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Existence of a $W_0^{1,1}$-solution for a semilinear Dirichlet problem with very singular convection term

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To Assunta (remembering the sharing "studio 107" years)

Abstract. *In this paper we prove the existence in* $\mathbf{W_0^{1,1}}$ *(not in BV) of a solution (in a very weak sense) for the boundary value problem* (1.1).

1. Introduction

In this paper we prove the existence in $\mathbf{W_0^{1,1}}$ of a solution (in a very weak sense) for the semilinear boundary value problem

$$\begin{cases} -\operatorname{div}(M(x)\nabla u) + A\,u|u| = -\operatorname{div}(u\,E(x)) + f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{1.1}$$

Here we assume that

$$f \in L^1(\Omega), \quad A \in \mathbb{R}^+ \tag{1.2}$$

and, on the singular convection term, that

$$E \in \left(L^2(\Omega)\right)^N, \tag{1.3}$$

where Ω is a bounded, open subset of $\mathbb{R}^N$, $N > 2$, and $M : \Omega \to \mathbb{R}^{N^2}$ is a matrix such that (for α, $\beta \in R^+$)

$$\alpha|\xi|^2 \leq M(x)\xi\xi, \quad |M(x)| \leq \beta. \tag{1.4}$$

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$$u \in W_0^{1,q}(\Omega) : \ -\operatorname{div}(M(x)\nabla u) + A\,u|u|^{\lambda-1} = -\operatorname{div}(u\,E(x)) + f(x),$$

with $f \in L^m(\Omega)$, $m > 1$, $E \in \left(L^r(\Omega)\right)^N$, $r > 2$, $\lambda > 1$ is studied in [8].

We point out that our assumptions (1.3), (1.2) on E, f are the minimal possible, so that even the existence of solutions in the sense of distributions can be lost

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and a very weak formulation (entropy solutions introduced in [7], following [5]) is needed.

We draw attention to the fact that we will prove the existence of a solution only belonging to the (nonreflexive) space $W_0^{1,1}(\Omega)\cap L^2(\Omega)$.

Nonlinear elliptic boundary value problems with $W_0^{1,1}(\Omega)$ solutions are studied in several papers; only we recall [13] and [12], where a semilinear problem with a $W_0^{1,1}(\Omega)\cap L^2(\Omega)$ solution is studied, with assumptions on the principal part of the operator similar to the ones of the paper [1] by Porzio–Pozio (see also [11]). Moreover radial examples show that the results of [13] and [12] are optimal.

2. Existence

2.1. Setting

We consider the following approximate Dirichlet problems

$$u_n \in W_0^{1,2}(\Omega):\ -\mathrm{div}(M(x)\nabla u_n) + A\, u_n|u_n| = -\mathrm{div}\left(\frac{u_n\, E_n}{1+\frac{1}{n}|u_n|}\right) + f_n, \quad (2.1)$$

where

$$E_n(x) = \frac{E(x)}{1+\frac{1}{n}|E|}, \quad f_n(x) = \frac{f(x)}{1+\frac{1}{n}|f|}.$$

Observe that

$$|E_n| \le |E|, \quad |f_n| \le |f|,$$

and

$$\left|\frac{u_n}{1+\frac{1}{n}|u_n|}\right| \le |u_n|\,.$$

Note that a weak solution u_n exists thanks to Schauder fixed point theorem (see also [8]), since, for every $n\in\mathbb{N}$, the nonlinear composition $\frac{u_n}{1+\frac{1}{n}|u_n|}$ of the solution u_n is a bounded one.

Moreover, since for every fixed n the function $f_n(x)$ and the vectorial field $E_n(x)$ are bounded, every u_n is bounded thanks to Stampacchia's boundedness theorem (see [15]).

2.2. Estimates

The following lemma is an improvement of a lemma of [6].

