that, in some cases, a Hilbert space like $L^2(\Omega)$ may be replaced, in some sense, by $L^2(\mathbb{T})$.

Proposition 5.4. *Let $(X,\mathcal{A},\nu)$ be a probability space, (if X is a Hausdorff topological space, we suppose that $\mathcal{A}$ contains the σ-algebra of all Borel sets). Fix a natural number N and assume that there are measurable functions*

$$\varphi\colon X \longrightarrow X,$$
$$\varphi_i\colon X \longrightarrow X, \;\; i=0,1,\ldots,N-1$$

such that φ is a N-to-1 map and φ_i are sections of φ. Assume finally there are positive numbers $P_0,\ldots,P_{N-1}$ such that $\sum_i P_i = 1$ and

$$\int_X f d\nu = \sum_{i=0}^{N-1} P_i \int_X (f\circ\varphi_i) d\nu.$$

Then, with the notation introduced in the previous subsections, there exists a measure isomorphism between X and Ω such that φ corresponds to σ and each φ_i corresponds to σ_i.

A proof of the following result is in [130, Corollary 3.2].

Theorem 5.5. *Every dyadic multiresolution analysis yields a representation ρ of $\mathcal{O}_2$ on $L^2(\mathbb{T})$.*

Composing representations with endomorphims is an effective way to obtain new representations. For instance, composing a permutative representation of a Cuntz algebra with a permutative endomorphim yields a representation which is still permutative. Analogously, it is a natural problem to look for automorphims or endomorphims that are compatible with wavelet representations. This leads us to formulate the following question.

Question 5.6. If π is a wavelet representation of $\mathcal{O}_2$ and $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2)$ is a Bogolubov automorphism, is $\pi \circ \lambda_u$ still a wavelet representation?

It is quite possible that the answer to the above question may be negative in general. In that case, one should try to to characterize those Bogolubov automorphisms λ_u such that $\pi \circ \lambda_u$ continues to be a representation coming from a wavelet.

6. Index and entropy of endomorphisms

Over the years Cuntz algebras have proved to be quite useful in the study of subfactors [108, 129], and several slightly different approaches have been followed to exhibit novel examples of subfactors and deal with both known and new examples.

One approach is to start with a unitary in $\mathcal{O}_n$, which is often selected for combinatorial and/or "aesthetic" reasons. Examples include solutions to the YB-equations [68], permutation matrices [69], Hadamard matrices, to name but few. Once a unitary $u \in \mathcal{O}_n$ is chosen, if λ_u is a proper endomorphism, in principle one may consider two different types of subfactors.[15] If λ_u restricts to an endomorphism of $\mathcal{F}_n$, one can consider the extension of $\lambda_u \restriction_{\mathcal{F}_n}$ to the weak closure of $\mathcal{F}_n$ with respect to the GNS representation of the unique trace. This yields a II_1 subfactor (here, $\mathcal{F}_n''$ is nothing but the unique hyperfinite factor of type II_1). Alternatively, one may look for a normal extension of λ_u to, say, the AFD factor of type $\mathrm{III}_{1/n}$ generated by $\mathcal{O}_n$ in the GNS representation associated with the KMS state ω and study the inclusion $\lambda_u(\mathcal{O}_n)'' \subset \mathcal{O}_n''$. Such an extension can always be found when $u \in \mathcal{F}_n^k$ for some k. These paths were first undertaken by Longo [142], Jones [126, Example 3.2], Akemann [18] (cf. [129, Example 5.1.6]), Conti and Pinzari [69]. Of course, once a subfactor is available, one may investigate the type of inclusion and the first problem is to compute its Jones index [125] or the type-III minimal index

$$[\mathcal{F}_n'' : \lambda_u(\mathcal{F}_n)'']$$
$$\mathrm{Ind}(\lambda_u) = \mathrm{Ind}(\lambda_u(\mathcal{O}_n)'' \subset \mathcal{O}_n'')$$

[15] Here one should be slightly more pedantic, in that there might be some proper endomorphism at the C^*-algebra level that extends to an automorphism of the corresponding weak closure, meaning that the subfactor is trivial in those cases.

where Ind denotes the index of the inclusion w.r.t. the unique minimal conditional expectation. Needless to say, all these index values belong to the famous set $\{4\cos^2(\pi/n) \mid n = 3, 4, 5, \ldots\} \cup [4, +\infty]$.

Recall that an endomorphism λ_u of $\mathcal{O}_n$ is called localized when $u \in \mathcal{U}(\mathcal{F}_n^k)$. It was shown in [142] that, for these endomorphisms, the Jones index $[\mathcal{F}_n'' : \lambda_u(\mathcal{F}_n)'']$ coincides with the index of the inclusion $\lambda_u(\mathcal{O}_n)'' \subset \mathcal{O}_n''$ w.r.t. the unique ω-invariant conditional expectation (whose existence is guaranteed by Takesaki's Theorem) and, moreover, it holds

$$\mathrm{Ind}(\lambda_u) \leq [\mathcal{F}_n'' : \lambda_u(\mathcal{F}_n)''] \leq n^{2(k-1)} .$$

It is worth stressing that for the above subfactors the property of being irreducible may vary depending on whether type II or type III cases are being considered.

Question 6.1. Find closed formulae to compute the above indices in terms of $u \in \mathcal{U}(\mathcal{F}_n^k) \simeq U(n^k)$.

See [69] for a number of general results in this direction. Moreover, if $u \in \mathcal{U}(\mathcal{F}_2^2)$, then $[\mathcal{F}_2'' : \lambda_u(\mathcal{F}_2)''] \in \{1, 2, 4\}$ [18]. It may be worth pointing out that the correspondence between unitaries and values of the index is not continuous. Actually, $\mathcal{U}_n^2 := \{u \in \mathcal{U}(\mathcal{F}_n^2) \mid \lambda_u \in \mathrm{Aut}(\mathcal{O}_n'')\}$ contains a dense open set in $\mathcal{U}(\mathcal{F}_n^2)$ [18]. Furthermore, the following is a long-standing conjecture by Longo.

Question 6.2. Is the index of a localized endomorphism a rational number? Is the index of an irreducible localized endomorphism always an integer?

For instance, this is certainly the case for the canonical endomorphism φ on $\mathcal{O}_n$, whose index is n^2, for $\varphi \simeq \mathrm{id} \oplus \cdots \oplus \mathrm{id}$ (n direct summands). More in general, in [74] the index is computed for any localized endomorphism $\lambda_u \in \mathrm{End}(\mathcal{O}_2)$ with $u \in \mathcal{P}_2^2$. In addition, further evidence that the conjecture might hold true is provided in [68], where endomorphisms associated with R-matrices are analyzed. There, it is also shown that the set of Jones indices obtained from R-matrices of any size is closed under multiplication. It is likely that this set contains $\mathbb{N}$, so one should worry whether it coincides with $\mathbb{N}$ or not.

Another approach is based on the interesting interplay between the general study of subfactors and the endomorphisms of the Cuntz algebras, which is recurrent in Izumi's work. We next recall it below. The starting point is that a finite-index properly infinite subfactor gives rise to a system of intertwining isometries for the system of sectors in the sense of Longo. In [116, 117] Izumi decided to undertake the opposite direction. Motivated by the principal graph (and the dual principal graph) of a subfactor, he assumed the existence of certain fusion rules of sectors and then deduced formulas for endomorphisms of a Cuntz algebra (which are not necessarily localised) realising the given fusion rules. Here an important role was played by a finite abelian group which gives rise to a system of equations whose solution determine the endomorphisms. From a deep analysis of these data, Izumi was then able to (re)construct interesting subfactors, including the E_6

subfactor and the Haagerup subfactor of index $(5+\sqrt{13})/2$. In this construction the ambient factor is obtained by passing to the weak closure in the GNS of some quasi-free KMS state of a Cuntz algebra (which in the aforementioned two examples is $\mathcal{O}_4$). This approach was really innovative since only two other methods for constructing subfactors were known before: by means of group actions or from AF-algebras arising from commuting squares. Thanks to this method Izumi also constructed several new examples of subfactors [122].

As already exemplified in Izumi's work, especially in the type-III setting, which is somewhat more flexible, the categorical structures associated to systems of endomorphisms of a Cuntz algebra, e.g. the fusion rules (modulo inners), are also worth studying.

Of course, one may also study the inclusion $\lambda_u(\mathcal{O}_n) \subset \mathcal{O}_n$ at C^*-algebra level by means of the Watatani index. However, this purely C^*-algebraic approach to index theory is somewhat more convoluted [118, 119].

Parallel to the study of indices/subfactors, there are also investigations on Voiculescu's topological entropy in the context of endomorphisms of Cuntz algebras. We briefly recall the definition of topological entropy from [171, Section 4]. Let $\mathcal{A}$ be a nuclear C^*-algebra and let $\alpha\colon \mathcal{A} \to \mathcal{A}$ be an endomorphism. Denote by $\mathrm{CPA}(\mathcal{A})$ the set of triples $(\phi, \psi, \mathcal{B})$, where $\mathcal{B}$ is a finite-dimensional C^*-algebra, $\phi\colon \mathcal{A} \to \mathcal{B}$, $\psi\colon \mathcal{B} \to \mathcal{A}$ are unital completely positive maps. For any $\epsilon > 0$ and any finite subset $\omega \subset \mathcal{A}$ (for brevity we write $\omega \in \mathcal{P}f(\mathcal{A})$), the completely positive ϵ-rank is defined by the following formula

$$\mathrm{rcp}(\omega, \epsilon) := \inf\{\mathrm{rank}(\mathcal{B}) \mid (\phi, \psi, \mathcal{B}) \in \mathrm{CPA}(\mathcal{A}), \|(\psi \circ \phi)(a) - a\| < \epsilon \ \text{ for } a \in \omega\}$$

We set

$$\begin{aligned} \mathrm{ht}(\alpha, \omega; \epsilon) &:= \limsup_{n\to\infty} \frac{\log \mathrm{rcp}(\omega \cup \alpha(\omega) \cup \ldots \cup \alpha^{n-1}(\omega); \epsilon)}{n} \\ \mathrm{ht}(\alpha, \epsilon) &:= \sup_{\epsilon>0} \mathrm{ht}(\alpha, \omega; \epsilon) \end{aligned}$$

Accordingly, the topological entropy of α is then defined as

$$\mathrm{ht}(\alpha) := \sup_{\omega \in \mathcal{P}f(\mathcal{A})} \mathrm{ht}(\alpha, \omega)\,.$$

A way to obtain a lower bound for the topological entropy is to consider a commutative C^*-algebra $\mathcal{C}$ of $\mathcal{A}$ that is invariant under α and using that $\alpha\restriction_{\mathcal{C}}$ is induced by a homeomorphism T of the spectrum of $\mathcal{C}$. Then, it holds $\mathrm{ht}(\alpha) \geq \mathrm{ht}(\alpha\restriction_{\mathcal{C}}) = \mathrm{h}_{\mathrm{top}}(T)$, [171].

In the context of Cuntz algebras, the first computation of the topological entropy is due to Choda [54], who showed that for the canonical endomorphism $\varphi\colon \mathcal{O}_n \to \mathcal{O}_n$ it is equal to $\log(n)$. Later, a systematic analysis of the topological entropy for permutative endomorphisms of the Cuntz algebra was performed by

Skalski and Zacharias in [164]. More precisely, they obtained a general bound for the entropy for localized endomorphisms [164, Theorem 2.2], namely

$$\mathrm{ht}(\lambda_u) \leq (k-1)\log(n) \qquad u \in \mathcal{U}(\mathcal{F}_n^k)\ .$$

(The bound is the logarithm of the square root of the analogous bound for the index that we saw before.) Then, the entropy of all the permutative endomorphisms of $\mathcal{O}_2$ of level 2 is computed. These are the endomorphisms λ_u, with u of the form $\sum_{\alpha,\beta} S_\alpha S_\beta^*$, α and β being multi-indices of length 2. For each of these cases, they considered the restriction of the endomorphism to a suitable possibly non-standard invariant MASA and showed that the restriction and the whole endomorphism have the same entropy. It is unknown whether the same holds for every localized endomorphism of the Cuntz algebras (see the introduction of [164]).

Question 6.3. Compute the topological entropy of permutative endomorphisms λ_u, with $u \in \mathcal{P}_n^k$, $(n,k)\ \neq (2,2)$.

Since the topological entropies of the endomorphisms considered in [164] are always 0 or $\log(2)$, the following question is entirely natural to raise.

Question 6.4. Is the topological entropy of permutative (resp.: localized) endomorphisms of $\mathcal{O}_n$ always of the form $\log(m)$, with $m \in \mathbb{N}$? Or even $\log(n^h)$, when $u \in \mathcal{P}_n^k$ (resp.: $u \in \mathcal{U}(\mathcal{F}_n^k)$), with $h \in \{0, \dots, k-1\}$?

The following question was already raised in [74].

Question 6.5. The endomorphisms considered in [164] have topological entropy equal to 0 exactly when they are automorphisms. The same is true for the endomorphisms restricted to the diagonal subalgebra $\mathcal{D}_2$. Is this a general fact, or a mere coincidence?

Question 6.6. Is there any definite relation between the values of the index and of the topological entropy of an endomorphism?

In [164] it is asked if the entropy of the permutative endomorphisms of $\mathcal{O}_n$ remains the same after restriction to the UHF subalgebra $\mathcal{F}_n$. One might also want to compute the entropy of the extension of the localized endomorphisms of $\mathcal{O}_p$ to $\mathcal{Q}_p$, $q \geq 2$ (see sections 9 and 10), when such extensions do exist [13].

7. Fixed points of endomorphisms

Fixed-point subalgebras of automorphisms and endomorphisms of $\mathcal{O}_n$ have been computed explicitly in some cases. For instance ${\mathcal{O}_n}^{\varphi} = \mathbb{C}$, but for the quasi-free automorphism λ_f of $\mathcal{O}_2$ one has ${\mathcal{O}_2}^{\lambda_f} \simeq \mathcal{O}_2$; it is also known that ${\mathcal{O}_2}^{\lambda_{-1}} \simeq \mathcal{O}_4$. This is stated (without proof) e.g. in [120, Example 5.8]. We include a proof for completeness.

Proposition 7.1. *The fixed-point algebra* ${\mathcal{O}_2}^{\lambda_{-1}}$ *is generated by* $\{S_i S_j\}_{1\leq i,j\leq 2}$ *and thus it is isomorphic to* $\mathcal{O}_4$*.*

Proof. It is clear that the S_iS_j are fixed points, so $A := C^*(S_iS_j \mid 1 \leq i,j \leq 2) \subseteq {\mathcal{O}_2}^{\lambda_{-1}}$. Concerning the opposite inclusion, notice that $S_iS_j(S_j)^*S_k^* \in A$, and thus $S_iS_j^* \in A$ for all i,j. Actually, by a similar argument, one gets that $\mathcal{F}_2 \subset A$. Taking the expectation onto the fixed point subalgebra, it is clear that any $x \in {\mathcal{O}_2}^{\lambda_{-1}}$ can be approximated in norm by elements in the span of the $S_\alpha S_\beta^*$ that are invariant, i.e. such that $|\alpha| + |\beta|$ is even. Now, there are two cases: if $|\alpha|$ and $|\beta|$ are both even, it is clear that $S_\alpha S_\beta^* \in A$; on the other hand, if $|\alpha|$ and $|\beta|$ are both odd, say r and s respectively, then $S_\alpha S_\beta^* = S_{\alpha_1\cdots\alpha_{r-1}}(S_{\alpha_r}S_{\beta_s}^*)S_{\beta_{s-1}\cdots\beta_1}^*$ and the conclusion is now clear. □

With a slightly heavier argument one can prove that the above proposition holds in general.

Proposition 7.2. *Let $\mu \in \mathbb{T}$ be a given primitive kth-root of unity, then the fixed-point algebra of the gauge automorphism $\lambda_{\mu 1} \in \mathrm{Aut}(\mathcal{O}_n)$ is generated by $\{S_{i_1}\cdots S_{i_k}\}_{1\leq i_1,\ldots,i_k\leq n}$ and is thus isomorphic to $\mathcal{O}_{n^k}$. On the other hand, if $\mu \in \mathbb{T}$ is not of the form $e^{2\pi i p/q}$ with $p,q \in \mathbb{Z}$, then the fixed-point algebra of $\lambda_{\mu 1} \in \mathrm{Aut}(\mathcal{O}_n)$ is precisely $\mathcal{F}_n$.*

For the proof of the second part observe that an element fixed by $\lambda_{\mu 1}$ is actually fixed by every gauge automorphism in that $\{\mu^n : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$.

This means that we have computed the fixed-point algebras for all the gauge automorphisms.

Question 7.3. What can be said about the fixed-point algebra under general quasi-free automorphisms. Can they be computed explicitly?

As a simple observation, a non-gauge quasi-free automorphism λ_u, $u \in \mathcal{U}(\mathcal{F}_n^1) \setminus \mathbb{T}1$ has always u, and thus $C^*(u)$, among its fixed points, which are thus nontrivial. But as the example of the flip-flop shows, there can be much more stuff.

