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Multiple positive solutions for some local and non-local elliptic systems arising in desertification models

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Abstract. *We consider a nonlinear elliptic system proposed in 2007 by E. Gilad, J. von Hardenberg, A. Provenzale, M. Shachak and E. Meron, in desertification studies. The system models the mutual interaction between the biomass b, the soil-water content w and the surface-water height h. The interactions with the plant environment may lead to some non-local terms which can be approximated by suitable local expressions. Various kinds of feedback processes arise. The change in environmental conditions can be simulated by the change of suitable parameters in the differential equations. Here we consider the case of Dirichlet boundary conditions. After describing some positive solutions corresponding to special values of the parameters, we prove the existence of positive solutions for the local and non-local system. We obtain some bifurcation diagrams showing, rigorously, its starting value and characterizing the supercritical (resp. subcritical) nature of the branch (something unnoticed before in the previous literature) according to a suitable parameters balance expression. Finally, we prove that if the precipitation datum $p(x)$ grows near the boundary of the domain $\partial\Omega$ as $d(x,\partial\Omega)^2$ then $h(x)$ grows, at most, as $d(x,\partial\Omega)^4$.*

1. Introduction

Equations and systems of reaction-diffusion type have been widely studied during the last fifty years both for their mathematical interest and the relevance in applications (population dynamics, combustion, chemical reactions, nerve impulses, etc.). See the books [30, 25] and the references therein for more informations.

Existence and uniqueness of solutions for parabolic systems are studied, and then the asymptotic behavior of solutions. From this point of view it is important the study of the existence (and multiplicity) of solutions of the associated stationary problem together with the stability of their solutions. Very often only positive solutions are interesting for the applications.

The first author and P. Kyriasopoulos studied in [15] an elliptic system arising in a dryland vegetation model suggested by Gilad et al. in [18] (other models can be found in [28, 3, 23]). This system was proposed through the modelling

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of appropriate ecosystems consisting of organisms that interact among themselves and with their environment. These interactions involve various kinds of feedback processes that may combine to form positive feedback loops and instabilities when environmental conditions change, and this can be simulated by the change of suitable parameters in the differential equations. Like the well-established activator-inhibitor principle in bio-chemical systems [24], the combination of these scale-dependent feedback mechanisms can induce instabilities that result in large-scale spatial patterns, which are similar to a wide variety of vegetation patterns observed in drylands, peatlands, savannas and undersea (see, e.g. the monograph [23]). Here we do not intend to enter in the very rich field of pattern formation but only to complete the mathematical analysis of these types of models already initiated in [15] (see also [22]). We also mention the mathematical study of the corresponding dynamical system (now given by a set of parabolic equations) made in [19, 20, 14]. Understanding the dynamics and stability/instability of spatially extended ecosystems has become an active field of research in the last two decades within communities of ecologists, environmental scientists, mathematicians and physicists.

The more general version of the system we shall consider in this paper is given by

$$\begin{cases} -\delta_b \Delta b = -\mu b + G_b b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w = -G_w w - \mathcal{E}_b w + \mathcal{I}_b h & \text{in } \Omega, \\ -\delta_h \Delta h^2 = -\mathcal{I}_b h + p & \text{in } \Omega, \end{cases} \tag{1.1}$$

where we suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with C^2 boundary and n is the outward pointing unit normal on $\partial\Omega$. The vertical variable usually arises in some extra terms in third equation representing the ground surface height for non-flat topographies but here they are neglected in order to get a more basic qualitative study. Here, b represents the biomass, w the soil-water content and h the surface-water height (after suitable non-dimensionalization). The growth rate G_b and the water uptake rate G_w are non-local terms given by

$$G_b(b,w) = \nu \int_\Omega g(x,y) w(y)\, dy \quad \text{and} \quad G_w(b) = \gamma \int_\Omega g(y,x) b(y)\, dy$$

where

$$g(x,y) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{|x-y|}{2\left[\sigma(1+\eta b)\right]^2} \right\} \text{ for } x, y \in \Omega. \tag{1.2}$$

