Rend. Mat. Appl. (7). Volume 42 (2021), 197 – 213

# Large solutions of semilinear elliptic equations with a Hardy potential

### **Catherine Bandle**

In memory of my dear friend and long-time collaborator Assunta Pozio

**Abstract.** Semilinear elliptic problems with solutions blowing up at the boundary are considered. The effect of a Hardy potential with a boundary singularity is discussed. Positive potentials reinforce the solution to blow up whereas negative prevent it. For the standard nonlinearities and sufficiently large potentials there exist solutions which are comparable to the blowing up solutions of the problem without Hardy potential. Near the boundary they depend only on the distance to the boundary, and the first order approximation is independent of the geometry. The precise estimates imply that those solutions are unique. The main tools used in this paper are the method of upper and lower solutions and boundary estimates for the blowup solutions without Hardy potential.

# 1. Introduction

Let  $\Omega$  be bounded smooth domain in  $\mathbb{R}^n$  and denote by  $\delta(x)$  the distance from  $x \in \Omega$  to the boundary. We consider problems of the type

$$\begin{cases} \Delta u + \frac{\mu}{\delta^2(x)} u = f(u) & \text{in } \Omega, \\ u(x) \to \infty & \text{as } x \to \partial\Omega, \end{cases}$$
(1.1)

where  $f: \mathbb{R} \to \mathbb{R}^+$  is a continuous and increasing function. The expression  $\frac{\mu}{\delta^2(x)}u$  is called the *Hardy potential*.

It is well-known that the problem without Hardy potential

$$\begin{cases} \Delta U_P = f(U_P) & \text{in } \Omega, \\ U_P(x) \to \infty & \text{as } x \to \partial\Omega, \end{cases}$$
(1.2)

possesses a solution provided the nonlinearity satisfies the Keller–Osserman condition

$$\int^{\infty} \frac{1}{\sqrt{F(s)}} ds < \infty \quad \text{where} \quad F' = f.$$
 (F-1)

<sup>2010</sup> Mathematics Subject Classification: 35J60, 35J67, 31B25.

Keywords: Boundary blow-up, singular coefficients, singular solutions, sub and super solutions, comparison principle.

<sup>©</sup> The Author(s) 2020. This article is an open access publication.

The solutions of Problem (1.2) are called *large solutions* because they dominate all bounded solutions. The boundary behavior of large solutions has been the object of many studies, cf. for instance [1, 3, 2, 7, 11] and the references cited therein. It turns out that the blowup rate is given by

$$\frac{\psi(U_P(x))}{\delta(x)} \to 1 \quad \text{as} \quad \delta(x) \to 0, \tag{1.3}$$

where

$$\psi(t) = \int_t^\infty \frac{ds}{\sqrt{2F(s)}}$$

If in addition

$$\liminf_{t \to \infty} \frac{\psi(\beta t)}{\psi(t)} > 1, \quad \forall \beta \in (0, 1)$$
 (F-2)

holds, then

$$\lim_{x \to \partial \Omega} \frac{U_P(x)}{\phi(\delta(x))} = 1, \quad \text{uniformly on} \quad \partial \Omega, \tag{1.4}$$

where  $\phi = \psi^{-1}(\delta)$ . It solves the one-dimensional problem

$$\phi''(x) = f(\phi(x)), \qquad \phi(x) \to \infty \quad \text{as} \quad x \to 0.$$

The functions f for which (F-2) holds will be called *strongly superlinear* nonlinearities. Typical examples of strongly superlinear nonlinearities are  $e^t$  and  $t^p$ , p > 1.

Problem (1.2) with weakly superlinear nonlinearities for which (F-2) is not satisfied, like  $f(t) = t(\log t)^{\alpha}$ ,  $\alpha > 2$ , has been treated in [7]. A similar boundary behavior has been observed there too.

In Problem (1.1) there is near the boundary a competition between the nonlinearity f(t) and the Hardy potential  $\frac{\mu}{\delta^2}u$ . In [6] the following theorem has been derived:

Let  $\partial \Omega \in C^{2+\gamma}$  for some  $\gamma \in (0,1)$ ,  $f(t) = t^p$ , p > 1 and  $-c^* < \mu < 0$  where

$$c^* := \frac{2(p+1)}{(p-1)^2}.$$

For any  $c \in C^{2+\gamma}(\partial\Omega)$ ,  $c \geq 0$ , there exists a unique solution of (1.1) such that

$$\lim_{\delta(x)\to 0} \left( \frac{u(x)}{\delta(x)^{\beta_{-}}} - c(x^{*}) \right) = 0, \qquad (1.5)$$

where  $x^* \in \partial \Omega$  is the projection of x on the boundary.

This solution is governed by the Hardy potential.

The aim of this paper is to study solutions governed by the nonlinearity. For instance we prove for  $f(t) = t^p$  and for  $-c^* < \mu$  that Problem (1.1) has a solution for which

$$\lim_{x \to \partial \Omega} \frac{u(x)}{\delta^{-\frac{2}{p-1}}(x)} = (c^* + \mu)^{\frac{1}{p-1}}.$$

It is larger than than those mentioned above. For positive  $\mu$  this result was already obtained by Du and Wei [10] who studied the problems with nonlinearities of the type  $\frac{u^p}{\delta^{\sigma}}$ .

A negative Hardy potential can also prevent the existence of a solution to (1.1) for example if  $f(t) = e^t$  as it was shown in [5].

Our main interest is to prove the existence and the boundary behavior of solutions u to Problem (1.1) which are comparable to  $U_P$  in the sense that  $0 < aU_P \leq u \leq bU_P$ , a < b. A similar investigation has been carried out in [15] where mostly the case of positive  $\mu$  has been considered. The assumptions on the nonlinearities are different. However for special cases the results coincide with ours.

The paper is organized as follows. Section 2 contains the major tools used in connexion with boundary singularities. In Section 3 the existence of large solutions is proved for  $\mu > 0$  whereby the monotone dependence of the domain plays a crucial role. The existence in the case  $\mu < 0$  is treated in Section 4. It applies to a smaller class of nonlinearities. Section 5 is concerned with the asymptotic behavior leading to a short discussion of the uniqueness in Section 6.