Lemma 2.1. *The sequence $\{u_n\}$ of the solutions of (2.1) satisfies the inequality*

$$\int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^2} \le \frac{1}{\alpha^2}\int_{\{k\le|u_n|\}} |E|^2 + \frac{2}{\alpha}\int_{\{k\le|u_n|\}} |f|, \quad \forall\, k\in\mathbb{R}^+. \quad (2.2)$$

Proof. Take $\left(\dfrac{|u_n|}{1+|u_n|} - \dfrac{k}{1+k}\right)^+ \dfrac{u_n}{|u_n|}$, $k \in \mathbb{R}^+$, as test function in (2.1). We have

$$\int_{\{k\le|u_n|\}} \frac{M(x)\nabla u_n\cdot\nabla u_n}{(1+|u_n|)^2} + A\int_{\{k\le|u_n|\}} u_n^2\left(\frac{|u_n|}{1+|u_n|} - \frac{k}{1+k}\right)$$
$$\le \int_{\{k\le|u_n|\}} \frac{|u_n|}{1+\frac{1}{n}|u_n|}|E_n|\frac{|\nabla u_n|}{(1+|u_n|)^2} + \int_{\{k\le|u_n|\}} \frac{|f_n|\,|u_n|}{1+|u_n|}.$$

Since $\frac{|u_n|}{1+|u_n|} \le 1$ we have, using (1.4) and the fact that $|f_n| \le |f|$,

$$\alpha\int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^2} + A\int_{\{k\le|u_n|\}} u_n^2\left(\frac{|u_n|}{1+|u_n|} - \frac{k}{1+k}\right)$$
$$\le \int_{\{k\le|u_n|\}} \frac{|E|\,|\nabla u_n|}{1+|u_n|} + \int_{\{k\le|u_n|\}} |f|\,,$$

so that (thanks to Young inequality),

$$\int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^2} + A\int_{\{k\le|u_n|\}} u_n^2\left(\frac{|u_n|}{1+|u_n|} - \frac{k}{1+k}\right)$$
$$\le \frac{1}{\alpha^2}\int_{\{k\le|u_n|\}} |E|^2 + \frac{2}{\alpha}\int_{\{k\le|u_n|\}} |f|$$

and, dropping a positive term,

$$\int_{\{k\le|u_n|\}} \frac{|\nabla u_n|^2}{(1+|u_n|)^2} \le \frac{1}{\alpha^2}\int_{\{k\le|u_n|\}} |E|^2 + \frac{2}{\alpha}\int_{\{k\le|u_n|\}} |f|.$$

□

Recall Stampacchia's definition of truncate:

$$T_k(s) = \begin{cases} s, & \text{if} \quad |s| \le k, \\ k\,\dfrac{s}{|s|}, & \text{if} \quad |s| > k, \end{cases}$$

The inequalities we will prove in the two lemmas below are improvements of results of [6, 7].

Lemma 2.2. *The sequence $\{u_n\}$ of the solutions of (2.1) satisfies the inequality*

$$\int_\Omega |\nabla T_k(u_n)|^2 \le \frac{k^2}{\alpha^2}\int_\Omega |E|^2 + k\,\frac{2}{\alpha}\int_\Omega |f|, \quad \forall\, k\in\mathbb{R}^+. \tag{2.3}$$

Proof. Take $T_k(u_n)$, $k \in \mathbb{R}^+$, as test function in (2.1). We have, using the ellipticity of the principal part and dropping a positive term,

$$\alpha\int_\Omega |\nabla T_k(u_n)|^2 \le \int_\Omega |u_n||E||\nabla T_k(u_n)| + k\int_\Omega |f|$$
$$\le k\int_\Omega |E||\nabla T_k(u_n)| + k\int_\Omega |f|\,.$$

Then the Young inequality yields (2.3). □

Lemma 2.3. *The sequence $\{u_n\}$ of the solutions of (2.1) satisfies the inequality*

$$\int_\Omega u_n^2 \le \frac{1}{A}\int_\Omega |f|, \tag{2.4}$$

Proof. Take $T_h(u_n)$, $h \in \mathbb{R}^+$, as test function in (2.1). Again we have

$$\alpha \int_\Omega |\nabla T_h(u_n)|^2 + A\int_\Omega u_n^2 |T_h(u_n)| \le h\int_\Omega |E||\nabla T_h(u_n)| + h\int_\Omega |f|\,.$$

Then the Young inequality yields

$$\frac{\alpha}{2}\int_\Omega |\nabla T_h(u_n)|^2 + A\int_\Omega u_n^2 |T_h(u_n)| \le h^2\frac{1}{2\alpha}\int_\Omega |E|^2 + h\int_\Omega |f|\,.$$

Now we drop the first term and we have, dividing by $h > 0$,

$$A\int_\Omega u_n^2\frac{|T_h(u_n)|}{h} \le h\frac{1}{2\alpha}\int_\Omega |E|^2 + \int_\Omega |f|\,.$$

The Fatou lemma, as $h \to 0$, yields (2.4). □

As a consequence of the estimate (2.4), we can state the following corollaries.