Moreover, $\mu(x) > 0$ represents the biomass loss rate,

$$\mathcal{E}_b(b) = \frac{\nu}{1+\rho b} \tag{1.3}$$

is the evaporation rate of the soil water, and

$$\mathcal{I}_b(b) = \alpha \frac{b + q/c}{b+q} \tag{1.4}$$

represents the infiltration rate of the surface water. Notice that the third equation involves nonlinear diffusion of porous medium type and the precipitation rate datum p which we will assume, for simplicity,

$$p \in C(\overline{\Omega}),\ p(x) > 0 \text{ for any } x \in \Omega. \tag{1.5}$$

All parameters $\alpha, \rho, \nu, q, \sigma$ are nonnegative and $c > 1$. With respect to the boundary conditions, we should point out that although in most of the previous papers it is assumed Neumann boundary conditions, $\dfrac{\partial b}{\partial n} = \dfrac{\partial w}{\partial n} = \dfrac{\partial h}{\partial n} = 0$ hold on $\partial\Omega$, a sharper modelling of some concrete example will require the use of other types of boundary conditions, among them, of course, Dirichlet conditions.

It is well-known ([18, 28, 23]) that the ecosystem response to decreasing rainfall, for example, may take the form of abrupt collapses to a nonproductive "desert state", or involve gradual desertification, consisting of a cascade of state transitions to sparser vegetation, or gradual vegetation retreat by front propagation. This explains the crucial role played by the data given by p. In a simplified framework (for very local purposes) it can be considered as a given positive constant (and the possible multiplicity of solutions according the values of p lead to different kinds of bifurcation diagrams). In a more spatially global framework p should be understood as a spatial given function $p(x)$ which may justify the occurrence of fronts separating regions in which $h > 0$ from parts in which $h = 0$. Let us mention that rigorous mathematical study of the qualitative behavior of solutions is a necessary complement of previous studies in which the use of computational tools for quite special cases (see a very complete source of references in [23]) leads to create some theories which require to be checked in each new special case of the parameters and other data.

In [15] the authors consider the situation of plant species with negligible below ground biomass. In this case it can be assumed that the root extension parameter is $\eta = 0$. And also that since the minimal root size of such plant specie goes to zero, non-local effects of the root systems are negligible. In such a situation we may replace $g(x, y)$ with the Dirac delta based on x and hence obtaining the following *local* coupled system

$$\begin{cases} -\delta_b \Delta b = -\mu(x) b + \upsilon w b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w = -\gamma b w - \mathcal{E}_b w + \mathcal{I}_b h & \text{in } \Omega, \\ -\delta_h \Delta h^2 = -\mathcal{I}_b h + p & \text{in } \Omega, \end{cases} \tag{1.6}$$

with the Neumann boundary conditions.

Then the authors do not study the full system (1.6) but only the case where infiltration feedback and soil-water diffusion are not present, i.e., when $\delta_w = \delta_h = 0$. In this case, they obtain existence of multiple positive solutions for $1 \le \mu(x) \le \bar{\mu}$ in terms of the parameter p. Moreover, they also get results concerning the free boundary of the surface-water solution component h in the case of the full system.

In this paper we study the full system in *both* the non-local and the (simplified) local versions. This study is greatly simplified by making the change of variable

$H = h^2$ (and then replacing again H by h) obtaining the system (now we change the boundary conditions)

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = G_b b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w + G_w w + \mathcal{E}_b w = \mathcal{I}_b h & \text{in } \Omega, \\ -\delta_h \Delta h + \mathcal{I}_b \sqrt{h} = p & \text{in } \Omega, \\ b = w = h = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

This choice of the boundary conditions will simplify (and somewhat modify) the problem with respect to Neumann boundary conditions, but we will show that most of the results hold for general (linear) boundary conditions of the third type

$$\frac{\partial b}{\partial n} + \omega(x) b = 0 \text{ on } \partial\Omega,$$

with $\omega(x) > 0$ smooth (see Remark 5.14).