### 2. Preliminaries

It is well-known that Problem (1.2) has a maximal and a minimal large solution which are ordered  $u_P \leq U_P$ .

The minimal large solution is obtained as  $\lim_{n\to\infty} u_n$  where  $u_n$  solves  $\Delta u_n = f(u_n)$  in  $\Omega$ ,  $u_n = n$  on  $\partial\Omega$ . For the construction of the maximal solution we consider a sequence  $\Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega$ , such that  $\bigcup_{1}^{\infty} \Omega_n = \Omega$ . Let  $U_n$  be a large solution in  $\Omega_n$ . In view of the Keller–Osserman a priori bound for large k the sequence  $\{U_n\}_{n>k}$  is uniformly bounded in any subset of  $\Omega$ . Then by compactness there is a subsequence which converges to the maximal solution U.

The uniqueness of large solutions has been studied in [3] and in [8]. In the last reference it was shown that in the ball there is a unique large solution if f is increasing and satisfies (F-1).

In connexion with Problem (1.1) the Hardy constant

$$C_H(\Omega) := \inf_{\mathcal{K}} \int_{\Omega} |\nabla v|^2 \, dx, \quad \mathcal{K} = \left\{ v \in W_0^{1,2}(\Omega) : \int_{\Omega} \delta^{-2}(x) v^2(x) \, dx = 1 \right\}$$

plays a crucial role. It turns out that for Lipschitz domains  $0 < C_H(\Omega) \le 1/4$ . For convex domains  $C_H(\Omega) = 1/4$  and for non-convex domains it can be arbitrarily small, cf. [9, 12].

As a consequence the problem

$$\Delta u + \frac{\mu}{\delta^2(x)}u = f(u), \quad u \in W_0^{1,2}(\Omega)$$
(2.1)

has no solution for  $\mu < C_H(\Omega)$ .

A function  $\overline{u} \in W^{1,2}(\Omega)$  is called a super solution if

$$\Delta \overline{u} + \frac{\mu}{\delta^2(x)} \overline{u} \le f(\overline{u}) \quad \text{in} \quad \Omega.$$

Similarly  $\underline{u}$  is a sub solution if the inequality sign is reversed.

The following comparison principle will be useful for our arguments.

COMPARISON PRINCIPLE. (i) Let  $\mu \leq C_H(\Omega)$  and let f(t) be increasing. If  $\underline{u}$  and  $\overline{u}$  are sub and super solutions in a domain  $\omega \subset \Omega$  such that  $\underline{u} \leq \overline{u}$  on  $\partial \omega$ , then  $\underline{u} \leq \overline{u}$  in  $\omega$ .

(ii) If  $\mu > C_H(\Omega)$ , then assertion (i) remains valid provided f(t) is increasing, convex with f(0) = 0 and  $\overline{u} > 0$ .

The proof of (i) is standard and follows by testing the inequality  $\Delta(\overline{u} - \underline{u}) + \frac{\mu}{\delta^2}(\overline{u} - \underline{u}) \leq f(\overline{u}) - f(\underline{u})$  with  $(\underline{u} - \overline{u})_+$ . The second assertion (ii) has been proved in [4] for power nonlinearities. Its extension to to convex functions is immediate. A related statement is found in [10]. The loss of positivity of the operator  $\Delta + \mu/\delta^2$  for large  $\mu$  is compensated by the nonlinearity.

The presence of the Hardy potential gives rise to a kind of threshold which is a special case of the Phragmen-Lindelöf alternative derived in [4]. It applies to positive subharmonic functions satisfying

$$\Delta h + \frac{\mu}{\delta^2(x)}h \ge 0$$
 in  $\{x \in \Omega : \delta(x) \le \delta_0\}$  for some positive  $\delta_0$ .

PHRAGMEN-LINDELÖF PRINCIPLE. Assume that  $\mu < 1/4$  and define

$$\beta_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}.$$

For any positive subharmonic function  $\underline{h}$  either of the following alternatives holds:

$$\limsup_{x \to \partial \Omega} \frac{\underline{h}}{\delta^{\beta_-}} > 0 \quad \text{or} \quad \limsup_{x \to \partial \Omega} \frac{\underline{h}}{\delta^{\beta_+}} < \infty.$$

For  $\mu < C_H(\Omega)$  this observation forces a solution of (1.1) to behave like

$$\limsup_{x \to \partial \Omega} \frac{u}{\delta^{\beta_-}} > 0.$$

Our existence proofs are based on the classical method of sub and super solutions which in this context has been used in [4] for power nonlinearities.

METHOD OF SUB AND SUPER SOLUTIONS. If there exist a sub and a super solution of (1.1) such that  $\underline{u}(x) \leq \overline{u}(x)$  in  $\Omega$ , then there exists a solution  $\underline{u} \leq u \leq \overline{u}$ .

# 3. Existence in the case $\mu > 0$

Throughout this section we assume (F - 1). We shall construct sub and super solutions by means of  $U_P$ . For the sub solution we need the assumption

$$U_P(x) > 0 \quad \text{in} \quad \Omega. \tag{3.1}$$

This is the case whenever f(0) = 0 or if the domain is not too large. If  $f(t) = e^t$ and  $U_P$  changes sign, we replace  $U_P$  by  $U_P + c$  and  $e^t$  by  $e^{-c}e^t$ .

For  $\mu > 0$ ,  $\underline{u} = U_P$  of (1.2) is a sub solution. We are looking for a super solution of the form  $\overline{u} = b(U_P + C)$ , where b > 1 and C > 0 will be determined later. It has to satisfy the inequality

$$\Delta b U_P + \frac{\mu}{\delta^2} b(U_P + C) \le f(b(U_P + C))$$
 in  $\Omega$ ,

or equivalently

$$\frac{\mu}{\delta^2} \le \frac{f(b(U_P + C))}{b(U_P + C)} - \frac{f(U_P)}{U_P + C} \quad \text{in} \quad \Omega.$$
(3.2)

Define

$$h(t) := \frac{f(t)}{t}$$

and assume that it satisfies the condition

h(t) is monotone increasing for  $t > t_0 \ge 0$ . (H-1)

The constant C is now chosen so large that  $U_P(x) + C > t_0$ .