Corollary 2.4. *Passing to a subsequence if necessary, we may assume the sequence $\{u_n\}$ converges weakly in $L^2(\Omega)$ to some u.*

Corollary 2.5. *Thanks to the estimate (2.4) we have*

$$\operatorname{meas}\{x \in \Omega : |u_n(x)| > k\} \le \frac{1}{A\,k^2}\int_\Omega |f|. \tag{2.5}$$

In the following lemma we prove an a priori bound (and more) in $W_0^{1,1}(\Omega)$ for the sequence $\{u_n\}$.

Lemma 2.6. *The sequence $\{u_n\}$ of the solutions of (2.1) satisfies the inequality*

$$\begin{aligned}\int_{\{k\le|u_n|\}} |\nabla u_n| \le &\Bigg[\int_{\{k\le|u_n|\}}\left(\frac{1}{\alpha^2}|E|^2 + \frac{2}{\alpha}|f|\right)\Bigg]^{\frac12}\\ &\cdot\left[\|1\|_{L^2(\Omega)} + \frac{1}{A}\,\|f\|_{L^1(\Omega)}\right]^{\frac12},\end{aligned} \tag{2.6}$$

for any $k \in \mathbb{R}^+$.

Proof. We write

$$\int_{\{k\le|u_n|\}}|\nabla u_n| = \int_{\{k\le|u_n|\}}\frac{|\nabla u_n|}{1+|u_n|}\,(1+|u_n|)$$

and then we use the Hölder inequality and the estimates (2.2), (2.4) so that

$$\begin{aligned}\int_{\{k\le|u_n|\}}|\nabla u_n| &\le \left[\frac{1}{\alpha^2}\int_{\{k\le|u_n|\}}|E|^2+\frac{2}{\alpha}\int_{\{k\le|u_n|\}}|f|\right]^{\frac12}\left[\int_\Omega(1+|u_n|)^2\right]^{\frac12}\\ &\le\left[\int_{\{k\le|u_n|\}}\left(\frac{1}{\alpha^2}|E|^2+\frac{2}{\alpha}|f|\right)\right]^{\frac12}\left[\|1\|_{L^2(\Omega)}+\frac{1}{A}\,\|f\|_{L^1(\Omega)}\right]^{\frac12}\end{aligned}$$

□

Remark 2.7. Note the importance of the assumption $A>0$: if $A=0$ it is not possible to prove the estimates (2.4) and (2.6), but only (2.2).

As a consequence of the estimate (2.6), we can state the following corollaries.

Corollary 2.8. *The sequence $\{u_n\}$ of the solutions of (2.1) is bounded in $W_0^{1,1}(\Omega)$.*

Proof. Take $k=0$ in (2.6). □

Remark 2.9. Rellich theorem implies the existence of a subsequence $\{u_{n_j}\}$ such that u_{n_j} converges strongly in $L^1(\Omega)$.

Thus it is possible to improve the statement of Corollary 2.4: passing to a subsequence if necessary, we may assume the sequence $\{u_n\}$ converges weakly in $L^2(\Omega)$ and strongly in $L^\rho(\Omega)$, $1\le\rho<2$, and a.e. to some u.

The next lemma improves the result of Corollary 2.8.

Lemma 2.10. *The sequence $\{u_n\}$ of the solutions of (2.1) is weakly compact in $W_0^{1,1}(\Omega)$.*

Proof. We will use the Dunford-Pettis theorem. Let X be a measurable subset of Ω. Then we have (using the inequality (2.6))

$$\begin{aligned}\int_X|\nabla u_n| &\le \int_X|\nabla T_k(u_n)|+\int_{\{k\le|u_n|\}}|\nabla u_n|\\ &\le |X|^{\frac12}\left[\int_\Omega|\nabla T_k(u_n)|^2\right]^{\frac12}+\left[\frac{1}{\alpha^2}\int_{\{k\le|u_n|\}}|E|^2+\frac{2}{\alpha}\int_{\{k\le|u_n|\}}|f|\right]^{\frac12}\\ &\qquad\cdot\left[\|1\|_{L^2(\Omega)}+\frac{1}{A}\,\|f\|_{L^1(\Omega)}\right]^{\frac12}\\ &\le |X|^{\frac12}\left[\frac{k^2}{\alpha^2}\int_\Omega|E|^2+k\,\frac{2}{\alpha}\int_\Omega|f|\right]^{\frac12}+\left[\int_{\{k\le|u_n|\}}\left(\frac{1}{\alpha^2}|E|^2+\frac{2}{\alpha}|f|\right)\right]^{\frac12}\\ &\qquad\cdot\left[\|1\|_{L^2(\Omega)}+\frac{1}{A}\,\|f\|_{L^1(\Omega)}\right]^{\frac12}\end{aligned}$$