Notice that the presence of $\sqrt{h}$ in the third equation of (1.7)) makes possible the existence of non-negative solutions with "dead core" where the solution annihilates (the above mentioned fronts originated by data $p(x)$ vanishing in some subregions). Examples of systems with such solutions were given time ago in [13, 26, 27].

The paper is organized as follows. Just for the sake of completeness, we deal, in Section 2, with a simple study of several particular cases of the system which may have some interest in the applications. The main results of this paper concern the rest of the sections of this paper. In Section 3 we prove existence of positive solutions for the local system and in Section 4 in the non-local case. In Section 5 we study, perhaps the deeper results of this paper, the existence of positive solutions when $\delta_b > 0$ and $\delta_w = 0$ getting both uniqueness and multiplicity results. In contrast to previous results dealing with Neumann boundary condition, we obtain some bifurcation diagrams showing rigorously its starting value (from the first eigenvalue λ_1 of a linear operator with the corresponding weights and with Dirichlet boundary conditions) and characterizing the supercritical (resp. subcritical) nature of the branch (something unnoticed before in the literature) according the positivity (resp. negativity) of the parameters balance expression $\nu(1-\rho)+\gamma$. In Section 6 we study the case in which $p(x)$ vanishes on $\partial\Omega$ completing the results of [15]. We show (for $\delta_h > 0$) that if $p(x)$ grows near $\partial\Omega$ as $d(x,\partial\Omega)^2$ then $h(x)$ grows, at most, as $d(x,\partial\Omega)^4$. In particular h is a "flat solution", in the sense that $h = \frac{\partial h}{\partial n} = 0$ on $\partial\Omega$, with $h > 0$ on Ω if $p > 0$ on Ω. Finally, Section 7 is devoted to recall the main conclusions and to state a few open problems.

2. The local system. Some particular cases

First we will consider some special cases of the general local system which may have some particular interest for the applications. The general version of the local system after one change of variable is

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = \nu w b (1-b) & \text{in } \Omega, \\ -\delta_w \Delta w + \gamma b w + \mathcal{E}_b w = \mathcal{I}_b \sqrt{h} & \text{in } \Omega, \\ -\delta_h \Delta h + \mathcal{I}_b \sqrt{h} = p(x) & \text{in } \Omega, \\ b = w = h = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

We recall that we have

$$\mathcal{E}_b(b) = \frac{\nu}{1+\rho b}, \qquad \mathcal{I}_b(b) = \alpha \frac{b + q/c}{b+q} \tag{2.2}$$

and thus $\mathcal{E}_b(b)$ is decreasing in b and

$$\frac{\nu}{1+\rho} \leq \mathcal{E}_b(b) \leq \nu \text{ if } b \in [0,1], \tag{2.3}$$

and that $\mathcal{I}_b(b)$ is increasing in b and for any $b \geq 0$

$$0 < \frac{\alpha}{c} < \mathcal{I}_b(b) < \alpha. \tag{2.4}$$

In this paper we are mainly interested in continuous solutions of system (2.1) which requires some assumptions on the data which are stronger than when one deals with other kind of weak solutions (see Remarks 2.3 and 5.13). We assume that μ is C^1 and for some $\overline{\mu} > 1$

$$1 \leq \mu(x) \leq \overline{\mu} \tag{2.5}$$

and $p(x) \geq 0$ is continuous on $\overline{\Omega}$.

We will use several well-known auxiliary results which give, at the same time, some basic properties of solutions. Their proof can be obtained even under conditions much more general than the above indicated framework, nevertheless we give here a short proof of them for the sake of completeness.