Consider first strongly superlinear nonlinearities f satisfying condition (F-2). By (1.4) there exists for given  $\epsilon > 0$  a small number  $\delta_0 > 0$  such that

$$(1-\epsilon)\phi(\delta) \le U_P(x) \le (1+\epsilon)\phi(\delta)$$
 in  $\Omega_{\delta_0} := \{x \in \Omega : \operatorname{dist}\{x, \partial\Omega\} < \delta_0\}.$ 

Hence if there exist b > 1 such that

$$\frac{\mu}{\delta^2} \le h[b(1-\epsilon)\phi(\delta) + bC] - h[(1+\epsilon)\phi(\delta) + C] \quad \text{in} \quad \Omega_{\delta_0}, \tag{3.3}$$

then (3.2) holds in  $\Omega_{\delta_0}$  and  $b(U_P + C)$  is a super solution. We claim that this is the case when h satisfies the additional condition:

for any 
$$t \ge t_0$$
,  $\frac{h(bt)}{h(t)} = \gamma(b) \to \infty$  as  $b \to \infty$ . (H-2)

This can be seen as follows. Set  $\zeta = (1 + \epsilon)\phi + C$ . Then  $(1 - \epsilon)\phi + C \ge (\frac{1-\epsilon}{1+\epsilon})\zeta$ . By (H-1) and (H-2)

$$h[b((1-\epsilon)\phi+C)] - h(\zeta) \ge h\left(b \ \frac{1-\epsilon}{1+\epsilon}\zeta\right) - h(\zeta) \ge \left\{\gamma\left(b \ \frac{1-\epsilon}{1+\epsilon}\right) - 1\right\}h(\zeta), \quad (3.4)$$

From (1.4) we have

$$\delta = \int_{\phi}^{\infty} \frac{dt}{\sqrt{2F(t)}} > \int_{\phi}^{\zeta} \frac{dt}{\sqrt{2F(t)}}$$

and by (H-1)

$$2(F(t) - F(t_0)) = 2\int_{t_0}^t h(s)s \, ds \le h(t)(t^2 - t_0^2).$$

Choosing t sufficiently large we obtain  $2F(t) \leq h(t)t^2$ . Let  $\phi(t) \geq t_0$ , then

$$\delta > \frac{1}{\sqrt{h(\zeta)}} \int_{\phi}^{\zeta} \frac{ds}{s} = \frac{\log \zeta \phi^{-1}}{\sqrt{h(\zeta)}} \iff h(\zeta) \ge \frac{(\log(1 + \epsilon + C/\phi))^2}{\delta^2}.$$

The constant b in (3.4) can be chosen so large that

$$\left\{\gamma\left(b\;\frac{1-\epsilon}{1+\epsilon}\right)-1\right\}h(\zeta)\geq\frac{\mu}{\delta^2}$$

Then (3.3) is satisfied and  $b(U_P(x) + C)$  is a super solution in  $\Omega_{\delta_0}$ . By possibly increasing b we can achieve that  $\overline{u} = b(U_P + C)$  is a super solution in the whole domain. Since  $\overline{u} \geq \underline{u} = U_P$ , the method of sub and super solutions leads to

**Theorem 3.1.** Let  $\mu > 0$  and assume (F-1)-(F-2),(H-1),(H-2) and (3.1). Then Problem (1.1) possesses a solution such that for b sufficiently large

$$U_P(x) \le u \le b(U_P + C).$$

We now consider the weakly superlinear nonlinearity

$$f(t) = t(\log t)^{\alpha}, \quad \alpha > 2.$$
(3.5)

The Keller-Osserman condition (F-1) is satisfied. However (F-2) does not hold. In [7] it was shown that the large solution of (1.2) satisfies

$$\lim_{\delta \to 0} \frac{U_P(x)}{\exp\left(\frac{\sigma}{\delta}\right)^{\sigma}} = e^{1/2},$$

where  $\sigma = \frac{2}{\alpha - 2}$ . Since  $U_P$  is positive we can use it as a sub solution. As before we look for a super solution of the form  $\overline{u} = bU_P$ . Write for short

$$v := \exp\left[\left(\frac{\sigma}{\delta}\right)^{\sigma} + \frac{1}{2}\right] = \exp A(\delta).$$

Then  $U_P$  satisfies in  $\Omega_{\delta_0}$ ,

$$(1-\epsilon)v \le U_p \le (1+\epsilon)v.$$

For small  $\delta \leq \delta_0$ 

$$h(bU_P) - h(U_P) \ge (\log b(1-\epsilon)v)^{\alpha} - (\log(1+\epsilon)v)^{\alpha}$$
$$= \{\log b(1-\epsilon) + A(\delta)\}^{\alpha} - \{\log(1+\epsilon) + A(\delta)\}^{\alpha}$$

Since  $\alpha > 2$ , we have  $-\sigma(\alpha - 1) = -2 - \sigma < -2$  and therefore

$$h(bU_P) - h(U_P) \ge g(b)\delta^{-\sigma(\alpha-1)} \ge g(b)\delta^{-2}$$
 in  $\Omega_{\delta_0}$ .

Since h(t) is monotone, (3.2) holds in  $\Omega$  for sufficiently large b,  $bU_P$  is therefore a super solution. Moreover by the positivity of  $U_P$ ,  $bU_P \ge U_P$ . This proves

**Corollary 3.2.** Assume (F-1) and (3.5). Then Problem (1.1) has a solution  $U_P \leq u < bU_P$  for b sufficiently large.

**Remark 3.3.** The corollary is valid for any nonlinearity f which is asymptotically equal to  $t(\log t)^{\alpha}$  in the sense that

$$\lim_{t \to \infty} \frac{t(\log t)^{\alpha}}{f(t)} = 1$$

This is due to the fact that the large solutions  $U_P$  have the same boundary behavior.