Thus

$$\lim_{\text{meas}(X)\to 0}\int_X |\nabla u_n| \leq \left[\int_{\{k\leq|u_n|\}}\left(\frac{1}{\alpha^2}|E|^2+\frac{2}{\alpha}|f|\right)\right]^{\frac{1}{2}}\left[\|1\|_{L^2(\Omega)}+\frac{1}{A}\,\|f\|_{L^1(\Omega)}\right]^{\frac{1}{2}},$$

where the right hand side is uniformly (with respect to n) small for k large, thanks to the absolute continuity of the Lebesgue integral and to the inequality (2.5).

Thus we proved that

$$\lim_{\text{meas}(X)\to 0}\int_X |\nabla u_n| = 0, \text{ uniformly with respect to } n.$$

□

Thus, at the present time, we proved the following convergence properties on the sequence $\{u_n\}$:

$$\begin{cases} u_n \text{ converges weakly to } u \text{ in } W_0^{1,1}(\Omega); \\ u_n \text{ converges weakly in } L^2(\Omega), \text{ strongly in } L^\rho,\ 1\leq\rho<2, \text{ a.e. to } u; \\ T_k(u_n) \text{ converges weakly in } W_0^{1,2}(\Omega) \text{ to } T_k(u), \text{ for every } k\in\mathbb{R}^+; \end{cases} \tag{2.7}$$

where the last convergence is a consequence of the previous ones and of (2.3).

Remark 2.11. We point out that, in the weak formulation of (2.1),

$$\int_\Omega M(x)\nabla u_n\nabla\varphi + A\int_\Omega u_n|u_n|\varphi = \int_\Omega \frac{u_n}{1+\frac{1}{n}u_n}E_n(x)\nabla\varphi + \int_\Omega f_n(x)\varphi, \tag{2.8}$$

$\forall\,\varphi\in Lip(\Omega)$, thanks to (2.7), it is possible to pass to the limit in 3 of the 4 terms: the only problematic term is the second one, since we did not prove the strong convergence of the sequence $\{u_n\}$ in $L^2(\Omega)$.

2.3. Entropy solutions

Since we are not able to pass to the limit (see to Remark 2.11) in

$$\int_\Omega u_n|u_n|\varphi, \quad \forall\,\varphi\in Lip(\Omega),$$

in order to give a meaning to the existence of solutions, we use the concept of *entropy solutions* which has been used in [7] for problems with convection terms and previously introduced in [5] to prove existence and uniqueness of solutions of nonlinear Dirichlet problems with L^1 data.

Definition 2.12. The function u is an *entropy solution* of (1.1), if

$$\begin{cases} u\in W_0^{1,1}(\Omega),\ T_k(u)\in W_0^{1,2}(\Omega): \\ \displaystyle\int_\Omega M(x)\nabla u\nabla T_k[u-\varphi] + A\int_\Omega u|u|\,T_k[u-\varphi] \\ \qquad\qquad \displaystyle\leq \int_\Omega u\,E(x)\nabla T_k[u-\varphi] + \int_\Omega f(x)\,T_k[u-\varphi] \end{cases} \tag{2.9}$$

for any $k \in \mathbb{R}^+$ and $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Theorem 2.13. *We assume* (1.2), (1.3), (1.4)*. Then there exists* u *entropy solution of* (1.1)*, in the sense of the above definition.*

Proof. For $k \in \mathbb{R}^+$ and $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, we use (recall (2.3)) $T_k[u_n - \varphi]$ as test function in (2.1) and we have

$$\begin{cases} T_k(u_n) \in W_0^{1,2}(\Omega) : \\ \displaystyle\int_\Omega M(x)\nabla u_n \nabla T_k[u_n - \varphi] + A\int_\Omega u_n|u_n| T_k[u_n - \varphi] \\ \qquad\qquad \displaystyle= \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\, E_n(x)\nabla T_k[u_n - \varphi] + \int_\Omega f_n(x)\, T_k[u_n - \varphi] \end{cases}$$

for any $k \in \mathbb{R}^+$ and $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