The regular representation ρ of $\mathbb{Z}_n$ induces a natural action of $\mathbb{Z}_n$ on $\mathcal{O}_{|\mathbb{Z}_n|} = \mathcal{O}_n$ by means of the corresponding Bogolubov automorphisms. Let E be the unitary matrix corresponding to $\rho(1)$. The corresponding Bogolubov automorphism λ_E of $\mathcal{O}_n$ is called the *cyclic automorphism*. When $n = 2$, the exchange automorphism is nothing but the flip-flop automorphism λ_f. We observe that $\mathcal{O}_{|\mathbb{Z}_n|}^{\mathbb{Z}_n} = \mathcal{O}_{|\mathbb{Z}_n|}^{\lambda_{\rho(1)}}$ and, by a previously mentioned result of Izumi and Pinzari, this algebra is isomorphic with $\mathcal{O}_{|\mathbb{Z}_n|}$. When $n = 2$, a companion result holds for the crossed product. More precisely, in [55, Theorem 2.1] it was shown that $\mathcal{O}_2 \rtimes_{\lambda_f} \mathbb{Z}_2$ is isomorphic with $\mathcal{O}_2$. In the following result we extend this to all n.

Theorem 7.4. *For any $n \geq 2$, the C^*-algebras $\mathcal{O}_n \rtimes_{\lambda_E} \mathbb{Z}_n$ and $\mathcal{O}_n$ are isomorphic.*

Proof. The C^*-algebra $\mathcal{O}_n \rtimes_{\lambda_E} \mathbb{Z}_n$ is generated by $\mathcal{O}_n$ and a unitary V such that $VS_iV^* = S_{i+1}$ for all $i = 1,\ldots,n-1$, $VS_nV^* = S_1$, $V^n = 1$. First we want to show that $V \in C^*(S_1, VS_1, \ldots, V^{n-1}S_1) \subseteq \mathcal{O}_n \rtimes_{\lambda_E} \mathbb{Z}_n$. Indeed, we have

$$\begin{aligned}
S_1S_1^*V^* + \sum_{i=1}^{n-1}(V^{i+1}S_1S_1^*(V^*)^i)^* &= S_1S_1^*V^* + \sum_{i=1}^{n-1} V^iS_1S_1^*V^{-i-1} \\
&= S_1S_1^*V^* + \sum_{i=1}^{n-1} V^iS_1S_1^*V^{-i}V^* \\
&= S_1S_1^*V^* + \sum_{i=1}^{n-1} S_{i+1}S_{i+1}^*V^* = V^*\,.
\end{aligned}$$

Now $V^iS_1V^{-i} = S_i$ and, thus, $\mathcal{O}_n \rtimes_{\lambda_E} \mathbb{Z}_n = C^*(S_1, VS_1, \ldots, V^{n-1}S_1)$. Finally, it suffices to show that $C^*(S_1, VS_1, \ldots, V^{n-1}S_1)$ is isomorphic with $\mathcal{O}_n$ and we are done. Indeed,

$$\begin{aligned}
S_1S_1^* + \sum_{i=1}^{n-1}(V^iS_1S_1^*(V^*)^i)^* &= S_1S_1^* + \sum_{i=1}^{n-1} V^iS_1S_1^*V^{-i} \\
&= S_1S_1^* + \sum_{i=1}^{n-1} S_{i+1}S_{i+1}^* = 1\,.
\end{aligned}$$

□

It should be mentioned that several fixed-point algebras and crossed products of both the Cuntz algebras and their tensor products were studied in [120, Section 5]. For example, Izumi showed that the crossed product $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_m$, where $\alpha(S_1) = S_1$ and $\alpha(S_2) = e^{2\pi i/m}S_2$, is isomorphic with $\mathcal{O}_2$. Similarly, the fixed-point subalgebra of $\bigotimes_{i=1}^p \mathcal{O}_2$, with p prime, under the cyclic permutation of the tensor components was also shown to be isomorphic with $\mathcal{O}_2$.

One can also examine fixed points under proper endomorphisms. In a recent paper by Conti and Lechner [68] the problem is addressed of characterizing the fixed-point algebra of Cuntz algebras under YB-endomorphisms, which include $\varphi = \lambda_F$. This fixed-point algebra can be trivial or not, but it can never be "too big" unless the unitary solution of the YBE is a root of 1, cf. Proposition 7.2. We report from the v1 of the arXiv version of [68].

Proposition 7.5. *Let $R \in \mathcal{U}(\mathcal{F}_n^2)$ be a unitary solution of the Yang-Baxter equation. If $\mathcal{O}_n^{\lambda_R}$ is simple and purely infinite then $R = \mu 1$, where $\mu \in \mathbb{T}$ is an m-th root of unity for some positive integer m.*

Proof. Suppose that the fixed point algebra is simple purely infinite. Then it is not contained in $\mathcal{F}_n$, and thus there exists some $x \in \mathcal{O}_n^{\lambda_R}$ with a non-zero spectral component $x^{(m)} \in \mathcal{O}_n^m$, for some $m > 0$. Now, necessarily one has $\lambda_R(x^{(m)}) = x^{(m)}$, and from this equality it follows easily from the YBE that

$x^{(m)}$ commutes with $C^*\{\varphi^h(R) \ : \ h \in \mathbb{N}_0\} \subset \mathcal{F}_n$ (cf. Lemma 7.6). Taking into account the fact that R is unitary we get

$$\begin{aligned}
&\|R\varphi(R)\cdots\varphi^{k+m-1}(R)x^{(m)}\varphi^{k-1}(R)^*\cdots\varphi(R)^*R^* - x^{(m)}\| \\
&\qquad = \|\varphi^k(R)\cdots\varphi^{k+m-1}(R)x^{(m)} - \varphi^{k-1}(R^*)\cdots R^*x^{(m)}R\cdots\varphi^{k-1}(R)\| \\
&\qquad = \|\varphi^k(R)\cdots\varphi^{k+m-1}(R)x^{(m)} - x^{(m)}\| \to 0
\end{aligned}$$

when $k \to \infty$. Pick $y, z \in \mathcal{O}_n^{\lambda_R}$ such that $yx^{(n)}z = 1$. Then,

$$\begin{aligned}
\|\varphi^k\big(R\cdots\varphi^{m-1}(R)\big) - 1\| &= \|\varphi^k(R)\cdots\varphi^{k+m-1}(R) - 1\| \\
&= \|y(\varphi^k(R)\cdots\varphi^{k+m-1}(R)x^{(m)} - x^{(m)})z\| \\
&\le \|\varphi^k(R)\cdots\varphi^{k+m-1}(R)x^{(m)} - x^{(m)}\| \, \|y\| \, \|z\| \longrightarrow 0
\end{aligned}$$

as $k \to \infty$. Since φ is unital and isometric, we get $R\cdots\varphi^{m-1}(R) = 1$. However, since $R^* \in U(\mathcal{F}_n^2)$ is also a unitary solution of the YBE, it follows that λ_{R^*} is not surjective and $\lambda_{R^*}^m = \lambda_{\varphi^{m-1}(R^*)\cdots\varphi(R^*)R^*}$ is not the identity, unless $R = \mu 1$ with $\mu^m = 1$. □

See [68, Section 8] for a more detailed analysis of the fixed-point algebra of YB-endomorphisms when $n = 2$.

As usual, we say that an endomorphism of a C^*-algebra is *ergodic* if the fixed-point algebra is trivial. The following property might come in useful to exhibit examples of ergodic endomorphisms of $\mathcal{O}_n$.

Lemma 7.6. *Let λ_u be an endomorphism of $\mathcal{O}_n$, then $\mathcal{O}_n^{\lambda_u}$ commutes with*

$$C^*\big((\lambda_u^k, \lambda_u^h),\ k, h \in \mathbb{N}_0\big)\ .$$

Proof. Let x be an element of $\mathcal{O}_n^{\lambda_u}$. If T belongs to $(\lambda_u^k, \lambda_u^h)$ for some h, $k \in \mathbb{N}_0$, then

$$Tx = T\lambda_u^k(x) = \lambda_u^h(x)T = xT\ .$$

□

In particular, if the C^*-algebra generated by the intertwiners is irreducible in $\mathcal{O}_2$, then λ_u is certainly ergodic.

We now generalise Proposition 7.5 to a suitable family of unitaries.

Lemma 7.7. *For any $k \in \mathbb{N}$, φ^k is an ergodic endomorphism of $\mathcal{O}_n$.*

Proof. The claim follows directly from the fact that $\bigcap_h \varphi^h(\mathcal{O}_2) = \mathbb{C}1$. □

Theorem 7.8. *Let u be in $\mathcal{U}(\mathcal{F}_n)$ such that $\varphi^i(u) \in C^*\big((\lambda_u^k, \lambda_u^h),\ k, h \in \mathbb{N}_0\big)$ for all $i \in \mathbb{N}_0$. If $\mathcal{O}_n^{\lambda_u}$ is simple and purely infinite then $u = \mu 1$, where $\mu \in \mathbb{T}$ is an m-th root of unity for some positive integer m.*

Proof. By similar arguments to those in the proof Proposition 7.5, but using Lemma 7.6 instead of the YBE, we deduce that $u\varphi(u)\cdots\varphi^{m-1}(u)=1$ for some $m\in\mathbb{N}$. This leads to

$$u=u\varphi(1)=u\varphi(u)\cdots\varphi^{m-1}(u)\varphi^m(u)=\varphi^m(u)$$

and thanks to Lemma 7.7 we are done. □

Remark 7.9. All nontrivial solutions R of the Yang-Baxter equation obviously satisfy the first hypothesis in the above theorem. More interestingly, the same is true for $\varphi(R)$ as well. Indeed, $\varphi^i(\varphi(R))=\varphi^{i+1}(R)$ belongs to the interwiner space $(\lambda^{i+2}_{\varphi(R)},\lambda^{i+2}_{\varphi(R)})$. This is in turn an easy application of the formula $\lambda^k_{\varphi(R)}=\lambda_{\varphi^k(R)\varphi^{k-1}(R)\cdots\varphi(R)}$, which can be proved by induction on $k\in\mathbb{N}_0$, along with the set equality $(\lambda_u,\lambda_u)=\{T:u^*Tu=\varphi(T)\}$. For the latter equality see also [70, Section 2]. In particular, we find that the fixed-point subalgebra $\mathcal{O}_n^{\lambda_{\varphi(R)}}$ cannot be both simple and purely infinite.

Another working example is $\varphi^2(R)$, where R is again a solution of the Yang-Baxter equation. Indeed, we now have $\varphi^i(\varphi^2(R))\in(\lambda^{i+4}_{\varphi^2(R)},\lambda^{i+4}_{\varphi^2(R)})$, which can be easily ascertained thanks to the equality $\lambda^k_{\varphi^2(R)}=\lambda_{\varphi^{k+1}(R)\varphi^k(R)\cdots\varphi^3(R)\varphi^2(R)}$ for any natural $k\geq 2$.

The unitary $\varphi(R)R$, too, satisfies the hypothesis. This is a consequence of the formula $\lambda^k_R=\lambda_{\varphi^{k-1}(R)\cdots\varphi(R)R}$, see [68].

For certain endomorphisms, the relationship between fixed elements and intertwiners becomes even more striking. Since $(\lambda_u,\lambda_u)=\{T\in\mathcal{O}_n\mid\varphi(T)=u^*Tu\}$ we get $(\lambda_u,\lambda_u)=\mathcal{O}_n^{\mathrm{Ad}(u)\varphi}$. In particular, for $u=F\in\mathcal{F}^2_n$ we find that

$$\mathcal{O}_n^{\lambda_{\varphi(F)}}=(\varphi,\varphi)=\mathcal{F}^1_n\ .$$

Obviously, there do exist ergodic *endomorphisms*: an example is the canonical endomorphism φ, which even enjoys the property of being a shift, i.e. $\bigcap_n\varphi^n(O_2)=\mathbb{C}1$. Nevertheless, the endomorphism φ fails to be *uniquely* ergodic[16], which rises the question of whether it is possible to exhibit uniquely ergodic endomorphisms as well. (The same question can of course be asked at the level of the UHF subalgebra.) It is also natural to exhibit some example of ergodic automorphisms as well. As pointed out to us by Izumi, a simple example of an ergodic automorphism of $\mathcal{O}_2$ is given by the Bernoulli shift on $\bigotimes_{k\in\mathbb{Z}}\mathcal{O}_2\simeq\mathcal{O}_2$. However, again this example falls short of being uniquely ergodic, for any product state $\otimes_{k\in\mathbb{Z}}\omega_k$, with $\omega_k=\omega\in\mathcal{S}(\mathcal{O}_2)$ for all $k\in\mathbb{Z}$ is shift-invariant.

Question 7.10. Discuss the existence of (uniquely) ergodic endomorphisms/automorphisms of the Cuntz algebras $\mathcal{O}_n$ and possibly "classify" them. When they do exist, provide some explicit examples.

[16] An endomorphism of a C^*-algebra is said to be uniquely ergodic if it admits a unique invariant state. For an uncountable family of such examples see *e.g* [85]. It is not difficult to see that a uniquely ergodic endomorphism is automatically ergodic. Moreover, if $\rho\in\mathrm{End}(\mathcal{A})$ is uniquely ergodic, with ω being its only invariant state, then for any $a\in\mathcal{A}$ the Cesàro means $\frac{1}{n}\sum_{k=0}^{n-1}\rho^k(a)$ converge to $\omega(a)1$ in norm.

Remark 7.11. It might be worth stressing that there are C^*-algebras whose automorphisms are never ergodic. The most notable example that should immediately spring to mind is $\mathcal{B}(\mathcal{H})$. Indeed, none of its automorphisms can be ergodic in that, as is known, these are all inner. Nevertheless, ergodic endomorphisms on $\mathcal{B}(\mathcal{H})$ do exist, see [42].

8. Physics, KMS states, and Noncommutative geometry

The interplay between physics and Cuntz algebras boasts a long-lasting history. Even before their formal definition in [78], the Cuntz algebras had in fact made an appearance in quantum physics. Indeed, Doplicher and Roberts systematically used multiplets of isometries satisfying the Cuntz relations since the early 1970s to describe the superselection sectors in algebraic quantum field theory, see *e.g.* [86]. To sum up, in that setting one starts with a C^*-algebra $\mathfrak{A}$ ("observables") and a suitable endomorphism $\rho \in \mathrm{End}(\mathfrak{A})$. An important step is then to write the action of ρ on the elements of $\mathfrak{A}$ as $\rho(a) = \sum_{i=1}^d S_i a S_i^*$, $a \in \mathfrak{A}$, with $d = d_\rho$, the so-called *statistical dimension* of ρ, where the S_i's are isometries with $\sum_{i=1}^d S_i S_i^* = 1$ that do not necessarily belong to $\mathfrak{A}$ but will sit in a larger C^*-algebra $\mathfrak{F}$ ("fields").[17] Thus, after their appearance the Cuntz algebras immediately began to be given much attention not least because of the role played in the theory alluded to above. On their way to obtain a novel duality theory for compact groups that would generalize the Tannaka-Krein theory, Doplicher and Roberts then also realized in [87] that the Cuntz algebras were examples of a more general construction which yields a C^*-algebra $\mathcal{O}_\rho$ out of an object of a ρ of a C^*-tensor category, and $\mathcal{O}_n$ corresponds to $\mathcal{H}_n$ the n-dimensional complex Hilbert space, thought of as an object of the C^*-tensor category of Hilbert spaces with morphims given by bounded linear maps. The inspiration provided by the Cuntz algebras proved soon after to be a successful tool to arrive at the sought duality theory, [88, 89]. More precisely, Cuntz algebras are all is needed to treat compact *Lie* groups. Indeed, any such group G can be viewed as a subgroup of $U(d)$, the group of unitary $d \times d$ matrices, for some $d \geq 1$. Therefore, there is a natural action of G on the Cuntz algebra $\mathcal{O}_d$ through Bogolubov automorphisms, and a corresponding fixed-point subalgebra, $\mathcal{O}_d^G$. Now if we denote by $\mathcal{T}_G$ the C^*-tensor category whose objects are the tensor powers of ρ, the faithful representation that embeds G into $U(d)$, and the arrows between them given by the intertwiners, the striking property is that the corresponding C^*-algebra $\mathcal{O}_\rho$ is just $\mathcal{O}_d^G$. Moreover, if G can be realized as a subgroup of $SU(d)$, the special unitary group, then the corresponding C^*-algebra $\mathcal{O}_d^G$ will be simple. For instance, the whole Cuntz algebra $\mathcal{O}_d$ corresponds to taking $G = \{e\} \subset U(d)$. This elegant theory most definitely contributed to attracting the interest of an even wider audience of mathematicians. on the Cuntz algebras.