### 4. Existence in the case $\mu < 0$ .

Let  $U_P$  be the maximal solution of (1.2). If  $U_P$  is negative in a subdomain of  $\Omega$ , we add a constant C such that  $U_P + C > 0$ . Then  $\overline{u} = U_P$  is a super solution of (1.1). The comparison principle implies that any solution u of (1.1) satisfies  $u \leq \overline{u}$ . On the other hand it follows from Phragmen-Lindelöf's principle that

$$\limsup_{x \to \partial \Omega} \frac{u}{\delta^{\beta_-}(x)} > 0.$$

This observation leads to some nonexistence results.

**Example 4.1.** Let  $f(t) = e^t$ . Near the boundary the solutions of (1.2) behave like  $(1+o(1)) \log \frac{2}{\delta^2}$ . Obviously  $\limsup_{x \to \partial \Omega} \frac{u}{\delta^{\beta-}(x)} = 0$ . Therefore (1.1) has no solution if  $\mu < 0$ . This was already observed in [5].

**Example 4.2.** Let  $f(t) = t^p$ . Then

$$\phi(\delta) = (c^*)^{\frac{1}{p-1}} \delta^{-\frac{2}{p-1}}$$
 where  $c^* = \frac{2(p+1)}{(p-1)^2}$ .

If a solution of (1.1) exists, we must have

$$\lim_{\delta \to 0} \delta^{-\frac{2}{p-1} - \frac{1}{2} + \sqrt{\frac{1}{4} - \mu}} > 0.$$

This implies that no solution exists if

$$\sqrt{\frac{1}{4} - \mu} > \frac{2}{p-1} - \frac{1}{2}$$

or, equivalently,  $\mu < -c^*$ .

Next we prove the existence of a large solution for power nonlinearities

$$f(t) = t^p \quad \text{if} \quad \mu > -c^*.$$

We choose  $\overline{u} = U_P$  as a super solution and  $\underline{u} = \max\{b(U_P - B), 0\}$  where b < 1and B is so large that the support of  $\underline{u}$  is in  $\Omega_{\delta_0}$ . In  $\Omega_{\delta_0} \cap \{\underline{u} > 0\}$  there holds

$$\frac{\Delta \underline{u}}{\underline{u}} + \frac{\mu}{\delta^2} - \underline{u}^{p-1} = \frac{U_P^p}{U_P - B} + \frac{\mu}{\delta^2} - b^{p-1} (U_P - B)^{p-1}$$
$$\geq U_P^{p-1} (1 - b^{p-1}) + \frac{\mu}{\delta^2}.$$

From (1.4) and Example 4.2 it follows that

$$(1-\epsilon)(c^*)^{\frac{1}{p-1}}\delta^{-\frac{2}{p-1}} \le U_P \quad \text{in} \quad \Omega_{\delta_0}.$$

Hence

$$U_P^{p-1}(1-b^{p-1}) \ge (1-\epsilon)^{p-1}c^*(1-b^{p-1})\delta^{-2}$$

Since by assumption  $c^* > -\mu$  we can find  $\epsilon$  and b sufficiently small such that  $U_P^{p-1}(1-b^{p-1})+\frac{\mu}{\delta^2}>0$ . Notice that if we choose  $\epsilon$  small, then  $\delta_0$  has to be small too. Hence  $\underline{u} = \max\{b(U_P - C), 0\}$  is a sub-solution in  $\Omega$  and consequently there exists a solution of (1.1) such that  $\underline{u} \leq u \leq \overline{u}$ .

Consider now more general function satisfying in addition to (F-1)

(i) 
$$f(t) \le t^p$$
 where  $\mu > -c^*$ ,  
(ii)  $f(t) \ge t^p$  where  $\mu < -c^*$ .  
(F-3)

**Theorem 4.3.** Assume  $\mu < 0$  and (F-1).

*i.* If in addition f satisfies (F-3)(i), then Problem (1.1) has a solution which near the boundary behaves like

$$b\,\delta^{-\frac{2}{p-1}} \le u(x) \le U_P(x)$$
 in  $\Omega_{\delta_0}$ ,

for some 0 < b < 1.

ii. If f satisfies (F-2) and (F-3)(ii), then (1.1) has no solution.

*Proof.* i. The sub solution constructed above for power nonlinearities is also a sub solution for the nonlinearity f(t) since

$$\Delta \underline{u} + \frac{\mu}{\delta^2} \underline{u} \ge \underline{u}^p \ge f(\underline{u}).$$

From the maximum principle it follows that  $U_P \ge \underline{u}$  where  $U_P$  is a solution of (1.2) with the nonlinearity f. The method of sub and super solutions applies and provides a solution of (1.1).

ii. Suppose that (1.1) has a solution. Let  $U_P$  be the maximal large solution of (1.2). From the comparison principle it follows that  $u \leq U_P$ . Near the boundary (1.4) implies that  $u \leq (1 + \epsilon)\phi$ . By (F-3)(ii),  $\phi(\delta) \leq c\delta^{-2/(p-1)}$ . This is incompatible with the Phragmen-Lindelöf principle. Hence (1.1) has no solution.

Let us now look at the problem with the weakly superlinear nonlinearity

$$f(t) = t(\log t)^{\alpha}$$
 with  $\alpha > 2$ .

The same type of arguments with the necessary modifications reveal that for sufficiently small  $b \ll 1$ ,  $b U_P$  is a sub solution. Hence we have

**Corollary 4.4.** Problem (1.1) has for  $f(t) = t(\log t)^{\alpha}$  and any  $\mu < 0$  a large solution of the type  $bU_P < u < U_P$  with small b.

**Problem 1.** Describe the most general class of nonlinearities which admit solutions behaving like

$$c_1 U_P(x) < u(x) < c_2 U_P(x)$$

near the boundary.