We write the equality as

$$\left[\begin{array}{l} \displaystyle\int_\Omega M(x)\nabla T_k[u_n - \varphi]\nabla T_k[u_n - \varphi] + \int_\Omega M(x)\nabla\varphi\nabla T_k[u_n - \varphi] \\ \displaystyle + A\int_\Omega \{u_n|u_n| - \varphi|\varphi|\} T_k[u_n - \varphi] + A\int_\Omega \varphi|\varphi|\, T_k[u_n - \varphi] \\ \displaystyle = \int_\Omega \frac{u_n}{1+\frac{1}{n}|u_n|}\, E_n(x)\nabla T_k[u_n - \varphi] + \int_\Omega f_n(x)\, T_k[u_n - \varphi], \end{array} \right.$$

(that is $I_1 + \cdots + I_4 = I_5 + I_6$) and we recall the convergences on E_n, f_n and (2.7), which allow to pass to the limit in I_2, I_4, I_5, I_6. In particular, note that $\nabla T_k[u_n - \varphi]$ weakly converges in L^2 to $\nabla T_k[u - \varphi]$ and that, for $k^* = k + \|\varphi\|_{L^\infty(\Omega)}$,

$$I_5 = \frac{u_n}{1+\frac{1}{n}|u_n|}\, E_n(x)\nabla T_k[u_n - \varphi] = \frac{T_{k^*}(u_n)}{1+\frac{1}{n}|u_n|}\, E_n(x)\nabla T_k[u_n - \varphi],$$

so that we can pass to the limit thanks to the Lebesgue theorem. Note that we can pass to the limit thanks to the weak lower semicontinuity and weak $W_0^{1,2}(\Omega)$ convergence of $T_k(u_n)$, in I_1, and Fatou lemma, in I_3. We obtain

$$\left[\begin{array}{l} \displaystyle\int_\Omega M(x)\nabla T_k[u - \varphi]\nabla T_k[u - \varphi] + \int_\Omega M(x)\nabla\varphi\nabla T_k[u - \varphi] \\ \displaystyle + A\int_\Omega \{u|u| - \varphi|\varphi|\} T_k[u - \varphi] + A\int_\Omega \varphi|\varphi|\, T_k[u - \varphi] \\ \displaystyle \le \int_\Omega u\, E(x)\nabla T_k[u - \varphi] + \int_\Omega f(x)\, T_k[u - \varphi] \end{array} \right.$$

and so, after simplifications, we prove (2.9). □

Open Problem 2.14. Recall we proved that ∇u, u^2 and $(u\,E)$ are summable function. Then is it possible to say that u is a distributional solution? That is $u \in W_0^{1,1}(\Omega) \cap L^2(\Omega)$ is a solution of

$$\int_\Omega M(x)\nabla u\nabla\varphi + A\int_\Omega u^2\,\varphi = \int_\Omega u\, E(x)\nabla\varphi + \int_\Omega f(x)\,\varphi, \quad \forall\varphi \in Lip(\Omega),$$

(see also Remark 2.11).

3. Regularizing effect of the interplay between coefficients

In this section we consider the boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + a(x)\, u|u| = -\mathrm{div}(u\, E(x)) + f(x) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{3.1}$$

where

$$f \in L^m(\Omega),\ m \geq 1, \tag{3.2}$$

$$|E|^2 \leq R\, a(x) \in L^1(\Omega), \quad \text{for some } R \in \mathbb{R}^+, \tag{3.3}$$

and we define the following approximate Dirichlet problems

$$\begin{aligned} &u_n \in W_0^{1,2}(\Omega): \\ &-\mathrm{div}(M(x)\nabla u_n) + a(x)u_n|u_n| = -\mathrm{div}\left(\frac{u_n}{1+\frac{1}{n}|u_n|}\, E_n\right) + f_n, \end{aligned} \tag{3.4}$$

The fact that the coefficient $a(x)$ belongs to $L^1(\Omega)$ does not change too much the framework with respect to the boundary value problem (2.1) (see also [14]); that is, for every $n \in \mathbb{N}$, there exists a bounded weak solution u_n.

The main feature of this section is the inequality $|E|^2 \leq R\, a(x)$, supposed in (3.3) (interplay between coefficients), which produces a regularizing effect on the solutions. The regularizing effect of the interplay between coefficients in nonlinear Dirichlet problems is studied in [2, 3, 4]; in particular, interplay between coefficients in Dirichlet problems with convection terms or drift terms is considered in [9, 10].