In what follows, though, we regrettably leave the themes outlined above to

[17]The C^*-algebra $\mathfrak{F}$ is the crossed product of $\mathfrak{A}$ by the category generated by the endomorphisms ρ as above in a sense that can be made precise.

introduce different instances of the interaction between Cuntz algebras and mathematical structures which are suggested by physics.

8.1. One-parameter groups of automorphisms and their KMS states

We start by recalling the definition of KMS state from [44]. Let A be a C^*-algebra and $\sigma\colon \mathbb{R} \to \mathrm{Aut}(A)$ a one-parameter group of automorphisms. An element $x \in A$ is said to be analytic if the map $t \mapsto \sigma_t(x)$ admits an extension to an analytic function from $\mathbb{C}$ to A. Now given $\beta > 0$, a linear functional $\psi\colon A \to \mathbb{C}$ is a KMS_β-state if it is a state satisfying the condition

$$\psi(xy) = \psi(y\sigma_{i\beta}(x))$$

for all analytic x and y in A.

This notion has deeply influenced the field of mathematical physics over the last fifty years or more. To take but one example, we would like to briefly report on the recent paper [168], where phase transitions for the Cuntz algebras have been focused on. Suppose that $a = a^* \in \mathcal{O}_n$ is a self-adjoint element such that $\sigma_t^a := \lambda_{e^{ita}}$ is a (continuous) one-parameter family of automorphisms of $\mathcal{O}_n$. It is then a natural problem to determine the set of β-KMS states for $\sigma_t^a, t \in \mathbb{R}$ for all inverse temperatures $\beta > 0$ and their corresponding types. In the case $n = 2$, $a = P_2 + P_{12}/2 + P_{112}/3 + \cdots \in \mathcal{D}_2$ an extensive analysys has been carried out in [168], showing the existence of phase transitions (see Theorem 1 therein). All these KMS-states are of type I_∞ except the unique β_0-KMS state, where β_0 is the only positive number such that $\sum_{k=1}^\infty \exp\Big(-\beta_0 \sum_{j=1}^k 1/j\Big) = 1$, that is of type III_1. Notice that if $a = 1$ (and any n) then σ_t^1 is nothing but the (periodic) gauge action, for which the existence of a unique KMS state has been known for a long time.

Actually, the study of the structure of the set of KMS states on a broad class of C^*-algebras generalising the Cuntz algebras for various dynamics and temperatures has become a very common trend, and the number of publications on this subject has since been increasing in an exponential way, so much so it is virtually impossible to cite all of them. Without any pretense of completeness, we refer the interested reader to [94, 81, 157, 139, 2, 49, 3, 22, 138].

8.2. Spectral triples and isometric isomorphisms

In noncommutative geometry, a wide area of research initiated by A. Connes and insightfully expounded in his classical treatise [58], a key role is undoubtedly played by spectral triples. Roughly speaking, these are structures that allow one to introduce the language of differential geometry in the setting of general C^*-algebras, thus providing a powerful means to generalize the techniques of the classical discipline to cover the more general situation in which there is no underlying manifold. More precisely, spectral triples truly represents a genuine generalization of the notion of compact Riemannian manifold in that under reasonable conditions a spectral triple on a commutative C^*-algebra always comes from such a manifold by virtue of a celebrated reconstruction theorem due to Connes, see [58]. Following that result, many authors have endeavoured to define spectral triples on particularly selected non-commutative C^*-algebras. For instance, a nice discussion of spectral triples for AF algebras (in particular UHF) appears in a paper by Christensen and Ivan [56]. Suitable versions of generalized spectral triples for the Cuntz algebras have been discussed by Carey-Neshveyev-Nest-Phillips-Rennie [51] (cf. [36]), and more recently by Goffeng-Mesland [104]. In the present section we are going to review a couple of spectral triples on the Cuntz algebras, which of late have come to the fore especially in relation to isometric isomorphisms, [72]. To ease the reading of the following material, we rather briefly recall what is meant by spectral triple, at least in the simplest situations, namely the so-called θ-summable spectral triples, [58].

Defintion 8.1. A (θ-summable) spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is assigned through a concrete C^*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space, and an unbounded self-adjoint operator D acting on $\mathcal{H}$ such that:

1. the set $\{a \in \mathcal{A} : [D, a] \text{ is densely defined and bounded}\}$ is norm dense in $\mathcal{A}$;
2. $(1 + D)^{-1}$ is a compact operator;
3. for any positive $t > 0$, e^{-tD^2} is a trace-class operator.

The operator D appearing in the definition above is commonly referred to as the Dirac operator of the spectral triple. Whenever one is given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, it is possible to define a pseudo-distance on the state space $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ by

$$d_D(\varphi, \varphi') \doteq \sup\left\{|\varphi(a) - \varphi'(a)| : a \in \mathcal{A} \text{ with } ||\,[a, D]\,|| \leq 1\right\}, \quad \varphi, \varphi' \in \mathcal{S}(A).$$

This is known as Connes' distance. Once a pseudo-metric is available, it is natural to introduce a notion of isometric automorphism. To the best of our knowledge, the first such notion can be traced back to work of Park, see *e.g.* [151], where an automorphism $\alpha \in \mathrm{Aut}(\mathcal{A})$ is said to be isometric with respect to the given spectral triple $(\mathcal{A}, \mathcal{H}, D)$ if there exists a unitary $U \in \mathcal{B}(\mathcal{H})$ that implements α, *i.e.* $\alpha(x) = UxU^*$ for any $x \in A$, and that commutes with the Dirac operator, *i.e.* $[U, D] = 0$. Compositions of isometric isomorphisms are seen at once to yield an isometric isomorphism. Therefore, the set of all isometric isomorphisms is actually a group, which we denote by $\mathrm{Iso}(A, \mathcal{H}, D)$. Now, it is not difficult to verify that any isometric isomorphism $\alpha \in \mathrm{Iso}(A, \mathcal{H}, D)$ automatically preserves the Connes' distance, namely

$$d_D(\varphi \circ \alpha, \varphi' \circ \alpha) = d_D(\varphi, \varphi'), \quad \text{for any } \varphi, \varphi' \in \mathcal{S}(\mathcal{A}).$$

It is quite natural to consider a wider notion of isometric isomorphism by only asking that the Connes' distance should be preserved. Curiously enough, it was not until the recent work [72] that this notion was analyzed, at least in the narrrower context of spectral triples on the Cuntz algebras. It is obvious that isometric isomorphisms with respect to this notion still form a group, which we denote by $\mathrm{ISO}(\mathcal{A},\mathcal{H},D)$, and the inclusion $\mathrm{Iso}(\mathcal{A},\mathcal{H},D)\subset\mathrm{ISO}(\mathcal{A},\mathcal{H},D)$ holds. For classical spectral triples, that is when $\mathcal{A}$ is a commutative C^*-algebra, the bigger group, $\mathrm{ISO}(\mathcal{A},\mathcal{H},D)$, actually reduces to the smaller, $\mathrm{Iso}(\mathcal{A},\mathcal{H},D)$. As shown in [72], the set equality $\mathrm{ISO}(\mathcal{A},\mathcal{H},D)=\mathrm{Iso}(\mathcal{A},\mathcal{H},D)$ follows from a celebrated result due to Myers and Steenrod that isometric homeomorphisms of a compact manifold are automatically smooth. The insight provided by the classical situation seems to raise the question as to whether the same group equality may hold true in non-commutative situations as well. In its full generality, this appears to be a very hard problem to tackle. Nevertheless, something is known for a couple spectral triples on the Cuntz algebras. The first has been introduced in [104] by Goffeng and Mesland. For the reader's convenience we outline its construction.

We simply denote by $(\mathcal{H},\pi,\xi)$ the GNS triple associated with the unique KMS state, ω, on the Cuntz algebra $\mathcal{O}_n$. For $\mu,\nu\in W_n$, we define $e_{\emptyset,\emptyset}=\xi$ and, for $\mu,\nu\neq\emptyset$,

$$e_{\mu,\nu}=\begin{cases} n^{|\nu|/2}S_\mu S_\nu^*\xi & t(\mu)\neq t(\nu)\\ n^{|\nu|/2}\sqrt{\frac{n}{n-1}}\,(S_\mu S_\nu^*-\frac{1}{n}S_{\underline{\mu}}S_{\underline{\nu}}^*)\xi & t(\mu)=t(\nu)\neq\emptyset\end{cases}$$

where, if $|\mu|\geq 1$, $t(\mu)$ denotes the last entry of the multi-index μ and $t(\emptyset)=\emptyset$, and $\mu=\underline{\mu}t(\mu)$ (if $|\mu|=1$, $\mu=t(\mu)$). The family $\{e_{\mu,\nu}:\mu,\nu\in W_n\}$ can be shown to span the set of vectors of $\mathcal{H}$.

The Dirac operator D_κ on $\mathcal{H}$ is then defined as

$$D_\kappa e_{\mu,\nu}=-(|\mu|+|\,|\mu|-|\nu|\,|)e_{\mu,\nu}\ .$$

Note that $\mathcal{H}$ decomposes into the direct orthogonal sum of

$$\mathcal{H}_{h,k}:=\mathrm{span}\{e_{\mu,\nu}:|\mu|=h\,\text{and}\,|\nu|=k,\ \text{with}\,h,k\in\mathbb{Z}_{\geq 0}\},$$

cf. [105], and on each of these subspaces the Dirac operator D_κ simply acts as the multiplication by $l_{h,k}:=-\,(h+|h-k|)$. Now $(\mathcal{O}_n,\mathcal{H},D_\kappa)$ is shown to be a θ-summable spectral triple in [104, 103].

The group of all isometries in the sense of Park has been computed in [72], where the authors proved it is actually given by the Bogolubov automorphism, that is

$$\mathrm{Iso}(\mathcal{O}_n,\mathcal{H},D_\kappa)=\{\lambda_u:u\in\mathcal{U}(\mathcal{F}_n^1)\}.$$

It is worth stressing that in this case $\mathrm{Iso}(\mathcal{O}_n,\mathcal{H},D_\kappa)$ is a compact group, being isomorphic with $U(n)$, as is always the case with groups of isometries of compact manifolds.

Unluckily, the group $\mathrm{ISO}(\mathcal{O}_n,\mathcal{H},D_k)$ turns out to be far more difficult to deal with not least because the Connes' distance is quite an elusive object, which can hardly ever be computed explicitly. We can nonetheless raise some related questions, whose answers are highly likely to shed light on the problem of computing $\mathrm{ISO}(\mathcal{O}_n,\mathcal{H},D_\kappa)$ to see whether it coincides with $\mathrm{Iso}(\mathcal{O}_n,\mathcal{H},D_\kappa)$.

Question 8.2.

- Is $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_k)$ a compact group? Is the intersection $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_k) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$ compact?
- Is it possible to come to an explicit formula for the Connes distance d_{D_κ} for some suitably chosen states (for instance for all vector states $\mathcal{V}_\pi$)?
- Is $(\mathcal{O}_n, \|[D_\kappa, \cdot]\|)$ a compact quantum metric space in the sense of Rieffel[18]?

The second spectral triple on $\mathcal{O}_n$ looked over in [72] is the so-called modular spectral triple, of which we sketch the definition below. To ease the notation, we keep identifying $\mathcal{O}_n$ with $\pi(\mathcal{O}_n)$, where $(\pi, \mathcal{H}, \xi)$ is the GNS triple associated with the KMS state ω, as above. The Dirac operator is now the operator D_ω still acting on $\mathcal{H}$ given by the logarithm of the modular operator, that is $D_\omega \doteq \log \Delta_\omega$; for more details see [51]. Unlike the first spectral triple we discussed, $(\mathcal{O}_n, \mathcal{H}, D_\omega)$ fails to be a θ-summable spectral triple. Even so, it does display some regularity, for it is a semi-finite spectral triple. However, we will refrain from giving a definition of what this means, but we refer the interested reader to [60, 53, 33]. Possibly because the spectral triple is not θ-summable, the group $\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\omega)$ is no longer compact. In fact, in [72] it is proved to coincide with $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n)$, the group of all automorphisms of $\mathcal{O}_n$ which preserve the *UHF* subalgebra $\mathcal{F}_n$. More interestingly, in this case $\mathrm{ISO}(\mathcal{O}_n, \mathcal{H}, D_\omega)$ can be seen to coincide with $\mathrm{Iso}(\mathcal{O}_n, \mathcal{H}, D_\omega)$, see [72, Theorem 4.6].

There is no reason to limit oneself to posing the problems above only in the setting of Cuntz algebras. On the contrary, we believe the existence of spectral triples for some C^*-algebras containing $\mathcal{O}_n$ is worth investigating as well. A case in point is certainly given by the so-called 2-adic ring C^*-algebra $\mathcal{Q}_2$, which contains $\mathcal{O}_2$ in a natural way, see Section 9 of the present work. To this end, it should be borne in mind that $\mathcal{Q}_2$ is a Pimsner algebra. It is then natural to give a close look at the construction of spectral triples for such algebras given in [157] to see whether reasonably nice spectral triples can be obtained in this way. Finally, we would like to end the present section by presenting more general problems one may want to attack.

- Study the automatic regularity property Iso = ISO for more spectral triples. Start computing Iso for the spectral triples for AF C^*-algebras defined by Christensen and Ivan, and then possibly ISO.
- Is it possible to define quantum isometry groups of a compact spectral triple (cf. Goswami)? Is it true that any compact matrix pseudogroup (in the sense of Woronowicz) arises as the quantum isometry group of a compact spectral triple?

[18] By definition, this means that d_{D_κ} is in fact a distance, to wit $d_{D_\kappa}(\varphi, \varphi')$ is finite for any $\varphi, \varphi' \in \mathcal{S}(\mathcal{O}_n)$, and the topology induced on $\mathcal{S}(\mathcal{O}_n)$ by d_{D_k} coincides with the weak* topology. See also [59, 158].

9. The 2-adic ring C^*-algebra

This section aims to acquaint the reader with the so-called 2-adic ring C^*-algebra along with several of its main properties, as they came to be pointed out in a series of recent papers by the present authors, see [10, 11, 12, 13] and also [14]. This C^*-algebra can be defined in many equivalent ways. Notably, it admits a description as both a Pimsner algebra and an Exel crossed product, cf. [140], where to the best of our knowledge it was addressed systematically for the first time, though it had appeared before elsewhere. However, the most relevant definition to our porposes is that which presents the algebra in terms of generators and relations. Thus the 2-adic ring C^*-algebra is the universal C^*-algebra $\mathcal{Q}_2$ generated by a unitary U and an isometry S_2 such that

$$S_2 U = U^2 S_2 \quad \text{and} \quad S_2 S_2^* + U S_2 S_2^* U^* = 1\,.$$

It is right in the aforementioned paper [140] by Larsen and Li that $\mathcal{Q}_2$ was shown to be simple, purely infinite and nuclear. As a consequence, it embeds into the Cuntz algebra $\mathcal{O}_2$. Needless to say, no explicit embedding of $\mathcal{Q}_2$ into $\mathcal{O}_2$ is known, nor does it seem an easy task to exhibit one. Furthermore, the Cuntz algebra $\mathcal{O}_2$ in turn embeds into $\mathcal{Q}_2$. More interestingly, an embedding can now be given in quite an explicit fashion. Indeed, by setting $S_1 \doteq US_2$ one immediately sees that the C^*-subalgebra generated by S_1 and S_2 is nothing but a copy of the Cuntz algebra $\mathcal{O}_2$ merely because $S_1S_1^* + S_2S_2^* = 1$. As of now, we will always think of $\mathcal{O}_2$ as given as a subalgebra of $\mathcal{Q}_2$ through the above identification, and we will simply write $\mathcal{O}_2 \subset \mathcal{Q}_2$ to refer to the inclusion thus obtained. Despite its very simple definition, some of the properties of the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$ are not thoroughly understood. In the sequel we intend to give a closer look at the state of the art. As will be clearer from what follows, the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$ seems to be somewhat tight. This raises the following question.

Question 9.1. Are there non-trivial intermediate C^*-subalgebras between $\mathcal{O}_2$ and $\mathcal{Q}_2$? How many?

Remark 9.2. Exhibiting examples of such subalgebras is not as easy as one might expect. For instance, the naive idea of taking C^*-algebras generated by $\mathcal{O}_2$ and non-trivial powers of U is bound to fail. Indeed, the C^*-algebra generated by $\mathcal{O}_2$ and U^2 is seen at once to coincide with $\mathcal{Q}_2$ because U can be written as $S_2^*U^2S_2$. More generally, $C^*(\mathcal{O}_2, U^n) = \mathcal{Q}_2$ for every $n \in \mathbb{Z}\setminus\{0\}$. We give a prove of this arguing by induction on n. To begin with, there is no lack of generality if we consider only positive integers. Let us denote by $\mathfrak{A}_n$ the C^*-algebra generated by $\mathcal{O}_2$ and U^n. Suppose we have proved the property for every power less or equal to $n-1$. If n is even, say $n = 2k$, then the equality $U^k = S_2^*U^{2k}S_2$ says that U^k lies in $\mathfrak{A}_n$, so $\mathcal{Q}_2 = \mathfrak{A}_k \subset \mathfrak{A}_n$. If n is odd, say $n = 2k+1$, the equality $US_2U^k = U^nS_2$ says that US_2U^k lies in $\mathfrak{A}_n$, which means $U^k = S_2^*U^*US_2U^k = S_1^*US_2U^k$ lies in $\mathfrak{A}_n$ as well, hence $\mathfrak{A}_n = \mathcal{Q}_2$. The above argument also shows that $C^*(S_2, U^{2^k}) = \mathcal{Q}_2$ for every $k \in \mathbb{N}$.