# 5. Asymptotic bevavior

#### 5.1. Ball

Let  $U_P(r)$  be a large radial solution of (1.2) in the ball  $B_R := \{x : |x| < R\}$  and let u(r) be a solution of (1.1) in  $B_R$ . It solves the ODE

$$u_{rr} + \frac{n-1}{r}u_r + \frac{\mu}{(R-r)^2}u = f(u) \text{ in } (0,R),$$
  
$$u_r(0) = 0, \quad \lim_{r \to R} u(r) = \infty.$$

Since  $U_P$  is positive in the neighborhood of the boundary we can write

$$u(r) = U_P(r)w(r)$$
 in  $(R - \epsilon, R)$ 

Then w satisfies in  $(R - \epsilon, R)$ 

$$w_{rr} + \frac{n-1}{r}w_r + 2\frac{(U_P)_r}{U_P}w_r = w\left\{h(wU_P) - h(U_P) - \frac{\mu}{(R-r)^2}\right\},$$

or equivalently

$$(w_r r^{n-1} U_P^2)_r = r^{n-1} U_P^2 wq(r)$$
 where  $q(r) = h(w U_P) - h(U_P) - \frac{\mu}{(R-r)^2}$ 

Integration from  $r_0 \in (R - \epsilon, R)$  to R yields

$$w(r) = w(r_0) + c(r_0) \int_{r_0}^r \frac{ds}{s^{n-1}U_P^2} + \int_{r_0}^r s^{n-1}U_P^2 wq \, ds \int_s^r \frac{d\xi}{U_P^2 \xi^{n-1}}$$

The change of variables  $s = R - \tilde{\delta}$  and  $\tilde{\xi} = R - \xi$  together with (1.4) implies that for strongly superlinear nonlinearities

$$w(R-\delta) = w(r_0) + w_1(\delta) + (1+\eta(\delta)) \int_{\delta}^{\delta_0} \phi^2(\tilde{\delta}) w(R-\tilde{\delta}) q(R-\tilde{\delta}) d\tilde{\delta} \int_{\delta}^{\tilde{\delta}} \frac{d\tilde{\xi}}{\phi^2(\tilde{\xi})}, \qquad (5.1)$$

where  $w_1$  and  $\eta$  are bounded and continuous functions in  $(0, \delta_0)$ . This observation leads to the following lemma which is the basic tool for our asymptotic estimates.

**Lemma 5.1.** Assume (F-1), (F-2) and  $\Omega = B_R$ . Let  $U_P$  and u be a solutions of (1.2) and (1.1), respectively. Suppose that

$$\exists \quad 0 < a < b \quad such \ that \quad aU_P(r) \le u(r) \le bU_P(r). \tag{5.2}$$

Then the integral

$$\int_0^{\delta_0} \phi^2(\tilde{\delta}) w \left\{ h(w(1+O(1))\phi) - h((1+O(1))\phi) - \frac{\mu}{\tilde{\delta}^2} \right\} d\tilde{\delta} \int_0^{\tilde{\delta}} \frac{d\tilde{\xi}}{\phi^2(\tilde{\xi})}$$

exists.

Since a < w < b the lemma holds only if the function

$$\mathcal{E}(\delta) := \left\{ h(w(1+O(1))\phi) - h((1+O(1))\phi) - \frac{\mu}{\delta^2} \right\} \phi^2(\delta) \int_0^\delta \frac{d\xi}{\phi^2(\xi)}$$

is integrable at 0.

Next we apply Lemma 5.1 to some special cases in order to obtain the asymptotic behavior of u(r) near the boundary.

#### 5.1.1. Special cases.

1.  $f(t) = t^p, \mu > -c^*$ . Then by Theorem 3.1 and Theorem 4.3 the condition (5.2) is satisfied.

$$\phi(\delta) = \left(\frac{c^*}{\delta^2}\right)^{1/(p-1)}, \qquad \phi^2(\delta) \int_0^\delta \frac{d\xi}{\phi^2(\xi)} = \frac{p-1}{p+3}\delta,$$
$$h(w(1+O(1))\phi) - h((1+O(1))\phi) = \frac{c^*}{\delta^2} [w^{p-1} - 1 + O(1)].$$

For  $\mathcal{E}(\delta)$  to be integrable at 0 we must have

$$\lim_{r \to R} w(r) = \left(\frac{c^* + \mu}{c^*}\right)^{\frac{1}{p-1}}$$

This implies

$$\lim_{r \to R} u(r)(R-r)^{2/(p-1)} = (c^* + \mu)^{1/(p-1)}.$$
(5.3)

For positive  $\mu$  this result is already contained in [10].

2.  $f(t) = e^t$  and  $\mu > 0$ .

Then by Theorem 3.1 the assumptions of Lemma 5.1 are satisfied. In this case  $\phi(\delta) = \ln \frac{2}{\delta^2}$ . We assume that  $\phi(\delta)$  is positive and therefore  $R < \sqrt{2}$ . Then

$$\phi^2(\delta) \int_0^\delta \frac{ds}{(\log\frac{2}{s^2})^2} = \frac{\phi^2}{\sqrt{2}} \int_\phi^\infty \frac{dt}{t^2 e^{t/2}} = \frac{\phi}{\sqrt{2}e^{\phi/2}} - \frac{\phi^2}{2^{3/2}} \int_\phi^\infty \frac{dt}{te^{t/2}} + \frac{\phi^2}{2^{3/2}} + \frac{\phi^2}{2^{3/2}} \int_\phi^\infty \frac{dt}{te^{t/2}} + \frac{\phi^2}{2^{3/2}} + \frac{\phi^2}{2^$$

The exponential integral is bounded from above by

$$E_1(\phi/2) := \int_{\phi}^{\infty} \frac{dt}{te^{t/2}} = e^{-\phi/2} \log\left(1 + \frac{2}{\phi}\right)$$
$$< e^{-\phi/2} \left(\frac{2}{\phi} - \frac{1}{2}\left(\frac{2}{\phi}\right)^2 + \frac{1}{3}\left(\frac{2}{\phi}\right)^3\right).$$

Hence

$$\phi^2(\delta) \int_0^\delta \frac{ds}{(\log\frac{2}{s^2})^2} \ge \frac{\delta}{2} + o(\delta).$$
(5.4)