Lemma 2.1. *Assume that $b \geq 0$ is a solution of (2.1). Then $0 \leq b(x) < 1$ on Ω.*

Proof. Indeed, let us prove, first, that $0 \leq b \leq 1$. If we define

$$A = \{x \in \Omega \mid 1 < b(x)\},$$

then we have $-\delta_b \Delta b + \mu(x) b < 0$ on A and $b = 1$ on ∂A. From the Maximum Principle, $b < 1$ on A, a contradiction. Moreover, if $b(x_0) = 1$ for some $x_0 \in \Omega$, then

$$0 \leq -\Delta b(x_0) = -\mu(x_0) b(x_0) + \nu w(x_0) b(x_0)(1 - b(x_0)) = -\mu(x_0) < 0,$$

which is a contradiction. □

Lemma 2.2. *The problem*

$$\begin{cases} -\Delta w + \beta(w) = h(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

where β is continuous and monotone increasing with $\beta(0) = 0$ and $h \in C(\overline{\Omega})$, $h \geq 0$ in Ω, has a solution $w \geq 0$. If $h > 0$ in Ω, then $w > 0$ in Ω and it is unique.

Proof. We use sub and supersolutions. As a supersolution we pick w^0 such that $-\Delta w^0 = h$ in Ω, $w^0 = 0$ on $\partial\Omega$, $w^0 \in C_0^1(\Omega)$ and $w^0 > 0$. For a subsolution let us choose D smooth such that $h(x) \geq c_1 > 0$ on D, and let $\mu_1 > 0$ the first eigenvalue of $-\Delta$ on D with eigenfunction $\psi_1 > 0$, i.e., $-\Delta\psi_1 = \mu_1\psi_1$ in D, $\psi_1 = 0$ on ∂D. We can show that

$$u_0 = \begin{cases} c\psi_1 & \text{on } D \\ 0 & \text{on } \Omega \setminus D, \end{cases}$$

for $c > 0$ small is a subsolution. Hence there exists a $C^1(\overline{\Omega})$ solution with $w \geq 0$. We do not have necessarily $w > 0$, but this is obviously the case if $h > 0$ in Ω. Uniqueness is proved by the usual monotonicity argument. □

Remark 2.3. As indicated before, the above auxiliary results hold under more general assumptions. For instance, the proof of Lemma 2.2 for $h \in L^1(\Omega, d)$ where $d = d(x, \partial\Omega)$ can be found in [16]. In this case the (unique) solution should be understood in the class of very weak solutions of problem (2.6).

We study next a few particular cases of system (2.1). The meaning of such particular cases sometimes can be easely understood in terms of the ecological model (see, e.g. Section 9.2.3 of [23]).

2.1. Solutions with $b \equiv 0$

In this case the system is reduced to

$$\begin{cases} -\delta_w \Delta w + \mathcal{E}_0 w = \mathcal{I}_0\sqrt{h} & \text{in } \Omega, \\ -\delta_h \Delta h + \mathcal{I}_0\sqrt{h} = p(x) & \text{in } \Omega, \\ w = h = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Since $p \geq 0$, the second equation has a unique solution $h(.;p) \geq 0$ (Lemma 2.2). For this $h(.;p)$ the first equation has a unique solution $w > 0$. Thus we have

Proposition 2.4. *There is a unique solution* $(\overline{w}, \overline{h})$ *with* $\overline{w} > 0$, $\overline{h} \geq 0$ *of system* (2.1) *with* $b \equiv 0$.

2.2. Solutions with $w \equiv 0$

Now the system is written as

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = 0 & \text{in } \Omega, \\ -\delta_h \Delta h + \mathcal{I}_b\sqrt{h} = p(x) & \text{in } \Omega, \\ b = h = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.8}$$

Now $b \equiv 0$ and the system (2.1) has only the solution $(0, 0, \overline{h})$ with $\overline{h} \geq 0$ the unique solution of

$$\begin{cases} -\delta_h \Delta h + \mathcal{I}_0\sqrt{h} = p(x) & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega. \end{cases}$$

Proposition 2.5. *The system* (2.1) *has a unique solution* $(0, 0, \overline{h})$ *with* $\overline{h} \geq 0$ *and* $w \equiv 0$.