Another related question is the following.

Question 9.3. Is it possible to recover $\mathcal{O}_2$ as the fixed-point subalgebra of $\mathcal{Q}_2$ with respect to a given action action of some group G, that is $\mathcal{Q}_2 = {\mathcal{O}_2}^G$?

Before moving on to next topics related to $\mathcal{Q}_2$, we would like to end this brief introduction to $\mathcal{Q}_2$ pointing out a problem which we believe natural. As is known, the tensor product $\mathcal{O}_2 \otimes \mathcal{O}_2$ is isomorphic with $\mathcal{O}_2$. We wonder whether this intriguing property holds for $\mathcal{Q}_2$ too.

Question 9.4. Is $\mathcal{Q}_2 \otimes \mathcal{Q}_2$ isomorphic with $\mathcal{Q}_2$?

9.1. Structure results

Every so often we will need to think of $\mathcal{Q}_2$ as a concrete algebra of operators on a Hilbert space. Among the many representations available, the so-called canonical representation has come in useful several times. In this representation $\mathcal{Q}_2$ is realized as the C^*-subalgebra of $\mathcal{B}(\ell^2(\mathbb{Z}))$ generated by the operators U and S_2 whose action on the canonical basis $\{e_k : k \in \mathbb{Z}\} \subset \ell^2(\mathbb{Z})$ is given by

$$Ue_k \doteq e_{k+1} \quad \text{and} \quad S_2 e_k \doteq e_{2k} \quad \text{for}\, k \in \mathbb{Z}\,.$$

This representation is irreducible, though its restriction to the Cuntz algebra $\mathcal{O}_2$ is not, see [10].

In Section 2 we saw that the diagonal subalgebra $\mathcal{D}_2$ is a maximal abelian self-adjoint subalgebra of the Cuntz algebra $\mathcal{O}_2$. This property is still true when the former algebra is understood as a subalgebra of $\mathcal{Q}_2$. In addition, $\mathcal{D}_2$ can be shown to be even a Cartan subalgebra of $\mathcal{Q}_2$, which is done in [10]. Now recall that two MASA's C_1 and C_2 in a C^*-algebra $\mathcal{A}$ are said to be *conjugate* if there exists $\alpha \in \mathrm{Aut}(A)$ such that $\alpha(C_1) = C_2$. Furthermore, such MASA's are called *inner conjugate* if they are conjugated as above through an inner automorphism. It would be nice to settle the following question, whose answer is known when $\mathcal{D}_2$ is thought of as a maximal abelian subalgebra of $\mathcal{O}_2$ rather than of $\mathcal{Q}_2$, [66].

Question 9.5. Are there MASA's in $\mathcal{Q}_2$ that are conjugate but not inner conjugate to $\mathcal{D}_2$? How many?

The C^*-subalgebra generated by U, which will be henceforth denoted by $C^*(U)$, is also worthy of consideration, not least because the von Neumann algebra $W^*(U)$ it generates in the canonical representation is maximal abelian as U is a multiplicity-free unitary. Furthermore, because the spectrum of U is the one-dimensional torus, we immediately see that $C^*(U)$ as an abstract algebra identifies with $C(\mathbb{T})$. In particular, not only is $C^*(U)$ not isomorphic with $\mathcal{D}_2$ but it also features markedly different properties, insofar as its spectrum is connected whereas the spectrum of $\mathcal{D}_2$ is totally disconnected. Even so, $C^*(U)$ is a maximal abelian selfadjoint subalgebra of $\mathcal{Q}_2$ as well, although it is not a Cartan subalgebra. Yet there does exist a unique conditional expectation E from $\mathcal{Q}_2$ onto $C^*(U)$. As one might expect, the subalgebra $C^*(U)$ has nothing to do with $\mathcal{O}_2$. This rather naive intuition can be given a much more precise statement. In fact, a result discussed in [10] says that $C^*(U)$ intersects $\mathcal{O}_2$ trivially, i.e. $C^*(U)\cap\mathcal{O}_2=\mathbb{C}1$. As is known, the two subalgebras generated by either Cuntz isometry are irreducible in $\mathcal{O}_2$, in the sense that their relative commutant is trivial. In light of the result recalled above, it is quite reasonable to expect this property to continue to hold true when $C^*(S_1)$ and $C^*(S_2)$ are regarded as subalgebras of $\mathcal{Q}_2$ instead. This is exactly the situation we are in, for it is proved in [10] that $C^*(S_i)'\cap\mathcal{Q}_2=\mathbb{C}1$, with $i=1,2$. As a consequence, no non-trivial inner automorphism can fix either subalgebra pointwise. The Cuntz algebra $\mathcal{O}_2$ is a fortiori irreducible in $\mathcal{Q}_2$, i.e. $\mathcal{O}_2'\cap\mathcal{Q}_2=\mathbb{C}1$. Having at one's disposal a MASA such as $C^*(U)$ or $\mathcal{D}_2$ is extremely useful when it comes to deciding whether an endomorphism of $\mathcal{Q}_2$ is surjective or not, [10, 13]. Indeed, if it preserves either, it is enough to study its surjectivity at the level of the MASA only. This is a quite straightforward application of the following general result, which we include with a proof since it is not to be easily found in the literature.

Proposition 9.6. *Let α be an injective endomorphism of a C^*-algebra $\mathcal{A}$. Let $\mathcal{B}\subset\mathcal{A}$ be a maximal abelian selfadjoint subalgebra of $\mathcal{A}$. If $\alpha(\mathcal{B})$ is strictly contained in $\mathcal{B}$, then α is not surjective.*

Proof. Set $\mathcal{C}\subset\mathcal{A}$ the C^*-subalgebra of those $x\in\mathcal{A}$ such that $\alpha(x)\in\mathcal{B}$. By hypothesis $\mathcal{B}\subset\mathcal{C}$ and $\mathcal{C}$ is an abelian subalgebra: since α is injective, two elements a_1, $a_2\in\mathcal{C}$ will commute if and only if $\alpha(a_1a_2)=\alpha(a_2a_1)$, which is of course true because $\mathcal{B}$ is commutative. By maximality, we must have $\mathcal{C}=\mathcal{B}$. In other terms, α cannot be surjective. □

We end this section by formulating a question to do with the interaction between the two MASA's in terms of automorphisms.

Question 9.7. An automorphism $\lambda\in\mathrm{Aut}(\mathcal{Q}_2)$ is called quasi-free if $\lambda(C^*(U))=C^*(U)$ and $\lambda(S_2)\in C^*(U)S_2$ (cf. [173]). For which such λ one has $\lambda(\mathcal{D}_2)=\mathcal{D}_2$? If $\lambda(\mathcal{D}_2)\neq\mathcal{D}_2$, does there exist a unitary $w\in\mathcal{Q}_2$ such that $\lambda(\mathcal{D}_2)=w\mathcal{D}_2w^*$ (actually this question makes sense for all automorphisms of $\mathcal{Q}_2$)? Likewise, if $\lambda\in\mathrm{Aut}(\mathcal{Q}_2)$ is any automorphism such that $\lambda(C^*(U))\neq C^*(U)$, does it exist a unitary $w\in\mathcal{Q}_2$ such that $\lambda(C^*(U))=wC^*(U)w^*$?

9.2. Extendable endomorphisms

The inclusion $\mathcal{O}_2\subset\mathcal{Q}_2$ has turned out to be rather rigid in many respects. This interesting if still vague property does become precise as soon as it is presented through its several incarnations. One is provided by dealing with the problem whether an endomorphism of the Cuntz algebra extends to $\mathcal{Q}_2$ or not. Unfortunately, we are still far from giving a complete solution to a question of this sort. Even so, not only do we have partial yet intriguing results, but we also have a general result on the uniqueness of the sought extension. Indeed, it is one on the main results obtained in [10] that if an (necessarily injective) endomorphism Λ of

$\mathcal{Q}_2$ acts trivially on $\mathcal{O}_2$, then Λ must be trivial itself, i.e. $\Lambda = \mathrm{id}_{\mathcal{Q}_2}$. In particular, given $\Lambda_1, \Lambda_2 \in \mathrm{End}(\mathcal{Q}_2)$ that coincide on the Cuntz algebra $\mathcal{O}_2$, if either is an automorphism, then $\Lambda_1 = \Lambda_2$. As a consequence, if we now start with and endomorphism of the Cuntz algebra, the above result says that it extends to $\mathcal{Q}_2$ uniquely provided that it has a surjective extension. At this point, one may be asking whether the hypothesis that the given extension is an automorphism can be dispensed with. In fact, this is still an open question, whose answer we would be inclined to believe is positive. All the same, however conclusive the result aimed at would be, it still would say nothing about the existence of extensions. This problem needs to be treated separately. An endomorphism or even an automorphism of $\mathcal{O}_2$ need not extend to $\mathcal{Q}_2$, as there are no general reasons for it to do so. On the other hand, there are no general reasons for it not to do so. In fact, extandable endomorphisms and automorphisms can be exhibited, which is what we intend to do next. All the same, endomorphisms of the Cuntz algebra are far more likely not to extend not least because $\mathcal{Q}_2$ is a more symmetric version of $\mathcal{O}_2$ where the Cuntz isometries are intertwined by U, that is $S_2U = US_1$.

Notable examples of endomorphisms and automorphisms that extend are the canonical endomorphism, the flip-flop and the gauge automorphisms. The most natural extension of the canonical endomorphism is that obtained by taking $\tilde{\varphi}$ given by the same expression as φ, i.e. $\tilde{\varphi}(x) = S_1xS_1^* + S_2xS_2^* = US_2xS_2^*U^* + S_2xS_2^*$, for every $x \in \mathcal{Q}_2$. In particular, the usual commutation rule $S_ix = \widetilde{\varphi}(x)S_i$, $x \in \mathcal{Q}_2$, continues to hold. Furthermore, the action of $\tilde{\varphi}$ on U is given by $\tilde{\varphi}(U) = S_1US_1^* + S_2US_2^* = U^2S_1S_1^* + U^2S_2S_2^* = U^2$. The endomorphism $\tilde{\varphi} \in \mathrm{End}(\mathcal{Q}_2)$ then turns out to be the unique extension of φ to $\mathcal{Q}_2$.

For all the Cuntz algebras $\mathcal{O}_n$ it is known that $(\varphi^k, \varphi^k) = \mathcal{F}_n^k$ for all $k \geq 1$. We next state and prove a result that says that the self-intertwiner space of $\widetilde{\varphi}$ remains the same for $\mathcal{Q}_2$ as well. This is actually done with $k = 1$ only, although there should be no major obstacles to extending it to every k.

Proposition 9.8. *The equalities below hold:*

- $\widetilde{\varphi}(\mathcal{Q}_2)' \cap \mathcal{Q}_2 = \mathcal{F}_2^1$,
- $(\mathcal{F}_2^1)' \cap \mathcal{Q}_2 = \tilde{\varphi}(\mathcal{Q}_2)$.

Proof. We start with the first equality. Clearly, $\tilde{\varphi}(\mathcal{Q}_2)' \cap \mathcal{Q}_2 \subset \varphi(\mathcal{O}_2)' \cap \mathcal{Q}_2$. Now it is easy to see that $\varphi(\mathcal{O}_2)' \cap \mathcal{Q}_2 = \mathcal{F}_2^1$. Indeed, for the inclusion $\varphi(\mathcal{O}_2)' \cap \mathcal{Q}_2 \subset \mathcal{F}_2^1$ it is enough to observe that any $x \in \varphi(\mathcal{O}_2)' \cap \mathcal{Q}_2$ must satisfy $S_i^*xS_jy = yS_i^*xS_j$ for all $y \in \mathcal{O}_2$ and thus $S_i^*xS_j \in \mathcal{O}_2' \cap \mathcal{Q}_2 = \mathbb{C}1$ for all $i, j \in \{1, 2\}$, which shows that $x = \sum_{i,j} S_iS_i^*xS_jS_j^* \in \mathcal{F}_2^1$. For the reverse inclusion, we need to recall the known equality $\mathcal{F}_2^1 = \varphi(\mathcal{O}_2)' \cap \mathcal{O}_2$, from which $\mathcal{F}_2^1 \subset \varphi(\mathcal{O}_2)' \cap \mathcal{Q}_2$ follows. Finally, all is left to do is prove that $\mathcal{F}_2^1$ is contained in $\tilde{\varphi}(\mathcal{Q}_2)' \cap \mathcal{Q}_2$. To this aim, it is enough to make sure that $\mathcal{F}_2^1$ commutes with $\tilde{\varphi}(U) = U^2$, which is shown by the computations

$$
\begin{aligned}
U^2S_1S_1^* &= S_1S_1^*U^2\\
U^2S_2S_2^* &= S_2S_2^*U^2\\
U^2S_1S_2^* &= S_1US_2^* = S_1S_2^*U^2\\
U^2S_2S_1^* &= S_2US_1^* = S_2S_1^*U^2\ .
\end{aligned}
$$

As for the second equality, we note that the first immediately yields the inclusion $\tilde{\varphi}(\mathcal{Q}_2) \subset (\mathcal{F}_2^1)' \cap \mathcal{Q}_2$. For the reverse inclusion, if $x \in \mathcal{Q}_2$ commutes with $\mathcal{F}_2^1$ then $S_iS_j^*x = xS_iS_j^*$ for $i, j = 1, 2$, whence $S_1^*xS_1 = S_2^*xS_2 \doteq y$. The conclusion is now reached by showing that $\tilde{\varphi}(y) = x$. Indeed,

$$\tilde{\varphi}(y) = S_1yS_1^* + S_2yS_2^* = S_1S_1xS_1^*S_1^* + S_2S_2xS_2^*S_2^* = (S_1S_1^* + S_2S_2^*)x = x.$$

□

As a straightforward consequence of the first equality, the endomorphism $\tilde{\varphi}$ is proper, i.e. non-surjective. In fact, this could also have been seen by noting that $\tilde{\varphi}$ restricts to $C^*(U)$ as a proper endomorphism, hence by maximality of $C^*(U)$ it cannot be surjective.

Remark 9.9. We also note that $fUf = S_1US_2^* + S_2S_1^*$ which leads to

$$
\begin{aligned}
fU^2f &= fUffUf = (S_1US_2^* + S_2S_1^*)(S_1US_2^* + S_2S_1^*) = S_1US_1^* + S_2US_2^*\\
&= \varphi(U) = U^2\ .
\end{aligned}
$$

The canonical endomorphism $\tilde{\varphi}$ of $\mathcal{Q}_2$ has no fixed points apart from scalars, exactly as the canonical endomorphism φ of $\mathcal{O}_2$. This is indeed a consequence of a much stronger result proved in [10] that $\tilde{\varphi}$ is even a *shift*, in the sense that $\bigcap_k \tilde{\varphi}^k(\mathcal{Q}_2) = \mathbb{C}1$.

The gauge automorphisms of $\mathcal{O}_2$ can be extended by taking $\{\tilde{\lambda}_z : z \in \mathbb{T}\} \subset \mathrm{Aut}(\mathcal{Q}_2)$, with $\tilde{\lambda}_z(U) = U$ for every $z \in \mathbb{T}$. Because each $\tilde{\lambda}_z$ is an automorphism, we see that $\tilde{\lambda}_z$ is actually the unique extension of λ_z. Moreover, all non-trivial extended gauge automorphisms are outer, see [10]; in particular, it readily follows that $\mathrm{Out}(\mathcal{Q}_2)$ is an uncountable group, and with some additional work it can be proved not to be abelian, [10, Theorem 6.21]. As a matter of fact, the group $\mathrm{Out}(\mathcal{Q}_2)$ is huge, for it contains all second countable locally compact groups. Indeed, as a consequence of the deep analysis by Kirchberg-Phillips [136], one has $\mathcal{O}_\infty \otimes \mathcal{Q}_2 \simeq \mathcal{Q}_2$ and thus the argument of Theorem 3.4 can be repeated *verbatim*. Nevertheless, $\mathrm{Out}(\mathcal{Q}_2)$ has thus far proved to be considerably less noncommutative than $\mathrm{Out}(\mathcal{O}_2)$, which is why we believe that the following question is likely to have a positive answer.