In [3] it was shown that  $U_P(\delta) = \log \frac{2}{\delta^2} (1 + \frac{n-1}{R}\delta + o(\delta))$  for  $\delta \to 0$ . Hence

$$h(wU_P) = \frac{\left(\frac{2}{\delta^2}\right)^{w(1+\omega)}}{w(1+\omega)\log\frac{2}{\delta^2}} \quad \text{where} \quad \omega = \frac{n-1}{R}\delta + o(\delta),$$

and

$$q(\delta) = \frac{2}{(1+\omega)\log\frac{2}{\delta^2}} \left\{ w^{-1} \left(\frac{2}{\delta^2}\right)^{w(1+\omega)} - \left(\frac{2}{\delta^2}\right)^{1+\omega} \right\} - \frac{\mu}{\delta^2}.$$

In view of (5.4) we have  $\mathcal{E}(\delta) > \frac{\delta}{2}q(\delta)$ . The integrability of  $\mathcal{E}(\delta)$  requires that  $\lim_{\delta \to 0} q(\delta) = 0$ . Set  $w = 1 + \eta(\delta)$  and  $y = \frac{2}{\delta^2}$ . Then

$$q(\delta) = \frac{2y^{1+\omega}}{(1+\omega)\log y} \left\{ \frac{y^{\eta(1+\omega)}}{1+\eta} - 1 \right\} - \frac{\mu}{2}y + o(\delta),$$

If  $\lim_{\delta \to 0} q(\delta) = 0$ , then

$$\eta(\delta) = \frac{\log\left(\frac{\mu}{4}\log\frac{2}{\delta^2} + 1\right)}{\log\frac{2}{\delta^2}} + \text{lower order terms.}$$

Consequently  $\lim_{\delta \to 0} \eta(\delta) = 0$  and  $\lim_{r \to R} w(r) = 1$ . Hence

$$\lim_{r \to R} \frac{u(r)}{\log \frac{2}{(R-r)^2}} = 1.$$
(5.5)

This result was already derived in [5] by a different method.

3.  $f(t) = t(\log t)^{\alpha}, \alpha > 2$ . By Corollary 4.4, Lemma 5.1 applies. In this case

$$U_P(\delta) = (1 + \mathrm{o}(1))\mathrm{e}^{(\frac{\sigma}{\delta})^{\sigma}}\mathrm{e}^{1/2}, \qquad \sigma = \frac{2}{\alpha - 2}$$

and

$$U_P^2(\delta) \int_0^\delta U_P^{-2}(\xi) d\xi \approx \text{cons. } \delta^{1+\sigma}.$$

Thus

$$\mathcal{E}(\delta) \approx c \left[ \left( \log w + \frac{1}{2} + \left(\frac{\sigma}{\delta}\right)^{\sigma} \right)^{\alpha} - \left(\frac{1}{2} + \left(\frac{\sigma}{\delta}\right)^{\sigma} \right)^{\alpha} - \frac{\mu}{\delta^2} \right] \delta^{1+\sigma}$$

The leading term is  $c\left(\frac{\sigma}{\delta}\right)^{\sigma(\alpha-1)}\log w$ . Since  $-\sigma(\alpha-1)+1+\sigma=-1$ ,  $\mathcal{E}(\delta)$  is integrable if  $\lim_{\delta\to 0}\log w=0$  which implies that  $\lim_{\delta\to 0}w(\delta)=1$  and

$$\lim_{r \to R} \frac{u(r)}{U_P(r)} = 1.$$
 (5.6)

#### 5.2. Annulus

In the annulus,  $A(R_0, R_1) := \{x : R_0 < |x| < R_1\}$  we consider the problem

$$\Delta \hat{u}(r) + \frac{\mu}{(r - R_0)^2} \hat{u}(r) = f(\hat{u}) \quad \text{in} \quad (R_0, R_1), \hat{u}(R_1) = 0, \ \hat{u}(r) \to \infty \quad \text{as} \quad r \to R_0.$$
(5.7)

According to [3] this problem without a Hardy potential has a solution  $\hat{U}_0(r)$  provided the Keller-Osserman condition (F-1) holds. Denote by  $\delta = r - R_0$  the distance to the inner boundary. It is well-known that under the assumption (F-2),

$$\lim_{\delta \to 0} \frac{\hat{U}_0(R_0 + \delta)}{\phi(\delta)} = 1.$$

The discussion for the solution of (1.1) in the ball can be transferred literally to  $\hat{u}(r)$  in the annulus. We therefore omit the details.

In short under the same conditions as for the ball there exists a solution such that  $0 < a\hat{U}_0 < \hat{u}(r) < b\hat{U}_0(r)$ . If (1.1) possesses in  $B_R$  a solution such that  $u(r) = U_P(r)w(r)$  with  $\lim_{r \to R} w(r) = w_0$ . then (5.7) has a solution of the form  $\hat{u}(r) = \hat{U}_0(r)\hat{w}(r)$  with  $\lim_{r \to R} \hat{w}(r) = w_0$ 

#### 5.3. General domains.

Throughout this section we assume that  $\Omega$  is a bounded domain satisfying an inner and outer sphere condition. In addition we require that  $\partial \Omega \in C^2$  and that the mean curvature H is well-defined.



Figure 1

**Theorem 5.2.** Let u(x) be a solution of (1.1) such that

$$0 < a U_P(x) \le u(x) \le b U_P(x) \quad for \ some \quad 0 < a < b.$$

If  $\mu < 0$ , we require in addition that f(0) = 0. Suppose that there is a large solution  $u_B(r)$  in the ball such that

$$\lim_{r \to R} \frac{u_B(r)}{\phi(R-r)} = w_0,$$

where  $w_0$  is independent of R. Then

$$u(x) = w_0 \phi(\delta(x)) (1 + o(1))$$
 as  $x \to \partial \Omega$ .

*Proof.* We shall distinguish between two cases:  $\mu > 0$  and  $\mu < 0$ .

(i)  $\mu > 0$ .