With the same proof of Lemma 2.3 it is possible to prove the following lemma.

Lemma 3.1. *The sequence $\{u_n\}$ of the solutions of (3.4) satisfies the inequality*

$$\int_\Omega a(x)\, u_n^2 \leq \int_\Omega |f|. \tag{3.5}$$

Lemma 3.2. *We assume (1.4), (3.2) with $m \geq \frac{2N}{N+2}$, (3.3). Then the sequence $\{u_n\}$ of the solutions of (3.4) satisfies the estimates*

$$\begin{aligned} &\{u_n\} \text{ is bounded in } W_0^{1,2}(\Omega); \\ &\{a(x)|u_n|^3\} \text{ is bounded in } L^1(\Omega). \end{aligned} \tag{3.6}$$

Proof. We use u_n as test function in the weak formulation of (3.4) and we have, thanks to the Young inequality and (3.5),

$$\begin{aligned}\alpha\int_\Omega|\nabla u_n|^2+\int_\Omega a(x)|u_n|^3 &\le \int_\Omega|u_n||E||\nabla u_n|+\int_\Omega|f||u_n|\\ &\le \frac{\alpha}{2}\int_\Omega|\nabla u_n|^2+\frac{1}{2\alpha}\int_\Omega u_n^2|E|^2+\int_\Omega|f||u_n|\\ &\le \frac{\alpha}{2}\int_\Omega|\nabla u_n|^2+\frac{R}{2\alpha}\int_\Omega a(x)u_n^2+\int_\Omega|f||u_n|\\ &\le \frac{\alpha}{2}\int_\Omega|\nabla u_n|^2+\frac{R}{2\alpha}\int_\Omega|f|+\int_\Omega|f||u_n|,\end{aligned}$$

which implies

$$\frac{\alpha}{2}\int_\Omega|\nabla u_n|^2+\int_\Omega a(x)|u_n|^3\le\frac{R}{2\alpha}\int_\Omega|f|+\int_\Omega|f||u_n|, \tag{3.7}$$

that is, a standard estimate for semilinear problems.

Thus, dropping the second (positive) integral, we deduce the first estimate in (3.6).

Then, in (3.7), dropping the first (positive) integral, thanks to the first estimate in (3.6) (just proved), we deduce the second estimate in (3.6). □

Now, thanks to the assumption (3.3), it is possible to strongly improve the existence result of Theorem 2.13. Indeed we will prove the existence of finite energy solutions.

Theorem 3.3. *We assume* (1.4)*,* (3.2)*,* (3.3)*. Then there exists a weak solution u of the boundary value* (3.1)*; that is a function u such that*

$$\begin{cases} u\in W_0^{1,2}(\Omega),\ a(x)\,u^2\in L^1(\Omega):\\ \displaystyle\int_\Omega M(x)\nabla u\nabla\varphi+\int_\Omega a(x)\,u|u|\,\varphi=\int_\Omega u\,E\,\nabla\varphi+\int_\Omega f\varphi\end{cases} \tag{3.8}$$

for any $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

Proof. As a consequence of the first statement in (3.6), passing to a subsequence if necessary, we may assume the sequence $\{u_n\}$ converges weakly in $W_0^{1,2}(\Omega)$ and a.e. to some u. Moreover, the second statement in (3.6) yields the strong convergence of $a(x)u_n|u_n|$ in $L^1(\Omega)$, thanks to the Vitali theorem. Indeed, since $a(x)u_n|u_n|$ converges a.e., we only need to prove the equi-integrability of the sequence $\{a(x)u_n|u_n|\}$: let X be a measurable subset of Ω and $k\in\mathbb{R}^+$, we have

$$\int_X|a(x)u_n^2|\le k^2\int_X a(x)+\frac{1}{k}\int_\Omega a(x)|u_n|^3.$$

Thus

$$\lim_{\mathrm{meas}(X)\to 0}\int_X|a(x)u_n^2|\le\frac{C_1}{k},\quad\forall\,k\in\mathbb{R}^+,$$

that is

$$\lim_{\operatorname{meas}(X)\to 0}\int_X |a(x)u_n^2| = 0, \quad \text{uniformly with respect to } n.$$