Question 9.10. Does $\{\widetilde{\alpha}_\theta : \theta \in \mathbb{R}\}$ (modulo inners) sit in the centre of $\mathrm{Out}(\mathcal{Q}_2)$, i.e.

$$\pi(\{\widetilde{\alpha}_\theta : \theta \in \mathbb{R}\}) \subset \mathcal{Z}(\mathrm{Out}(\mathcal{Q}_2))\ ,$$

where $\pi : \mathrm{Aut}(\mathcal{Q}_2) \to \mathrm{Out}(\mathcal{Q}_2)$ is the canonical projection?

Since it might be quite demanding to arrive at a full description of the whole group $\mathrm{Out}(\mathcal{Q}_2)$, the following question seems a natural preliminary problem to face. To do so, we need to single out a distinguished automorphism of $\mathcal{Q}_2$, which we denote by λ_{-1}. This is defined on the generators as $\lambda_{-1}(S_2) := S_2$ and $\lambda_{-1}(U) := U^*$.

Question 9.11. Does $\mathrm{Aut}_{C^*(S_2)}(\mathcal{Q}_2)$ reduces to $\{\mathrm{id}, \lambda_{-1}\} \simeq \mathbb{Z}_2$?

An equivalent way to recast the above question is to decide whether

$$\mathcal{U}_2 := \{V \in \mathcal{U}(\mathcal{Q}_2) \mid V^2 S_2 = S_2 V, S_2 S_2^* + V S_2 S_2^* V^* = 1\}$$

coincides with $\{U^{2k+1} : k \in \mathbb{Z}\}$.

Going back to the gauge automorphisms, the fixed-point algebra under their extended action on $\mathcal{Q}_2$, which will be henceforth denoted by $\mathcal{Q}_2^{\mathbb{T}}$, is a remarkable subalgebra of $\mathcal{Q}_2$ that plays a privileged role. As a matter of fact, it is nothing but $C^*(\mathcal{D}_2, U) = C^*(\mathcal{F}_2, U) \subset \mathcal{Q}_2$. In addition, it enjoys nice properties as an abstract C^*-algebra. For example, it is a Bunce-Deddens algebra. Quite interestingly, all fixed-point subalgebras under the action of a given gauge automorphism can also be described completely, as is done below in terms of either p-adic C^*-algebras (discussed later on) or $\mathcal{Q}_2^{\mathbb{T}}$ itself.

Proposition 9.12. *If $\mu \in \mathbb{T}$ is a primitive k-th root of unity, then the fixed-point algebra ${\mathcal{Q}_2}^{\tilde{\lambda}_\mu}$ is generated by S_2^k and U and it is thus isomorphic to $\mathcal{Q}_{2^k}$. On the other hand, if $\mu \in \mathbb{T}$ has infinite order, then the fixed-point algebra of $\lambda_{\mu 1} \in \mathrm{Aut}(\mathcal{Q}_2)$ is precisely the Bunce-Deddens algebra $\mathcal{Q}_2^{\mathbb{T}}$.*

Proof. The first claim can be proved in much the same way as in Proposition 7.2, *mutatis mutandis*. It also follows that ${\mathcal{Q}_2}^{\tilde{\lambda}_\mu} = C^*({\mathcal{O}_2}^{\lambda_\mu}, U)$. The isomorphism then follows from the fact that $C^*(S_2^k, U) = C^*(S_2^{k-1} S_1, U)$ and from Proposition 10.1. The last claim can be proved again by the same argument as for $\mathcal{O}_2$. □

Up to rescaling, the gauge action is nothing but the periodic modular action with respect to the canonical KMS-state on $\mathcal{O}_n$. In a very similar fashion, the extended gauge automorphisms should still admit a modular interpretation in terms of the state of $\mathcal{Q}_2$ obtained by composing the unique trace of the Bunce-Deddens algebra with the faithful conditional expectation from $\mathcal{Q}_2$ onto $\mathcal{Q}_2^{\mathbb{T}}$ obtained by averaging the gauge action, cf. [81, Proposition 4.2].

The flip-flop λ_f, too, extends uniquely. Its unique extension $\tilde{\lambda}_f$ is given by $\tilde{\lambda}_f(U) = U^*$. The automorphism $\tilde{\lambda}_f$ continues to be outer at the level of $\mathcal{Q}_2$ as well. More generally, it is proved in [10] that any automorphism of $\mathcal{Q}_2$ that maps U to its adjoint is necessarily outer. Unlike λ_f, very little is known about its extension $\tilde{\lambda}_f$. For instance, it is a well-known fact that $\mathcal{O}_2^{\lambda_f}$ is isomorphic with $\mathcal{O}_2$ itself, see e.g. [55]. In fact, an explicit description of $\mathcal{Q}_2^{\tilde{\lambda}_f}$ is not available yet, nor is it known whether $\mathcal{Q}_2^{\tilde{\lambda}_f}$ is isomorphic with $\mathcal{Q}_2$. All elements of $\mathcal{O}_2^{\lambda_f}$ and $(U+U^*)/2$

are obviously invariant under $\tilde{\lambda}_f$, as are those of the form $xU^h + \tilde{\lambda}_f(x)U^{-h}$, $x \in \mathcal{O}_2$ and $h \in \mathbb{Z}$. In other terms, we do have the two inclusions

$$C^*(\mathcal{O}_2^{\lambda_f}, (U+U^*)/2) \subseteq C^*(xU^h + \tilde{\lambda}_f(x)U^{-h} \mid x \in \mathcal{O}_2, h \in \mathbb{Z}) \subseteq \mathcal{Q}_2^{\tilde{\lambda}_f}$$

but we do not know if either is actually an equality. To sum up, a good many natural problems concerning $\tilde{\lambda}_f$ boil down to answering the questions gathered below.

Question 9.13.

- Is it possible to provide a concrete description of $\mathcal{Q}_2^{\tilde{\lambda}_f}$?
- Are $\mathcal{Q}_2$ and $\mathcal{Q}_2^{\tilde{\lambda}_f}$ isomorphic? If so, are there any isomorphisms ψ such that $\psi(\mathcal{O}_2) \subset \mathcal{O}_2$, or $\psi(\mathcal{O}_2) = \mathcal{O}_2^{\lambda_f}$, or that even extend the Choi-Latremoliere isomorphism between $\mathcal{O}_2$ and $\mathcal{O}_2^{\lambda_f}$?

One could also ask analogous questions for the crossed product of $\mathcal{Q}_2 \rtimes_{\tilde{\lambda}_f} \mathbb{Z}_2$ as compared to $\mathcal{O}_2 \rtimes_{\lambda_f} \mathbb{Z}_2$.

The examples of extendable endomorphisms and automorphisms could be multiplied, see [13]. However, general endomorphisms or automorphisms typically will not extend. For example, a strong result in the negative is given in [10], where gauge automorphisms, flip-flop, and their compositions are proved to be the only extendable Bogolubov automorphisms.

Unlike the Cuntz algebras, a general structure theorem for the endomorphisms of $\mathcal{Q}_2$ is still missing, and it might well be chimerical to reach. Moreover, the lack of such a characterization is highly likely to be a major obstacle to working out the extension problem if too much generality is allowed. The following question is nevertheless worth asking.

Question 9.14. Is it possible to give a complete explicit description of all endomorphisms and automorphisms of $\mathcal{O}_2$ that extend to $\mathcal{Q}_2$?

One might also ask whether extendable automorphisms or proper endomorphisms, extend to automorphisms and proper endomorphisms, respectively.

At this point, we should point out that an answer, albeit not completely satisfactory, to the above question has already been given in [10]. Indeed, given a unitary V in $\mathcal{O}_2$ the corresponding endomorphism $\lambda_V \in \mathrm{End}(\mathcal{O}_2)$ extends to $\mathcal{Q}_2$ if and only if there exists a unitary W in $\mathcal{Q}_2$ such that the two equations $WS_2 = S_2$ and $S_2VUWV^* = UWUS_2$ are satisfied. For every such a unitary W there is a corresponding extension $\tilde{\lambda}_{V,W}$, whose action on U is given by $\tilde{\lambda}_{V,W}(U) = VUWV^*$. Straightforward examples of extendable automorphisms are provided by inner automorphism of the form $\mathrm{Ad}(u)$ with $u \in N_{\mathcal{Q}_2}(\mathcal{O}_2)$. In this case $\mathrm{Ad}(u)$ certainly restricts to the Cuntz algebra as λ_v, where $v \in \mathcal{U}(\mathcal{O}_2)$ is nothing but $u\tilde{\varphi}(u^*)$. However, even in such a simple situation the corresponding W is not as simple. Indeed, from the equality $\mathrm{Ad}(u)(U) = uUu^* = vUWv^*$ we see that $W = U^*v^*uUu^*v = U^*\tilde{\varphi}(u)U\tilde{\varphi}(u^*)$, which allows us to rewrite $\mathrm{Ad}(u)$ in the following way

$$\mathrm{Ad}(u) = \tilde{\lambda}_{u\tilde{\varphi}(u^*),U^*\tilde{\varphi}(u)U\tilde{\varphi}(u^*)}$$

Endomorphisms and automorphisms of $\mathcal{Q}_2$ are interesting irrespective of whether they come from the Cuntz algebra. Obviously, it is not difficult to provide examples of automorphisms of $\mathcal{Q}_2$ that do not come from $\mathcal{O}_2$. Phrased differently, there are automorphisms that do not leave the Cuntz algebra globally invariant. The possibly easiest examples are provided by inner automorphisms, as shown below.

Example 9.15. The inner automorphism $\mathrm{Ad}(U)$ does not leave $\mathcal{O}_2$ globally invariant. Since $US_1 = S_2U$, we have

$$\mathrm{Ad}(U)S_1 = US_1U^* = S_2 = US_1, \quad \mathrm{Ad}(U)S_2 = US_2U^* = S_1U^* = U^*S_2\ .$$

Hence, $\mathrm{Ad}(U)(\mathcal{O}_2)$ is not contained in $\mathcal{O}_2$, because clearly $S_1U^* \notin \mathcal{O}_2$. Even more can be said. Indeed, $\mathrm{Ad}(U)(\mathcal{F}_2)$ is not contained in $\mathcal{O}_2$ either. This is seen as easily as before, since for instance $\mathrm{Ad}(U)(S_1S_2^*) = US_1S_2^*U^*$ does not belong to O_2 although $S_1S_2^*$ belongs to $\mathcal{F}_2$. Given that $US_1S_2^*U^* = S_2US_2^*U^* = S_2US_1^*UU^* = S_2US_1^*$, if $US_1S_2^*U^*$ were in $\mathcal{O}_2$, then $U = S_2^*S_2US_1^*S_1$ would in turn be in $\mathcal{O}_2$, which is not.

Answering the following question is a rather ambitious task.

Question 9.16. Is it possible to give a complete description of all endomorphisms of $\mathcal{Q}_2$ in a similar spirit of $\mathcal{O}_2$?

Again, an admittedly rather partial answer has been given in [10], where it is pointed out that for any given $\Lambda \in \mathrm{End}(\mathcal{Q}_2)$ there exists a unique unitary u_Λ in $\mathcal{Q}_2$ such that $\Lambda(S_i) = u_\Lambda S_i$, $i = 1, 2$. For instance, with $\Lambda = \mathrm{Ad}(U)$ we simply have $\Lambda(S_i) = US_iU^* = U^*S_i$, $i = 1, 2$, and so $u_\Lambda = U^*$. However, unlike what happens with $\mathcal{O}_2$, the correspondence $\mathrm{End}(\mathcal{Q}_2) \ni \Lambda \mapsto u_\Lambda \in \mathcal{U}(\mathcal{O}_2)$ fails to be surjective, as its image does not even exhaust $\mathcal{U}(\mathcal{O}_2)$. Furthermore, it is not clear whether the correspondence is injective. Obviously, proving that it is injective amounts to answering the following question in the positive.

Question 9.17. If $\Lambda_i \in \mathrm{End}(\mathcal{Q}_2)$, $i = 1, 2$ and $\Lambda_1 \restriction_{\mathcal{O}_2} = \Lambda_2 \restriction_{\mathcal{O}_2}$, does it follow that $\Lambda_1 = \Lambda_2$?

Going back to the problem of extending endomorphisms from $\mathcal{O}_2$ to $\mathcal{Q}_2$, an hitherto promising strategy is to focus on selected classes of endomorphisms of the Cuntz algebra rather than to face the problem in its full generality.

Question 9.18. Discuss extendability for endomorphisms of the following kind.

1. Diagonal automorphisms,
2. Permutative endomorphisms,

3. Localized endomorphisms,
4. Endomorphisms associated to unitaries in the Thompson groups $\mathcal{S}_2$.

Diagonal automorphisms have been addressed in [11], where their extendability is completely characterized in the localized case. Permutative endomorphisms λ_u, with $u \in \mathcal{P}_2^k$, are dealt with in [12] for $k = 2$ and in [13] for $k > 2$. However, the question whether the permutative (outer) automorphisms of $\mathcal{O}_2$ constructed in [75] at level $k = 4$ extend to automorphisms/endomorphisms of $\mathcal{Q}_2$ is still open. Apart from Bogolubov and diagonal automorphisms, localized endomorphisms still lack a general analysis. Notably, an intriguing subclass is certainly given by those unitaries R in $\mathcal{F}_2^2$ that satisfy the Yang-Baxter equation.

Among the several problems only hinted at above, perhaps the most important is how many diagonal/localized/permutative outer automorphisms of $\mathcal{Q}_2$ exist other than those already found. There follows a list of specific open problems.

Question 9.19.

If $d \in \mathcal{U}(\mathcal{D}_2)$ and λ_d is extendable, is λ_d necessarily the composition of a gauge and a diagonal inner automorphisms?

If $d \in \mathcal{U}(\mathcal{D}_2)$, $dS_2 = S_2$ and λ_d is extendable, is $d = 1$?

In particular, if $d \in \mathcal{U}(\mathcal{D}_2)$, $dS_2 = S_2$, $dS_1^k = S_1^k$, $k \geq 3$ and λ_d is extendable, is $d = 1$?

If $u \in \mathcal{U}(\bigcup_k \mathcal{F}_2^k)$ and $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2)$ is extendable, is it the composition of the flip-flop, a gauge and a localized inner automorphisms?

If $u \in \mathcal{U}(\mathcal{F}_2)$ and $\lambda_u \in \mathrm{Aut}(\mathcal{O}_2)$ is extendable, is it the composition of the flip-flop, a gauge and an inner automorphisms (the latter is necessarily an element in $\mathrm{Inn}(\mathcal{O}_2) \cap \{\lambda_v \mid v \in \mathcal{U}(\mathcal{F}_2)\} = \{\mathrm{Ad}(v) \mid v \in \mathcal{U}(\mathcal{F}_2)\}$) ?

9.3. Automorphisms preserving notable subalgebras

We start by racalling a definition from group theory. The normalizer of a subgroup $S \subset G$ of a group G is the biggest subgroup $N_G(S)$ of G containing S and in which S is normal. The normalizer $N_G(S)$ can also be defined as $N_S(G) \doteq \{g \in G \mid gSg^{-1} = S\}$.

Analogously, given an inclusion of C^*-algebras $\mathcal{B} \subset \mathcal{A}$, the (unitary) normalizer of $\mathcal{B}$ relative to $\mathcal{A}$, $N_{\mathcal{A}}(\mathcal{B})$ is the group of all unitaries in $\mathcal{A}$ whose adjoint action leaves $\mathcal{B}$ globally invariant, namely

$$N_{\mathcal{A}}(\mathcal{B}) := \{u \in \mathcal{U}(\mathcal{A}) : u\mathcal{B}u^* = \mathcal{B}\}.$$

For a general C^*-algebra inclusion it is no easy task to compute the corresponding normalizer so much so only few such computations have been made, see *e.g.* [155, 154]. Moreover, in [13] the normalizer of the diagonal $\mathcal{D}_2$ relative to the inclusion in the 2-adic ring C^*-algebra $\mathcal{Q}_2$ has been described thoroughly. It turns out that every unitary $u \in \mathcal{Q}_2$ that normalizes $\mathcal{D}_2$ factors as $u = dP$, where d is a

unitary in the diagonal $\mathcal{D}_2$ and P is a unitary in $\mathcal{Q}_2$ of the form $\sum_{i=1}^N S_{\alpha_i} U^{k_i} S^*_{\beta_i}$ with $\sum_{i=1}^N S_{\alpha_i} S^*_{\alpha_i} = \sum_{i=1}^N S_{\beta_i} S^*_{\beta_i} = 1$. The unitaries of the form P above make up a group denoted by $\mathcal{W}$, which is there called the extended Thompson group since it is bigger than $\mathcal{S}_2 = V$. As is known, the elements of the Thompson group correspond to taking $k_i = 0$, for every $i = 1, 2, \ldots, N$, and represent the discrete component of the normalizer of the diagonal $\mathcal{D}_2$ in the Cuntz algebra $\mathcal{O}_2$.
The inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$ has proved much harder to deal with, and the normalizer $N_{\mathcal{Q}_2}(\mathcal{O}_2)$ is not known yet. All we know is the C^*-algebra generated by $N_{\mathcal{Q}_2}(\mathcal{O}_2)$ does not coincide with $\mathcal{Q}_2$, see [13, Theorem 7.4]. This suggests the following question.