This case has already be treated in [15]. We present the proof for the sake of completeness. Let  $\underline{x}$  be an arbitrary point on the boundary  $\partial\Omega$  and let  $B_R \subset \Omega$ 

be a ball such that  $\underline{x} \in \partial B_R$  (see Figure 1). We choose R so small that such a ball can be inscribed at every boundary point  $\overline{x}$ . Denote by  $\delta_B$  the distance of  $x \in B_R$ to  $\partial B_R$ . Then since  $\delta_B \leq \delta$  and  $\mu > 0$ 

$$0 = \Delta u_B + \frac{\mu}{\delta_B^2} u_B - f(u_B) \ge \Delta u_B + \frac{\mu}{\delta^2} u_B - f(u_B) \quad \text{in} \quad B_R \cap \Omega, \tag{5.8}$$

where  $u_B$  is the solution of (1.1) in  $B_R$ . By shifting the ball slightly inside  $\Omega$  and by the comparison principle it follows that  $u \leq u_B$  in  $B_R$ .

Consider now an annulus  $A(R_0, R_1)$  centered at the origin such that  $\underline{x} \in \{|x| = R_0\}$  and  $B_{R_0} \cap \Omega = \emptyset$ . Moreover we assume that  $A(R_0, R_1)$  contains  $B_{R_0}$  (see Figure 1). The radii are chosen such that this holds for every  $\overline{x} \in \partial \Omega$ . Let  $u_A(r)$  be a solution of (5.7). Let  $\delta_A(x) = \text{dist}\{x, \partial B_{R_0}\}$ . Then for  $x \in \Omega \cap A(R_0, R_1)$  we have  $\delta_A \geq \delta$  and therefore

$$0 = \Delta u_A + \frac{\mu}{\delta_A^2} u_A - f(u_A) \le \Delta u_A + \frac{\mu}{\delta^2} u_A - f(u_A) \quad \text{in} \quad \Omega \cap A(R_0, R_1).$$
(5.9)

The functions  $u_A$  is a lower solution of (1.1). Similarly by shifting the inner boundary of  $A(R_0, R_1)$  slightly away from  $\partial\Omega$  and keeping in mind that  $u(x) \to \infty$ as  $x \to \partial\Omega$  we get  $u \ge u_A$  in  $A(R_0, R_1) \cap \Omega$ .

The claim is a consequence of  $u_A \leq u$  in  $A(R_0, R_1) \cap \Omega$  and  $u \leq u_B$  in  $B_R$  and the fact that  $u_B(r) = w_0 \phi(R-r)(1+o(1))$  for  $r \to R$  and  $u_A(r) = w_0 \phi(r-R_0)(1+o(1))$  for  $r \to R_0$ .

(ii)  $\mu < 0$ .

In this case  $u_B$  and  $u_A$  cannot serve as local super and sub solutions. We follow a device used in [6].

We start with the upper solution. Set  $\tilde{u}(\delta) := u_B(R - \delta)$ . It satisfies

$$\begin{split} \tilde{u}_{\delta\delta} &- \frac{(n-1)}{R-\delta} \tilde{u}_{\delta} + \frac{\mu}{\delta^2} \tilde{u} = f(\tilde{u}) \quad \text{in} \quad (0,R) \\ \lim_{\delta \to 0} \tilde{u}(\delta) &= \infty, \quad \tilde{u}_{\delta}(R) = 0. \end{split}$$

Choose R so small that  $\overline{u}(x) := \tilde{u}(\delta(x))$  is twice differentiable in the parallel set  $\Omega_R := \{x \in \Omega : \delta(x) < R\}$ . Then

$$\Delta \overline{u} = \tilde{u}_{\delta\delta} + \tilde{u}_{\delta} \Delta \delta$$

and

$$\Delta\delta(x) = H(\overline{x})(n-1) + O(\delta),$$

where H is the mean curvature of  $\partial \Omega$  at the projection <u>x</u> of x. Thus

$$\Delta \overline{u} + \frac{\mu}{\delta^2} \overline{u} - f(\overline{u}) = \overline{u}_{\delta} \left( H(n-1) + O(\delta) + \frac{n-1}{R-\delta} \right) \quad \text{in} \quad \Omega_R.$$

If R is sufficiently small, the expression in the bracket is positive. Since  $\overline{u}_{\delta} < 0$  it follows that  $\overline{u}$  is an upper solution in  $\Omega_R$ . Any positive constant is a super solution. Since  $\overline{u}_{\delta}(R) = 0$ 

$$\overline{u}_{\Omega}(x) = \begin{cases} \overline{u} & \text{in } \Omega_R, \\ \overline{u}(R) & \text{in } \Omega \setminus \Omega_R \end{cases}$$

is a weak super solution in  $\Omega$ . By the assumption  $u \leq b U_P$  we can choose R so small that  $u_B(0) > u(x)$  somewhere. Hence the comparison principle shows that  $\overline{u} \geq u(x)$  in  $\Omega_R$ .

In order to construct a lower solution we proceed as follows. Consider a solution  $u_A(r)$  of (5.7) with  $R_1 = R_0 + R$ . Set  $\delta = r - R_0$  and  $\tilde{u}(\delta) = \hat{U}(R_0 + \delta)$ . For small R the function  $\underline{u}(x) = \tilde{u}(R_0 + \delta(x))$  satisfies

$$\Delta \underline{u} + \frac{\mu}{\delta^2} \underline{u} - f(\underline{u}) = \underline{u}_{\delta}((n-1)H - \frac{n-1}{R_0 + \delta} + O(\delta)) \quad \text{in} \quad \Omega_R.$$

Let  $R_0$  and R be so small that the expression in the bracket is negative in (0, R). Since  $\underline{u}_{\delta}$  is negative,  $\underline{u}$  is a lower solution in  $\Omega_{\rho_0}$ . Since f(0) = 0,  $\underline{u}$  can be extended to  $\Omega$  as a sub solution as follows

$$\underline{u}_{\Omega} = \begin{cases} \underline{u} & \text{in } \Omega_R, \\ 0 & \text{in } \Omega \setminus \Omega_R, \end{cases}$$

By the standard arguments varying slightly  $\delta$  it follows that  $\underline{u} \leq u \leq \overline{u}$  The conclusion now follows from the fact that the first order asymptotic behavior of the large solutions in balls and annuli are the same.

#### 6. Uniqueness

The precise asymptotic behavior of the large solutions gives rise to uniqueness results. The poof is standard and has often been used in the context of problems with boundary blow up. A different approach is found in [15] where similar results have been obtained for  $\mu > 0$ .