Moreover the inequality

$$\begin{aligned}\left|\frac{u_n}{1+\frac{1}{n}|u_n|}\,E_n\right| &\le \sqrt{a(x)}|u_n|\frac{|E|}{\sqrt{a(x)}}\\ &\le \frac{1}{2}a(x)|u_n|^2+\frac{1}{2}\frac{|E|^2}{a(x)} \le \frac{1}{2}a(x)|u_n|^2+\frac{1}{2}\,R\end{aligned}$$

and the strong convergence of $a(x)u_n|u_n|$ in $L^1(\Omega)$, thanks to the Vitali theorem, yield to the convergence of $\frac{u_n}{1+\frac{1}{n}|u_n|}\,E_n$ in $L^1(\Omega)$.

Thus it is possible to pass to the limit in the weak formulation of (3.4) and we obtain (3.8). □

3.1. A second interplay

Now, briefly, we consider a second interplay: instead of (3.3), we assume

$$|f(x)| \le Q\,a(x) \in L^1(\Omega), \quad \text{for some } Q \in \mathbb{R}^+, \tag{3.9}$$

and we study the behaviour of the sequence $\{u_n\}$.

Lemma 3.4. *Under the assumptions* (1.4), (3.2), *with* $m = 1$, $E \in (L^r(\Omega))^N$, *with* $r > N$, (3.9), *the sequence* $\{u_n\}$ *is bounded in* $L^\infty(\Omega)$.

Proof. Take $\left(\frac{|u_n|}{1+|u_n|}-\frac{k}{1+k}\right)^+\frac{u_n}{|u_n|}$, $k \ge \sqrt{Q}$, as test function in (3.4). We use the assumption (3.9) and we have

$$\begin{aligned}&\alpha\int_{\{k\le|u_n|\}}\frac{|\nabla u_n|^2}{(1+|u_n|)^2}+\int_{\{k\le|u_n|\}}a(x)u_n^2\left(\frac{|u_n|}{1+|u_n|}-\frac{k}{1+k}\right)\\ &\le \int_{\{k\le|u_n|\}}\frac{|u_n|}{1+\frac{1}{n}|u_n|}|E_n|\frac{|\nabla u_n|}{(1+|u_n|)^2}+\int_{\{k\le|u_n|\}}\frac{Q\,a(x)}{1+\frac{1}{n}Q\,a(x)}\left(\frac{|u_n|}{1+|u_n|}-\frac{k}{1+k}\right)\\ &\le \frac{\alpha}{2}\int_{\{k\le|u_n|\}}\frac{|\nabla u_n|^2}{(1+|u_n|)^2}+\int_{\{k\le|u_n|\}}|E|^2+\int_{\{k\le|u_n|\}}Q\,a(x)\left(\frac{|u_n|}{1+|u_n|}-\frac{k}{1+k}\right).\end{aligned}$$

Note that we work with $k \ge \sqrt{Q}$, so that $u_n^2 \ge Q$ in the subset $\{|u_n| \ge k\}$. Then the above inequalities yield, dropping a positive contribution,

$$\frac{\alpha}{2}\int_{\{k\le|u_n|\}}\frac{|\nabla u_n|^2}{(1+|u_n|)^2} \le \int_{\{k\le|u_n|\}}|E|^2.$$

Here the Stampacchia theorem on $\log(1+|u_n|)$, thanks to the assumption $E \in \big(L^r(\Omega)\big)^N$, with $r > N$, gives the boundedness in $L^\infty(\Omega)$ of the sequence $\{\log(1+|u_n|)\}$; that is the boundedness of the sequence $\{u_n\}$. □

Thanks to the a priori bound of the previous lemma, it is easy to prove that the sequence $\{u_n\}$ is also bounded in $W_0^{1,2}(\Omega)$.

Thus it is not difficult to pass to the limit in (3.4) and to prove the following theorem.

Theorem 3.5. *Under the assumptions* (1.4), (3.2), *with* $m = 1$, $E \in \big(L^r(\Omega)\big)^N$, *with* $r > N$, (3.9), *there exists a bounded weak solution of the boundary value problem* (3.1), *that is a function* u *such that*

$$\begin{cases} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ \displaystyle\int_\Omega M(x)\nabla u \nabla\varphi + \int_\Omega a(x)\, u|u|\, \varphi = \int_\Omega u\, E\, \nabla\varphi + \int_\Omega f\varphi \end{cases} \tag{3.10}$$

for any $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

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