Question 9.20. What is $N_{\mathcal{Q}_2}(\mathcal{O}_2)$?

As for the above question, we ought to mention that at the moment no unitary in $\mathcal{Q}_2 \setminus \mathcal{O}_2$ is known which normalizes the Cuntz algebra $\mathcal{O}_2$, and the normalizer $N_{\mathcal{Q}_2}(\mathcal{O}_2)$ might thus reduce to $\mathcal{U}(\mathcal{O}_2)$. We would like to keep track of this curious circumstance by raising the following question.

Question 9.21. Is there some $u \in \mathcal{U}(\mathcal{Q}_2) \setminus \mathcal{O}_2$ such that $\mathrm{Ad}(u) \in \mathrm{Aut}(\mathcal{Q}_2, \mathcal{O}_2)$?

The above question can also be posed for other inclusion of C^*-algebras, for instance when the larger algebra is a p-adic ring C^*-algebra (see Section 10).

Question 9.22. Describe $N_{\mathcal{D}_n}(\mathcal{Q}_n)$ for $n > 2$.

Since for any $n \geq 2$ the diagonal $\mathcal{D}_n$ is isomorphic with the C^*-algebra of continuous functions on the Cantor set K_n, the following couple of questions are quite natural.

Question 9.23.

- Provide an intrinsic characterization of those homeomorphisms of the Cantor set X_n, $n \geq 2$ implemented by the adjoint action of unitaries in the normalizer $N_{\mathcal{D}_n}(\mathcal{Q}_n)$ (for $n = 2$, this is the subgroup generated by those homeomorphisms implemented by unitaries in the Thompson group $\mathcal{S}_2 \simeq V$ and the odometer);

- building upon the work of Nekrashevych, is it always possible to realize $V_d(G)$ as defined therein as a normalizer in the corresponding Cuntz-Pimsner algebra?

Obviously, there is no reason to limit oneself to considering inner automorphisms. On the contrary, studying all automorphisms, whether they are inner or outer, preserving a subalgebra can be quite a rewarding task, which is why we also single out the following problems.

Question 9.24. What can be said about $\mathrm{Aut}(\mathcal{Q}_2, \mathcal{O}_2)$, $\mathrm{Aut}(\mathcal{Q}_2, \mathcal{D}_2)$ and $\mathrm{Aut}_{\mathcal{D}_2}(\mathcal{Q}_2)$?

Known elements in $\mathrm{Aut}(\mathcal{Q}_2, \mathcal{O}_2)$ are $\lambda_{t1}, t \in \mathbb{T}$, λ_f and $\mathrm{Ad}(u), u \in \mathcal{U}(\mathcal{O}_2)$. It is a nice question to answer to see if these generate the whole group. Note that these automorphisms all commute with one another (modulus inner automorphisms), which suggests the question below.

Question 9.25. Is the image of $\mathrm{Aut}(\mathcal{Q}_2, \mathcal{O}_2)$ in $\mathrm{Out}(\mathcal{Q}_2)$ an abelian group?

In [62, 64] a notion of Weyl group taken from the theory of compact Lie groups was introduced and the relative theory developed for the inclusion of the diagonal subalgebras $\mathcal{D}_n$ in the Cuntz algebras $\mathcal{O}_n$. In particular, since the diagonal subalgebra $\mathcal{D}_2$ is still a MASA of $\mathcal{Q}_2$, the quotient group $\mathrm{Aut}(\mathcal{Q}_2, \mathcal{D}_2)/\mathrm{Aut}_{\mathcal{D}_2}(\mathcal{Q}_2)$ would also be worth studying. Furthermore, what was done in the aforementioned papers is also meaningful for the inclusion of $C^*(U)$ in $\mathcal{Q}_2$. Indeed, since $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2) \subset \mathrm{Aut}(\mathcal{Q}_2)$ is maximal abelian (cf. [10, Theorem 6.27]), one may define a sort of Weyl group as the quotient

$$\mathcal{W}(\mathcal{Q}_2, C^*(U)) \doteq \frac{N_{\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)}(\mathrm{Aut}(\mathcal{Q}_2))}{\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)} \ .$$

In [10] $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$ has been proved to be isomorphic with the loop group $C(\mathbb{T}, \mathbb{T})$ through the isomorphism $C(\mathbb{T}, \mathbb{T}) \ni h \in \beta_h \in \mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$, where β_h is determined by $\beta_h(U) = U$ and $\beta_h(S_2) = h(U)S_2$. We observe that the (class of the) extension of the flip-flop automorphism $\widetilde{\lambda}_f$ is a non-trivial element of this group. Indeed, $\widetilde{\lambda}_f \circ \beta \circ \widetilde{\lambda}_f(U) = U$ for all $\beta \in \mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$. The same is true also for (the class of) λ_{-1}, although this is not big news as λ_{-1} is equal to $\mathrm{Ad}(U) \circ \widetilde{\lambda}_f$, and so it defines the same element as $\widetilde{\lambda}_f$ in the quotient. In order to study the above quotient , it becomes a priority to answer the question whether one may describe in explicit way the normalizer of $\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)$ in $\mathrm{Aut}(\mathcal{Q}_2)$. To this end, the following proposition, which is probably known, comes in useful.

Proposition 9.26. *Let $A \subseteq B$ be a unital inclusion of C^*-algebras. Then*

1. $\mathrm{Aut}(B, A) \subseteq N_{\mathrm{Aut}_A(B)}(\mathrm{Aut}(B))$;
2. *if* $B^{\mathrm{Aut}_A(B)} = A$ *one has* $\mathrm{Aut}(B, A) = N_{\mathrm{Aut}_A(B)}(\mathrm{Aut}(B))$;
3. *if A is a MASA in B, then* $B^{\mathrm{Aut}_A(B)} = A$ *and* $\mathrm{Aut}(B, A) = N_{\mathrm{Aut}_A(B)}(\mathrm{Aut}(B))$.

For the proof, the reader is referred to e.g. [11, Prop. 3.1].

Since $C^*(U)$ is maximal abelian in $\mathcal{Q}_2$, the above proposition implies that $\mathcal{Q}_2^{\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)} = C^*(U)$. Thus we have shown that indeed one has

$$\mathcal{W}(\mathcal{Q}_2, C^*(U)) = \frac{\mathrm{Aut}(\mathcal{Q}_2, C^*(U))}{\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)} \ .$$

Despite the rewriting above, the Weyl group $\mathcal{W}(\mathcal{Q}_2, C^*(U))$ remains at the moment quite an elusive object essentially because a complete description of $\mathrm{Aut}(\mathcal{Q}_2, C^*(U))$ is not available. Even so, we do know that

$$\mathcal{W}(\mathcal{Q}_2, C^*(U)) \supset \{[\widetilde{\lambda}_f], \{[Ad(U_z)]\}_{z\in I_2}\} \doteq G_{C^*(U)} .$$

where, for any $z \in \mathbb{T}$, $U_z \in \mathcal{B}(\ell^2(\mathbb{Z}))$ is the unitary acting on the canonical basis as $U_z e_k := z^k e_k$, and $I_2 \subset \mathbb{T}$ is the set of all roots of unity of order a power of 2. Obviously for the inclusion above to make sense, one has to work with the concrete copy of $\mathcal{Q}_2$ given by its image through the canonical representation.

We would like to end this discussion with some questions.

Question 9.27. Is $N_{\mathcal{Q}_2}(C^*(U))$ equal to $\langle \mathcal{U}(C^*(U)), \{U_z\}_{z\in I_2}\rangle$?

Question 9.28. Is $\mathcal{W}(\mathcal{Q}_2, C^*(U))$ equal to $G_{C^*(U)}$?

An outer version the Weyl group may also be defined, namely

$$\mathcal{W}_\pi(\mathcal{Q}_2, C^*(U)) \doteq \frac{N_{\pi[\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)]}(\mathrm{Out}(\mathcal{Q}_2))}{\pi[\mathrm{Aut}_{C^*(U)}(\mathcal{Q}_2)]}$$

where $\pi : \mathrm{Aut}(\mathcal{Q}_2) \to \mathrm{Out}(\mathcal{Q}_2)$ is the canonical projection onto the quotient.

Question 9.29. Is $\mathcal{W}_\pi(\mathcal{Q}_2, C^*(U)) = \{1, \widetilde{\lambda}_f\}$?

9.4. On the ergodic properties of a class of endomorphisms of $\mathcal{Q}_2$

This section is devoted to discussing some interesting ergodic properties enjoyed by a countable class of endomorphisms of $\mathcal{Q}_2$ when restricted to suitable subalgebras. These are the endomorphisms

$$\theta_{2k+1} := \widetilde{\lambda_f} \circ \chi_{2k+1}$$

where $\widetilde{\lambda_f}$ is the extension to $\mathcal{Q}_2$ of the flip-flop automorphism of $\mathcal{O}_2$ and χ_{2k+1}, $k \in \mathbb{Z}$, is given by by $\chi_{2k+1}(S_2) := S_2$, and $\chi_{2k+1}(U) := U^{2k+1}$, *cf.* [10, Section 6.1]. From now on we will always assume $2k+1 \neq \pm 1$.

We start by noting that $C^*(U)$ is invariant under the action of each θ_{2k+1} as $\theta_{2k+1}(U) = U^{-(2k+1)}$. The behaviour of θ_{2k+1} at the level of $C^*(U)$ is summerized in the next result, which should be known.

Proposition 9.30. *The fixed-point subalgebra $C^*(U)^{\theta_{2k+1}}$ is trivial. Moreover, the restriction of θ_{2k+1} to $C^*(U)$ is not uniquely ergodic.*

Proof. After identifying $C^*(U)$ with $C(\mathbb{T})$, it is straightforward to realize that our endomorphism is the Koopman operator associated with the continuous map T of $\mathbb{T}$ to itself given by $T(z) = z^{-(2k+1)}$, that is $\theta_{2k+1}(f) := f \circ T$, for any $f \in C(\mathbb{T})$. We will show that given $f \in C(\mathbb{T})$ such that $f(z^n) = f(z)$, $z \in \mathbb{T}$, implies f

is constant if $n \neq \pm 1$. This is actually true for any $f \in L^2(\mathbb{T}, \mathrm{d}z)$. Indeed, we can expand such an f as $f(z) = \sum_{l\in\mathbb{Z}} a_l z^l$, where $a_l := \int_{\mathbb{T}} f(z) z^{-l} \mathrm{d}z$, and the convergence of the series is understood with respect to the L^2-norm. The fixed-point equation then reads as $\sum_{l\in\mathbb{Z}} a_l z^{nl} = \sum_{l\in\mathbb{Z}} a_l z^l$, which easily implies $a_l = 0$ for any $l \neq 0$.

As for the second part of the statement, note that all $2k$-roots of unity are fixed points of T. This means that the Dirac measures δ_z are T-invariant if $z^{2n} = 1$. In particular, there are infinite T-invariant measures, and thus the dynamical system cannot be uniquely ergodic. □

We next note that the diagonal subalgebra $\mathcal{D}_2$ is invariant as well. However, the behaviour of the restriction of θ_{2k+1} to $\mathcal{D}_2$ is quite different as opposed to what happens for $C^*(U)$. For instance, when k is even $\mathcal{D}_2^{\theta_{2k+1}}$ is never trivial as shown below.

Example 9.31. By definition we have $\theta_{2k+1}(S_2) = S_1$ and $\theta_{2k+1}(S_1) = U^{-(2k+1)} S_1$. If we recall the short notation $P_\alpha := S_\alpha S_\alpha^*$ for the projections in $\mathcal{D}_2$, it is easy to see that

$$\begin{aligned}
&\theta_{2k+1}(P_{22}) = P_{11} \\
&\theta_{2k+1}(P_{11}) = \begin{cases} P_{22} & \text{if } k \text{ is even} \\ P_{21} & \text{if } k \text{ is odd} \end{cases} \\
&\theta_{2k+1}(P_{12}) = \begin{cases} P_{21} & \text{if } k \text{ is even} \\ P_{22} & \text{if } k \text{ is odd} \end{cases} \\
&\theta_{2k+1}(P_{21}) = P_{12}.
\end{aligned}$$

Therefore, if k is even, then $\theta_{2k+1}(P_{11} + P_{22})$ is a fixed element of $\mathcal{D}_2$.

We next collect a series of formulas that come in useful to prove that our endomorphisms act periodically on all projections P_α in $\mathcal{D}_2$.

Lemma 9.32. *For every $h \in \mathbb{N}$ the following formulas hold:*

$$\begin{aligned}
(\theta_{2k+1})^{2^h}(U) &= U^{(2k+1)^{2^h}}, \\
(\theta_{2k+1})^{2^h}(S_2) &= U^{-2k\prod_{i=1}^{h-1}(1+(2k+1)^{2^i})} S_2, \\
(\theta_{2k+1})^{2^h}(S_1) &= U^{2k\prod_{i=1}^{h-1}(1+(2k+1)^{2^i})} S_1 \ .
\end{aligned}$$

Proof. It is a matter of a direct induction on h. □

We are now in a position to show that the orbit of each projection P_α under θ_{2k+1} is finite. More precisely, we bound the cardinality of any such orbit from above by $2^{|\alpha|}$.

Proposition 9.33. *For any multi-index $\alpha \in W_2$, we have $(\theta_{2k+1})^{2^{|\alpha|}}(P_\alpha) = P_\alpha$.*

Proof. It is an induction on $|\alpha|$. The basis of the induction has been carried out in Example 9.31. Let us then move on to the inductive step. We assume the property has been verified for all multi-indices of length a given $|\alpha|$, and prove it continues to hold for any multi-index $\beta = (i, \alpha)$, $i = 1, 2$, of length $|\alpha| + 1$. We have:

$$\begin{aligned}
(\theta_{2k+1})^{2^{|\alpha|+1}}(P_\beta) &= (\theta_{2k+1})^{(2^{|\alpha|})}\big((\theta_{2k+1})^{2^{|\alpha|}}(S_i P_\alpha S_i^*)\big) \\
&= (\theta_{2k+1})^{(2^{|\alpha|})}\big((\theta_{2k+1})^{2^{|\alpha|}}(S_i)\big)P_\alpha(\theta_{2k+1})^{(2^{|\alpha|})}\big((\theta_{2k+1})^{2^{|\alpha|}}(S_i^*)\big) \\
&= U^{(-1)^{i+1}2k\prod_{i=1}^{|\alpha|}(1+(2k+1)^{2^i})} S_i P_\alpha S_i^* U^{(-1)^i 2k\prod_{i=1}^{|\alpha|}(1+(2k+1)^{2^i})} \\
&= S_i P_\alpha S_i^* = P_\beta
\end{aligned}$$

where we used that $2^{|\alpha|+1}$ divides $2k\prod_{i=1}^{|\alpha|}(1+(2k+1)^{2^i})$ since it is the product of $|\alpha|+1$ even factors, and $U^{2^{|\beta|}}P_\beta U^{-2^{|\beta|}} = P_\beta$. $\square$

Remark 9.34. Example 9.31 makes it clear that in general the length of a θ_{2k+1}-orbit of a projection P_α may be shorter than $2^{|\alpha|}$, at least when k is even.

Theorem 9.35. *For any $x \in \mathcal{D}_2$, the Cesàro averages $\frac{1}{N+1}\sum_{n=0}^{N}(\theta_{2k+1})^n(x)$ converges in norm to an element in $\mathcal{D}_2^{\theta_{2k+1}}$.*

Proof. Since the linear operators $T_N := \frac{1}{N+1}\sum_{n=0}^{N}(\theta_{2k+1})^n$ are contractions, *i.e.* $\|T_N\| \leq 1$ for every N, it is enough to verify the statement only on projections P_α as their linear span is a dense *-subalgebra of $\mathcal{D}_2$. If we define $Q_\alpha := \frac{1}{2^{|\alpha|}}\sum_{i=0}^{2^{|\alpha|}-1}(\theta_{2k+1})^i(P_\alpha)$, an application of Proposition 9.33 yields

$$\lim_{N\to\infty} T_N(P_\alpha) = Q_\alpha.$$

$\square$

Borrowing the terminology from [1], the above theorem can be recast by saying that the restriction of θ_{2k+1} is *uniquely ergodic with respect to the fixed-point subalgebra*. In the aforementioned paper, the condition is characterized in the following way: an endomorphism Φ of a C^*-algebra $\mathcal{A}$ is uniquely ergodic w.r.t. the fixed-point subalgebra $\mathcal{A}^\Phi$, namely the norm limit of the Cesàro averages $\frac{1}{N+1}\sum_{n=0}^{N}\Phi^n(x)$ exists for any $x \in \mathcal{A}$, if and only if any state ω on $\mathcal{A}^\Phi$ has a unique extension to a Φ-invariant state of the whole C^*-algebra $\mathcal{A}$, *cf.* [1, Theorem 3.2]. Note that when the fixed-point algebra is trivial, *i.e.* $\mathcal{A}^\Phi = \mathbb{C}$, the definition boils down to unique ergodicity.