The uniqueness of the solutions of Problem (1.2) has been studied by various authors, s. for instance [13, 1, 8] and the references cited therein. It turns out that for general bounded domains  $\Omega$  with smooth boundary Problem (1.2) has a unique large solution for the nonlinearities

 $t^p$  (p > 1),  $e^t$  and  $t(\log t)^{\alpha}$   $(\alpha > 2)$ .

Less is known for (1.1). Results for power nonlinearities have been obtained in [10] The case of  $e^t$  has been treated in [5] and a different class has been considered in [15].

**Theorem 6.1.** If h(t) is monotone, then (1.1) has at most one positive solution for which  $\lim_{x\to\partial\Omega} \frac{u(x)}{U_P(x)} = w_0$ . *Proof.* Let  $u_1$  and  $u_2$  be two different solutions. Since  $u_2$  is positive we can set  $u_1 = vu_2$ . Then

$$\Delta v + 2\left(\nabla v \cdot \frac{\nabla u_2}{u_2}\right) + \frac{v}{u_2}\Delta u_2 + \frac{\mu}{\delta^2}v = h(vu_2)v.$$

Hence

$$\Delta v + 2\left(\nabla v \cdot \frac{\nabla u_2}{u_2}\right) = \left\{h(u_2 v) - h(u_2)\right\}v.$$

Suppose that  $u_1 < u_2$  and therefore 0 < v < 1 in  $\Omega^-$ . By our assumption v = 1 on  $\partial\Omega$  and therefore  $\Omega^- \subset \Omega$ . In  $\Omega^- v$  satisfies  $\Delta v + 2\left(\nabla v \cdot \frac{\nabla u_2}{u_2}\right) < 0$  and on  $\partial\Omega^-$  we have v = 1. By the maximum principle v > 1 in  $\Omega^-$ . This is a contradiction and consequently  $u_2 \ge u_1$  in  $\Omega$ . By interchanging the role of  $u_1$  and  $u_2$  it follows that  $u_1 \ge u_2$  which establishes the assertion.

We end with an overview of the solutions for Problem (1.1) for power nonlinearities  $f(t) = t^p$ , p > 1. Define

$$u \sim g(\delta) \iff \frac{1}{c} < \frac{u(x)}{g(\delta(x))} < c, c > 1 \text{for} \delta \to 0$$

We introduce the notion of small, moderate, intermediate, large and very large solutions as follows

s-solution	$u \sim \mathrm{o}(\delta^{\beta_{-}}),$
m-solution	$u\sim\delta^{eta},$
i-solution	$u=\mathrm{o}(\delta^{-2/(\mathrm{p}-1)})\text{and}\frac{\mathrm{u}(\mathbf{x})}{\delta^{\beta}}\to\infty\text{as}\delta\to0,$
L-solution	$u \sim \delta^{-2/(p-1)},$
vL-solution	$\frac{u}{\delta^{-2/(p-1)}} \to \infty  \text{as}  \delta \to 0.$

Notice that for  $\mu > 0$ , s- and m-solutions don't make sense.

	s-sol.	m-sol.	i-sol.	L-sol-	vL-sol.
$-c^* < \mu < 0$	0	$\infty$	?	1	0
$0 < \mu$	-	-	?	1	0

For radial solutions in the ball the situation is as follows.

	s-sol.	m-sol.	i-sol.	L-sol-	vL-sol.
$-c^* < \mu < 0$	0	$\infty$	0	1	0
$0 < \mu$	-	-	0	1	0

## References

- Bandle C., Marcus M.: Large solutions of semilinear elliptic equations: existence, uniqueness and asymptotic behaviour, J. Anal. Math. 10, 9–24 (1992)
- [2] Bandle C., Marcus M.: Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup at the boundary, Ann. Henri Poincaré 12, 155–171 (1995)
- [3] Bandle C., Marcus M.: On second order effects in the boundary behaviour of large solutions of semilinear elliptic problems, Differ. Integral Equ. 11, 23–34 (1998)
- [4] Bandle C., Moroz V., Reichel W.: 'Boundary blowup' type sub-solutions to semilinear elliptic equations with Hardy potential, J. London Math. Soc. 77, 503–523 (2008)
- [5] Bandle C., Moroz V., Reichel W.: Large solutions to semilinear elliptic equations with Hardy potential and exponential nonlinearity, In: Around the Research of Vladimir Maz'ya, II. Partial Differential Equations, pp. 1–22, Springer (2010).
- [6] Bandle C., Pozio M.A.: Posiive solutions of semilinear elliptic problems with a Hardy potential, P. Roy. Soc. Edinb. A 150, 1791–1804 (2020)
- [7] Cirstea F. C., Du Y.: Large solutions of elliptic equations with a weakly superlinear nonlinearity, J. Anal. Math. 103, 261–277 (2007)
- [8] Costin O., Dupaigne L., Goubet O.: Uniqueness of large solutions, J. Math. Anal. Appl. 395 806–8012 (2012)
- [9] Davies E. B.: The Hardy constant, Quart. J. Math. Oxford Ser. (2) 46, 417-431 (1995)
- [10] Y. Du, L. Wei: Boundary behavior of positive solutions to nonlinear elliptic equations with Hardy potential, J. London Math. Soc. (2) 91 (2015), 731-749.
- [11] Ghergu M., Rădulescu V.D.: Nonlinear PDEs. Mathematical models in Biology, Chemistry and Population Genetics. Springer Verlag Berlin Heidelberg, 2012.
- [13] Marcus M., Véron L.: Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations, Ann. Henri Poincaré 14, 237–274 (1997)
- [14] Marcus M., Nguyen P.-T.: Moderate solutions of semilinear elliptic equations with Hardy potential, Ann. Henri Poincaré 34, 69–88 (2014)
- [15] Marcus M.: Large solutions of semilinear equations with Hardy potential, arXiv:2006.02993 (2020)

Received: 10 September 2020/Accepted: 19 December 2020/Published online: 22 February 2021

Departement Mathematik und Informatik, Universität Basel, Spiegelgasse 1, CH-4051 Basel, Switzerland

c.bandle@gmx.ch

**Open Access.** This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.