Going back to our endomorphims θ_{2k+1}, it is not clear whether $\mathcal{D}_2^{\theta_{2k+1}}$ may not be trivial also for odd values of k. At any rate, if $\mathcal{D}_2^{\theta_{2k+1}}$ turned out to be trivial for odd values of k, Theorem 9.35 would tell us the restriction of θ_{2k+1} to $\mathcal{D}_2$ is actually unique ergodic. Therefore, we deem the following question worth answering.

Question 9.36. Is the fixed-point subalgebra $\mathcal{D}_2^{\theta_{2k+1}}$ trivial if k is odd?

Examples of uniquely ergodic maps on the Cantor set, which is here conceived of as the spectrum of $\mathcal{D}_2$, are certainly known. One is the so-called odometer (see Example 9.39 in Section 9.6). Here we limit ourselves to mentioning that at the algebra level the odometer is nothing but the restriction to $\mathcal{D}_2$ of the inner automorphism $\mathrm{ad}(U)$. Note that $\mathcal{D}_2^{\mathrm{ad}(U)} = \mathbb{C}$ as follows from $C^*(U)' \cap \mathcal{O}_2 = \mathbb{C}$ (see [10, Corollary 2.12]).

9.5. Permutative representations

The canonical representation is perhaps the most remarkable example of a permutative representation of $\mathcal{Q}_2$. Permutative representations of Cuntz algebras have been studied by many authors. The notion has then been extended to $\mathcal{Q}_2$ in [12]. The definition is easily guessed: a representation $\rho : \mathcal{Q}_2 \to \mathcal{B}(\mathcal{H})$ is said to be permutative if there exists an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of the Hilbert space $\mathcal{H}$ such that $\rho(S_2)e_n = e_{\sigma_2(n)}$ and $\rho(U)e_n = e_{\tau(n)}$, for every $n \in \mathbb{N}$, where σ_2 is an injection and τ a bijection of $\mathbb{N}$ such that $\sigma \circ \tau = \tau^2 \circ \sigma_2$ and $\mathbb{N}$ decomposes as the disjoint union of $\sigma_2(\mathbb{N})$ and $\tau \circ \sigma_2(\mathbb{N})$. Obviously the restriction of ρ to the Cuntz algebra $\mathcal{O}_2$ is still a permutative representation. Conversely, it is a result proved in [12] that if a permutative representation π of $\mathcal{O}_2$ extends to $\mathcal{Q}_2$ then it also admits permutative extensions. In addition, all permutative extensions are unitarily equivalent to one another. The canonical representation of $\mathcal{Q}_2$ plays a privileged role. For instance, it is (up to unitary equivalence) the only permutative representation in which τ has exactly one orbit. Moreover, it is the only permutative representation that restricts to $\mathcal{O}_2$ as a reducible representation. In fact, every irreducible permutative representation of $\mathcal{O}_2$ automatically extends to a (necessarily irreducible) representation of $\mathcal{O}_2$, and apart from the canonical representation every irreducible permutative representation of $\mathcal{Q}_2$ restricts to $\mathcal{O}_2$ as an irreducible representation. Finally, the decomposition of a permutative representation ρ of $\mathcal{Q}_2$ depends only upon its restriction to the Cuntz algebra. More precisely, in [12] a permutative representation ρ is proved to be completely reducible if and only if its restriction $\rho \restriction_{\mathcal{O}_2}$ is regular in the sense of Bratteli-Jorgensen. As a result of the analysis carried out in [12], one also has that two irreducible permutative representations of $\mathcal{Q}_2$ are unitarily equivalent if and only if their restrictions to $\mathcal{O}_2$ are, which is yet another way in which the rigidity of the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$ becomes apparent. We have ample reason to expect the property to continue to hold even when the hypothesis that our representations are permutative is dispensed with. Despite our attempts, though, we have not been able to prove that, which is why we single out the problem in the following question.

Question 9.37. Given ρ_1 and ρ_2 irreducible representations of $\mathcal{Q}_2$ such that $\rho_1 \restriction_{\mathcal{O}_2}$ and $\rho_2 \restriction_{\mathcal{O}_2}$ are unitarily equivalent, is it true that ρ_1 and ρ_2 are already so at the level of $\mathcal{Q}_2$?

A permutative irreducible representation of $\mathcal{Q}_2$ remains irreducible when restricted to $\mathcal{O}_2$ unless it is unitarily equivalent to the canonical representation. The hypothesis that the representation is permutative, however, might well turn out to be dispensable.

Question 9.38. Given any irreducible representation ρ of $\mathcal{Q}_2$ that is not equivalent to the canonical representation, is it true that $\rho\restriction_{\mathcal{O}_2}$ is still irreducible?

A positive answer to the last question would provide a rather strong characterization of the canonical representation as the sole irreducible representation of $\mathcal{Q}_2$ whose restriction to $\mathcal{O}_2$ acts reducibly instead.

9.6. More representations of $\mathcal{Q}_2$

Apart from permutative representations, other interesting representations of $\mathcal{Q}_2$ are known. We list here some of them which we believe might have a role to play in relation to some of the problems we have raised.

Example 9.39 (Bernoulli Representations)**.** Here we describe the (unique) extensions to $\mathcal{Q}_2$ of the representations of $\mathcal{O}_2$ considered in [146], therein called *Bernoulli representations*. Let $K = \prod_{i=1}^{\infty}\{1,2\}$, $p_1 > 0$, $p_2 > 0$ such that $p_1 + p_2 = 1$ and consider the odometer map T

$$\begin{aligned} &T : K \to K \\ &(1x) \mapsto (2x) \\ &(2_k 1x) \mapsto (1_k 2x) \end{aligned}$$

where $x = (x_1, x_2, \ldots) \in K$. Now consider the measure ν on K determined by

$$\nu([u_1, \ldots, u_m]) = p_{u_1} \cdots p_{u_m} \ .$$

where $[u_1, \ldots, u_m] = \{x = (x_1, x_2, \ldots) \in K \mid x_i = u_i \text{ for } i = 1, \ldots, m\}$.

We now recall the definition of the Bernoulli representation $\pi_{p_1,p_2} \colon \mathcal{O}_2 \to \mathcal{B}(L^2(K,\nu))$. The two isometries $\pi_{p_1,p_2}(S_1)$ and $\pi_{p_1,p_2}(S_2)$ are defined in [146] as follows

$$(\pi_{p_1,p_2}(S_1)f)(x_1, x_2, \ldots) = \begin{cases} p_1^{-1/2} f(x_2, \ldots) & \text{if } x_1 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$(\pi_{p_1,p_2}(S_2)f)(x_1, x_2, \ldots) = \begin{cases} p_2^{-1/2} f(x_2, \ldots) & \text{if } x_1 = 2 \\ 0 & \text{otherwise} \end{cases}$$

Mori, Suzuki and Watatani showed that the isometries $\pi_{p_1,p_2}(S_1)$ and $\pi_{p_1,p_2}(S_2)$ are pure. This implies that there exists a unique extension to $\mathcal{Q}_2$. We want to describe explicitly this extension by exhibiting a formula for $\pi_{p_1,p_2}(U)$. First of all, we define a function

$$\begin{aligned} &m : K \to \mathbb{R} \\ &m(1x) \doteq (p_2/p_1)^{1/2} \\ &m(2_k 1x) \doteq (p_1/p_2)^{(k-1)/2} \qquad k \geq 1 \ . \end{aligned}$$

We now define $\pi_{p_1,p_2}(U)$ as follows

$$(\pi_{p_1,p_2}(U)f)(x) \doteq m(x)f(T(x)) .$$

Let us check that $\pi_{p_1,p_2}(U)$, $\pi_{p_1,p_2}(S_2)$ and $\pi_{p_1,p_2}(S_1)$ satisfy the usual relations. First of all we observe that the following equality holds

$$\begin{aligned}
\pi_{p_1,p_2}(US_2)f(x) &= m(x)(\pi_{p_1,p_2}(S_2)f)(Tx) \\
&= \begin{cases} {p_2}^{-1/2}m(x)f(y_2,\dots) & \text{if } y_1 = 2 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} {p_2}^{-1/2}(p_2/p_1)^{1/2}f(x_2,\dots) & \text{if } x_1 = 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \pi_{p_1,p_2}(S_1)f(x)
\end{aligned}$$

where $y = Tx$. All is left to do is check the remaining relation

$$\begin{aligned}
&\pi_{p_1,p_2}(US_1^kS_2)f(x) = m(x)(\pi_{p_1,p_2}(S_1^kS_2)f)(Tx) \\
&= \begin{cases} m(x){p_2}^{-1/2}{p_1}^{-k/2}m(x)f(y_{k+2},\dots) & \text{if } y_1 = \dots = y_k = 1, y_{k+1} = 2 \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} {p_2}^{-k/2}{p_1}^{-1/2}m(x)f(y_{k+2},\dots) & \text{if } x_1 = \dots = x_k = 2, x_{k+1} = 1 \\ 0 & \text{otherwise} \end{cases} \\
&= \pi_{p_1,p_2}(S_2^kS_1)f(x)
\end{aligned}$$

Remark 9.40. These representations enjoy the following properties

1. the representation are irreducible for all p_1 and p_2, [135, Theorem 2.8],
2. π_{p_1,p_2} is unitarily equivalent to $\pi_{p'_1,p'_2}$ if and only if $p_1 = p'_1$, [146, Theorem 3.1],
3. $\pi_{p_1,p_2}(S_\alpha)$ has no eigenvectors for all words in $\{1,2\}$, [146, Proposition 3.3],
4. $\pi_{p_1,p_2}(S_1)$ and $\pi_{p_1,p_2}(S_2)$ are pure isometries, see the proof of [146, Proposition 3.3]

Question 9.41. Are the Bernoulli representations permutative?

Example 9.42. This representation is in some sense a restriction of the representation considered in [133]. Since [133] studies some permutative representations of $\mathcal{O}_\infty$, while here we deal with $\mathcal{O}_2$ and $\mathcal{Q}_2$, we choose a *smaller* Hilbert space so that the range of the projection $S_1S_1^* + S_2S_2^*$ is the whole Hilbert space. Let $K = \prod_{i=1}^\infty \{1,2\}$, and consider the maps $\beta_1, \beta_2 \colon K \to K$ defined as $\beta_1(x) = (1x)$,

$\beta_2(x) = (2x)$, where $x \in K$. We have the following representation of $\pi_\beta\colon \mathcal{O}_2 \to \mathcal{B}(\ell_2(K))$

$$\pi_\beta(S_1)e_a \doteq e_{\beta_1(a)}$$
$$\pi_\beta(S_2)e_a \doteq e_{\beta_2(a)}$$

where $a \in K$.

Remark 9.43. One can show that

$$\cap_k \pi_\beta(S_1^k)(\ell_2(K)) = \mathbb{C}\{e_{\underline{2}}\} \text{ and } \cap_k \pi_\beta(S_2^k)(\ell_2(K)) = \mathbb{C}\{e_{\underline{1}}\}.$$

Moreover, $\pi_\beta(S_1)$ and $\pi_\beta(S_2)$ have eigenvectors $e_{\underline{1}}$ and $e_{\underline{2}}$, respectively. Thus π_β gives rise to a representation of $\mathcal{Q}_2$ (actually there are many extensions as in the case of the canonical representation) determined by this formula

$$\pi_\beta(U)e_a \doteq e_{T(a)}$$

where $T\colon K \to K$ is the odometer considered above. It is clear, that this representation is permutative by its very definition.

9.7. Pure states on $\mathcal{Q}_2$

In [77] Cuntz faced the problem of when a pure state of the diagonal $\mathcal{D}_2$ has only one pure extension to $\mathcal{Q}_2$. He showed that any such a state will in general fail to have a unique pure extension. However, evaluations at the so-called *irrational points* in $\{0,1\}^{\mathbb{N}}$ do uniquely extend to pure states of $\mathcal{O}_2$. Questions of the same sort have been addressed in [12] for the inclusion $C^*(U) \subset \mathcal{Q}_2$, although the results there obtained are not conclusive. Even so, it is known that evaluations ω_z at a point $z \in \mathbb{T}$ which is a root of unity of order 2^n, $n \in \mathbb{N}$, do not extend uniquely. In fact, any such state displays uncountably many pure extensions, [12, Proposition 8.5]. It is not clear what may happen in general when z is not such a root. Nevertheless, if z is not a root of unity of order $(2^h - 1)2^k$, $h, k \in \mathbb{N}$, the corresponding state does extend uniquely to a pure state [12, Proposition 8.6]. In particular, if z is not a root of unity, then the corresponding evaluation state uniquely extends to a pure state of $\mathcal{Q}_2$. The discussion in [12, Section 8] thus leaves the following question open.

Question 9.44. How many pure extensions does ω_z have for those values of z not covered above?

The above question is also worth asking for the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$. Now, unlike $C^*(U)$, pure states on $\mathcal{O}_2$ are too many to be treated completely, which means it is natural to start tackling the problem with suitable classes of pure states. This is done in [12], where some interesting if partial results are proved for vector states associated with permutative representations of $\mathcal{O}_2$. Perhaps as an effect of the inclusion $\mathcal{O}_2 \subset \mathcal{Q}_2$ being rigid in many respects, we have only been able to provide examples of pure states of the Cuntz algebra $\mathcal{O}_2$ that uniquely extend to $\mathcal{Q}_2$. For instance, any vector state associated with an irreducible permutative representation π in which both $\pi(S_1)$ and $\pi(S_2)$ are pure isometries will uniquely extend to a pure state of $\mathcal{Q}_2$, [12, Theorem 8.1]. At any rate, pureness of $\pi(S_1)$ and $\pi(S_2)$ is actually not needed, for any vector state associated with either $P(1)$ or $P(2)$ (see [12] for the notation) still extends uniquely to a pure states of $\mathcal{Q}_2$, [12, Theorem 8.4]. At this point, the following circle of questions is natural to ask.

Question 9.45. Given a pure state ω on $\mathcal{O}_2$, how many extensions to a pure state of $\mathcal{Q}_2$ does ω have? Is it true that $\mathcal{O}_2 \subset \mathcal{Q}_2$ has the unique extension property? If not, find an example of a pure state of $\mathcal{O}_2$ with at least two pure extensions to $\mathcal{Q}_2$. Discuss some special cases, e.g. $\omega = \omega_\xi$ is the unique extension to $\mathcal{O}_2$ of a pure state of $\mathcal{D}_2$ corresponding to an irrational point ξ in the spectrum (see [77]). If ξ is not irrational, and ω_ξ is one of the corresponding pure state extensions to $\mathcal{O}_2$, how many pure extensions of ω_ξ to $\mathcal{Q}_2$ do exist? Can an extendable irreducible representation π of $\mathcal{O}_2$ have at least two inequivalent extensions?

Recall that a Cartan subalgebra $A \subset B$ is said to be C^*-diagonal if every pure state of A extends uniquely to a (necessarily pure) state of B. Examples are given e.g. in [172, Remark 3.5]. In light of the result by Cuntz recalled above, it is obvious that $\mathcal{D}_2$ cannot be C^*-diagonal in $\mathcal{Q}_2$ (since it already fails to be so in $\mathcal{O}_2$). Even so, we do believe the following problem is worth raising.

Question 9.46. Given a pure state on $\mathcal{D}_2$, how many extensions to $\mathcal{Q}_2$ do exist?

We would like to end the present section by pointing out that the questions above can also be formulated for the p-adic C^*-algebras $\mathcal{Q}_p$, which are the topic the next chapter is focused on.

9.8. KMS states on $\mathcal{Q}_2$

As proved in [149], the Cuntz algebras $\mathcal{O}_n$ have only one KMS state (at inverse temperature $\beta = \log(n)$) with respect to the (2π-periodic) action of $\mathbb{R}$ given by the gauge automorphisms. This is nothing but the composition $\tau \circ E$, where E is the canonical conditional expectation from $\mathcal{O}_n$ onto $\mathcal{F}_n$, and τ the unique tracial state on the UHF algebra $\mathcal{F}_n$. One might wonder if $\mathcal{Q}_2$ has a KMS for the action of the (extended) gauge automorphisms and if it is unique as well. As for the existence of such a state, the answer is positive, see e.g. [13, Proposition 6.8]. As one would expect,this is also the unique KMS state with respect to such dynamic. This follows from [81, Proposition 4.2], or even from the much stronger results contained in [2][19]. Finally, given the KMS state ω on $\mathcal{Q}_2$, it would be interesting to determine the type of the von Neumann algebra $\pi_\omega(\mathcal{Q}_2)''$, cf. [121]. Again, questions of this sort can be posed for the p-adic C^*-algebras $\mathcal{Q}_p$ as well.

[19] We would like to thank N. Stammeier for pointing out to us the references along with an argument for proving the uniqueness of the KMS state.

10. The p-adic ring C^*-algebras

The C^*-algebra $\mathcal{Q}_2$ is actually part of a countable family of C^*-algebras, known as p-adic C^*-algebras, which in turn belong to quite a rich class of C^*-algebras, the so-called boundary quotients of right LCM semigroups, see [15] and the references therein. Indeed, associated with any natural number $p \geq 2$, there is a *p-adic C^*-algebra* $\mathcal{Q}_p$ which by definition is the universal C^*-algebra generated by a unitary u and a proper isometry s_p such that

$$s_p u = u^p s_p \text{ and } \sum_{k=0}^{p-1} u^k s_p s_p^* u^{-k} = 1 \,.$$

In order to simplify the notation, we prefer to write u rather than u_p, hoping that this will not cause confusion. As the Cuntz algebra $\mathcal{O}_2$ can be naturally embedded into $\mathcal{Q}_2$, so each Cuntz algebra $\mathcal{O}_p$, $p \geq 2$, embeds into $\mathcal{Q}_p$ through the injective endomorphism ι defined on the generating isometries as $\iota(S_k) := u^k s_p$, for $k = 1, 2, \ldots, p-1$, and $\iota(S_p) := s_p$.

Now the p-adic C^*-algebras can all be realized as subalgebras of a larger C^*-algebra that arises from considering the whole $\mathbb{N}^\times$, the multiplicative monoid associated with $\{1, 2, \ldots\}$. This is the universal C^*-algebra $\mathcal{Q}_\mathbb{N}$ generated by isometries $\{s_n\}_{n\in\mathbb{N}^\times}$ and a unitary u such that, for any $n, m \in \mathbb{N}^\times$,

$$s_m s_n = s_{nm}, \quad s_n u = u^n s_n, \quad \text{and} \quad \sum_{k=0}^{n-1} u^k s_n s_n^* u^{-k} = 1.$$

Note that s_1 is nothing but the identity[20]. Much more information on both $\mathcal{Q}_p$'s and $\mathcal{Q}_\mathbb{N}$ can be found in the paper where these were originally introduced, [81]. Here we limit ourselves to recalling that all the above C^*-algebras are mutually non-isomorphic. This can be shown by explicit computation of their K-groups, see *e.g.* [81] and the discussion in [30].

The algebras $\mathcal{Q}_p$ and $\mathcal{Q}_\mathbb{N}$ admit a distinguished representation acting on $\ell_2(\mathbb{Z})$: the so-called canonical representation. This is defined by the following formulas

$$s_p e_k = e_{pk} \qquad u e_k = e_{k+1} \qquad \forall k \in \mathbb{N} \,.$$

The following result somehow parallels a well-known result about embeddings of Cuntz algebras.

Proposition 10.1. *For all integers $p \geq 2$, $h \geq 2$, $k \geq 1$, there exists an embedding $\Phi_{p^{hk},h} \colon \mathcal{Q}_{p^{hk}} \to \mathcal{Q}_{p^k}$ given by $\Phi_{p^{hk},h}(s_{p^{hk}}) = s_{p^k}^h$ and $\Phi_{p^{hk},h}(u) = u$. In particular, for every integer $k \geq 1$ there exists an embedding $\Psi_{p^k} \colon \mathcal{Q}_{p^k} \to \mathcal{Q}_p$ given by $\Psi_{p^k}(s_{p^k}) = s_p^k$ and $\Psi_{p^k}(u) = u$.*

[20] We warn the reader that s_1 is not the same as S_1, the Cuntz isometry.

Proof. As for the map $\Phi_{p^{hk},h}$ it suffices to notice that $s_{p^k}^h = s_{p^{hk}}$. This can be checked by means of the canonical representation

$$\Phi_{p^{hk},h}(s_{p^{hk}})e_n = s_{p^k}^h e_n = s_{p^k}^{h-1} e_{p^k n} = e_{p^{hk} n} = s_{p^{hk}} e_n \qquad \forall n \in \mathbb{Z}.$$

Similarly, for Ψ_{p^k} the claim follows from the fact that $s_{p^k} = s_p^k$

$$\Psi_{p^k}(s_{p^k})e_n = s_p^k e_n = e_{p^k n} = s_{p^k} e_n \qquad \forall n \in \mathbb{Z}.$$

$\square$

Corollary 10.2. *For any $k \geq 2$, there is an embedding*

$$\Phi_{p^p,p} \circ \cdots \circ \Phi_{p^{p^k},p} : \mathcal{Q}_{p^{p^k}} \to \mathcal{Q}_p\,.$$

Of course, in this respect there is no uniqueness and one may find several such embeddings. It would be nice to combine this embedding with the embedding of the canonical copy of $\mathcal{O}_{2^k} \subset \mathcal{Q}_{2^k}$ into the canonical copy of $\mathcal{O}_2 \subset \mathcal{Q}_2$.

Question 10.3. Do the results obtained for $\mathcal{Q}_2$ in [140, 10, 11, 12, 13] carry over to $\mathcal{Q}_p$ for all positive integers p and $\mathcal{Q}_{\mathbb{N}}$ (cf. the introduction of [140])?

For instance, in [10, Theorem 4.5] $\mathrm{Aut}_{\mathcal{O}_2}(\mathcal{Q}_2)$ was shown to be trivial. This actually continues to be true in $\mathcal{Q}_p$ as well. More precisely, as a consequence of [15, Remark 5.1], any $\alpha \in \mathrm{End}(\mathcal{Q}_p)$ such that $\alpha(x) = x$ for any $x \in \mathcal{O}_p$ is necessarily the identity, that is $\alpha = \mathrm{id}_{\mathcal{Q}_p}$. Moreover, we also provide a result that generalizes to $\mathcal{Q}_p$ the criterion due to Larsen and Li for a representation of $\mathcal{O}_2$ to extend to a representation of $\mathcal{Q}_2$ after raising some a few questions.

Question 10.4. To which extent do the results proved for $\mathcal{Q}_2$ extend to general boundary quotient C^*-algebras $\mathcal{Q}(G \rtimes_\theta P)$ (for the notation see *e.g.* [15]) ? In particular:

- Closely examine fine properties of the group $\mathrm{Out}(\mathcal{Q}(G \rtimes_\theta P))$, cf. [10];
- What can be said about the automorphisms of the C^*-algebras $\mathcal{Q}(G \rtimes_\theta P)$ that fix the diagonal pointwise? Study the group $\mathrm{Aut}_{\mathcal{D}}(\mathcal{Q}(G \rtimes_\theta P))$, cf. [11];
- Provide a definition of quasi-free automorphisms for Cuntz-Pimsner-Nica algebras (i.e. C^*-algebras generated by suitable systems of Hilbert bimodules), along with motivating examples.

We can now move on to the announced generalization of [140, Prop. 4.1].

Theorem 10.5. *Let $\pi\colon \mathcal{O}_p \to \mathcal{B}(H)$ be a representations of the Cuntz algebra generated by p isometries $T_0, \ldots, T_{p-1}$. Then there exists a representation $\tilde{\pi}\colon \mathcal{Q}_p \to \mathcal{B}(H)$ such that $\tilde{\pi}(u^i s_p) = \pi(T_i)$ for $i = 0, \ldots, p-1$ if and only if the unitary parts in the Wold decomposition of $\pi(T_0)$ and $\pi(T_{p-1})$ are unitarily equivalent.*

Proof. ($\Rightarrow$) Consider a representation $\tilde{\pi}\colon \mathcal{Q}_p \to \mathcal{B}(H)$. If we set $\pi(T_i) \doteq \tilde{\pi}(u^i s_p)$ for $i = 0, \dots, p-1$ we get a representation of $\mathcal{O}_p$. Now by the definition of $\mathcal{Q}_p$ we see that

$$\tilde{\pi}(u)\pi(T_{p-1})\tilde{\pi}(u^*) = \tilde{\pi}(uu^{p-1}s_p)\tilde{\pi}(u)^* = \tilde{\pi}(u^p s_p) = \tilde{\pi}(s_p u)\tilde{\pi}(u)^* = \pi(T_0)\ .$$

As the two isometries are unitarily equivalent it follows that their unitary parts in the Wold decomposition are unitarily equivalent.

($\Leftarrow$) Now suppose that we are given a representation of $\mathcal{O}_p$ such that unitary parts of $\pi(T_0)$ and $\pi(T_{p-1})$ in the Wold decomposition are unitarily equivalent. In particular, we have a unitary operator

$$W_0\colon\ \cap_{i\geq 1}\,\pi(T_{p-1})^i(H) \to \cap_{i\geq 1}\pi(T_0)^i(H)\ .$$

We may extend W to a partial isometry $W\colon H \to H$ by setting W equal to 0 on the orthogonal space to $\cap_{i\geq 1}\pi(T_{p-1})^i(H)$. We observe that this partial isometry satisfies

$$W\pi(T_{p-1}) = \pi(T_0)W\ .$$

Consider the sequence

$$V_k = \sum_{i=0}^{p-2}(\pi(T_{i+1}T_i^*) + \sum_{j=1}^{k}\pi(T_0^j T_{i+1}T_i^*(T_{p-1}^j)^*)) \quad k \in \mathbb{N}\ .$$

We claim that the strong limit

$$V \doteq s - \lim_k V_k$$

exists. First of all we have that

$$\begin{aligned}
V_k^* V_k &= \left(\sum_{i=0}^{p-2}(\pi(T_{i+1}T_i^*) + \sum_{j=1}^{k}\pi(T_0^j T_{i+1}T_i^*(T_{p-1}^j)^*))\right)^* \\
&\qquad \times \left(\sum_{i=0}^{p-2}(\pi(T_{i+1}T_i^*) + \sum_{j=1}^{k}\pi(T_0^j T_{i+1}T_i^*(T_{p-1}^j)^*))\right) \\
&= \left(\sum_{i=0}^{p-2}\pi(T_i T_{i+1}^* + \sum_{j=1}^{k}T_{p-1}^j T_i T_{i+1}^*(T_0^j)^*)\right) \\
&\qquad \times \left(\sum_{i=0}^{p-2}\pi(T_{i+1}T_i^* + \sum_{j=1}^{k}T_0^j T_{i+1}T_i^*(T_{p-1}^j)^*)\right) \\
&= \sum_{i=0}^{p-2}\pi(T_i T_i^* + \sum_{j=1}^{k}T_{p-1}^j T_i T_i^*(T_{p-1}^j)^*) \leq 1
\end{aligned}$$

which implies that $\|V_k\|^2 = \|V_k^* V_k\| \leq 1$. Now let $\xi \in H$. For $k \geq h$, we have

$$\begin{aligned}
\|V_k\xi\|^2 &= \sum_{i=0}^{p-2}(\|\pi(T_{i+1}T_i^*)\xi\|^2 + \sum_{j=1}^{k}\|\pi(T_0^j T_{i+1}T_i^*(T_{p-1}^*)^j)\xi\|^2) \\
&= \sum_{i=0}^{p-2}(\|\pi(T_{i+1}T_i^*)\xi\|^2 + \sum_{j=1}^{h}\|\pi(T_0^j T_{i+1}T_i^*(T_{p-1}^*)^j)\xi\|^2 \\
&\quad + \sum_{j=h+1}^{k}\|\pi(T_0^j T_{i+1}T_i^*(T_{p-1}^*)^j)\xi\|^2) \\
&= \|V_h\xi\|^2 + \|V_k\xi - V_h\xi\|^2 \ .
\end{aligned}$$

Now the sequence $\|V_k\xi\|^2$ is bounded (by $\|\xi\|^2$) and monotone increasing, so it convergences. This implies that $V_k\xi$ is a Cauchy sequence and this ends the proof of the claim.

It can be easily seen that also the strong limit of V_k^* exists and that this limit yields the adjoint of V. By continuity of multiplication (on bounded subset) with respect to the strong operator topology, it follows that

$$V^*V = s - \lim_k V_k^* V_k \quad \text{ and } VV^* = s - \lim_k V_k V_k^*$$

are orthogonal projections onto $[\cap_{i\geq 1}\pi(T_{p-1}^i(H)]^\perp$ and $[\cap_{i\geq 1}\pi(T_0^i(H)]^\perp$, respectively. From this discussion follows that $\mathfrak{U} = V + W$ is a unitary operator. We claim that by setting $\tilde{\pi}(u) = \mathfrak{U} \doteq V + W$ and $\tilde{\pi}(s_p) \doteq \pi(T_0)$ we get the desired extension of our representation. In the first place we want to show that

$$\mathfrak{U}^i\pi(T_0) = \pi(T_i) \quad \text{ for } i = 0, \ldots, p-2$$

It is enough to prove that $\mathfrak{U}\pi(T_i) = \pi(T_{i+1})$ for $i = 0, \ldots, p-2$. Since $WT_j = 0$ for $j = 0, \ldots, p-2$ and $W\pi(T_{p-1}) = \pi(T_0)W$, we see that

$$\mathfrak{U}\pi(T_j) = (V + W)\pi(T_j) = V\pi(T_j) = \pi(T_{j+1})$$

where we used that $V_k\pi(T_j) = \pi(T_{j+1})$ for all $k \in \mathbb{N}$. Clearly, from this relation we see that

$$\sum_{i=0}^{p-1}\mathfrak{U}^i\pi(T_0T_0^*)\mathfrak{U}^{-i} = 1 \ .$$

Now, we are left to check that $\mathfrak{U}^p\pi(T_0) = \pi(T_0)\mathfrak{U}$. As

$$\mathfrak{U}^p\pi(T_0) = \mathfrak{U}\pi(T_{p-1}) = (V + W)\pi(T_{p-1}) = VT_{p-1} + WT_{p-1} = VT_{p-1} + T_0W$$

and

$$T_0\mathfrak{U} = T_0V + T_0W$$

we only need to check that $V\pi(T_{p-1}) = \pi(T_0)V$. It is enough to show that $\pi(T_0)V_{k-1} = V_k\pi(T_{p-1})$. Indeed,

$$\begin{aligned} V_k\pi(T_{p-1}) &= \sum_{i=0}^{p-2}\sum_{j=1}^{k}\pi(T_0^jT_{i+1}T_i^*(T_{p-1}^*)^{j-1}) \\ &= \pi(T_0)\left(\sum_{i=0}^{p-2}\sum_{j=1}^{k}\pi(T_0^{j-1}T_{i+1}T_i^*(T_{p-1}^*)^{j-1})\right) \\ &= \pi(T_0)\left(\sum_{i=0}^{p-2}(\pi(T_{i+1}T_i^*)+\sum_{j=0}^{k-1}\pi(T_0^jT_{i+1}T_i^*(T_{p-1}^*)^j))\right) \end{aligned}$$

where we used the convention $(T_0)^0 = 1$. This ends the proof.

□

The following result is a generalisation of [10, Theorem 3.16].

Corollary 10.6. *There is no unital conditional expectation from $\mathcal{Q}_p$ to $\mathcal{O}_p$.*

Proof. We give a proof by contradiction. Suppose that such a conditional expectation does exist and denote it by E. We shall work in any representation in which the isometry T_{p-1} is pure, like the interval picture [15]. We want to show that $E(U) = U$ and thus reach a contradiction. Note that, for all $i = 0, \ldots, p-2$ we have

$$E(U)(T_iT_i^*) = E(UT_iT_i^*) = E(T_{i+1}T_i^*) = T_{i+1}T_i^* \ .$$

and

$$\begin{aligned} E(U)(T_{p-1}^nT_iT_i(T_{p-1}^*)^n) &= E(UT_{p-1}^nT_iT_i(T_{p-1}^*)^n) = E(T_0^nT_{i+1}T_i(T_{p-1}^*)^n) \\ &= T_0^nT_{i+1}T_i(T_{p-1}^*)^n \ . \end{aligned}$$

The above computations show that $E(U) = U$ (since the relations amount to stating that U and $E(U)$ agree on the orthogonal complement of $\cap_{i\geq 1}T_{p-1}^i(T_{p-1}^*)^i$) and thus we are done. □

Parallel to the quasi-free action of $U(n)$ on $\mathcal{O}_n$, there has been some study of natural actions of compact quantum groups (such as $U_q(n)$, $SU_q(n)$) on $\mathcal{O}_n$, see e.g. Konishi-Nagisa-Watatani [137], Paolucci [150], Carey-Paolucci-Zhang [52], Gabriel [99] to mention but a few. At this stage, it is not clear if $\mathcal{Q}_n$ may be acted upon non-trivially by some of the quantum groups above. As a matter of fact, it is already an interesting problem to come up with explicit natural actions of classical Lie groups on $\mathcal{Q}_n$. A study of this sort would of course require a better grasp of what $\mathrm{Aut}(\mathcal{Q}_n)$ and $\mathrm{Out}(\mathcal{Q}_n)$ actually are.

Acknowledgements. V. A. is grateful to the Swiss National Science Foundation for the support. R. C. acknowledges partial support by Sapienza Università di Roma.

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Received: 19 November 2020/Accepted: 29 January 2021/Published online: 2 February 2021

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