2.3. Solutions with $h \equiv 0$

It is clear that this is only possible if $p(x) \equiv 0$ as well. In this case the system (2.1) is reduced to

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = \nu w b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w + \gamma b w + \mathcal{E}_b w = 0 & \text{in } \Omega, \\ b = w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

But since $b \geq 0$, the second equation and the Maximum Principle give $w \equiv 0$ and the first one $b \equiv 0$. Hence

Proposition 2.6. *If $p(x) \equiv 0$ the system (2.1) has only the trivial solution.*

2.4. Solutions with $\delta_h \equiv 0$

Now we have $\mathcal{I}_b \sqrt{h} = p(x)$ and the resultant system is

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = \nu w b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w + \gamma b w + \mathcal{E}_b w = p(x) & \text{in } \Omega, \\ b = w = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.10}$$

which can be written as

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = \nu w b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w = p(x) - \gamma b w - \mathcal{E}_b w & \text{in } \Omega, \\ b = w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Let us see first if the system is *cooperative* (in the sense of the associate dynamical system; see, e.g., [25] and [29]). First

$$\frac{\partial}{\partial w}(\nu w b(1-b)) = \nu b(1-b) \geq 0,$$

since $b \in [0,1]$. For the second equation we need

$$\frac{\partial}{\partial b}(p(x) - \gamma b w - \mathcal{E}_b w) = -\gamma w + \frac{\nu \rho w}{(1+\rho b)^2} \geq 0, \ \forall b \in [0,1],$$

i.e., $\nu\rho > \gamma(1+\rho b)^2$, which holds if

$$\nu\rho > \gamma(1+b)^2,$$

for any $b \in [0,1]$, in particular if

$$\nu\rho > 4\gamma. \tag{2.12}$$

Notice that in the framework of *activator/inhibitor reaction-diffusion systems* this means that b is an activator, see [24]. Now let us check that $(b_0, w_0) = (0,0)$ is a subsolution and that $(b^0, w^0) = (1, w^*)$, with $w^* > 0$ the unique solution of the linear problem

$$\begin{cases} -\delta_w \Delta w^* + \gamma w^* + \mathcal{E}_1 w^* = p(x) & \text{in } \Omega, \\ w^* = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

is a supersolution. Indeed,

$$\begin{cases} -\delta_b \Delta b_0 + \mu(x) b_0 - \nu w_0 b_0 (1 - b_0) = 0 \\ -\delta_b \Delta b^0 + \mu(x) b^0 - \nu w_0 b^0 (1 - b^0) = \mu(x) > 0 \\ -\delta_w \Delta w_0 + \gamma b_0 w_0 + \mathcal{E}_{b_0} w_0 - p(x) = -p(x) \leq 0 \\ -\delta_w \Delta w^0 + \gamma b^0 w^0 + \mathcal{E}_{b^0} w^0 - p(x) = 0 \end{cases}$$

and then there exists at least a solution $(\overline{b}, \overline{w})$ such that $0 \leq \overline{b} \leq 1$ and $0 \leq \overline{w} \leq w^*$.

Proposition 2.7. *Under condition* (2.12) *there exists at least a solution* $(\overline{b}, \overline{w})$ *of the system* (2.1) *with* $\delta_h = 0$.

Remark 2.8. Apparently the usual "concavity" condition giving uniqueness of solutions for systems is not satisfied.

If condition (2.12) is not satisfied we should follow a different approach, e.g., coupled sub and supersolutions or Schauder fixed point theorem.

If (2.12) is not satisfied we use Schauder's fixed point theorem with $T : K \to K$, $K = [0, 1] \times [0, W^*] \subset C(\overline{\Omega})^2$ defined by $T(b, w) = (B, W)$ given by

$$\begin{cases} -\delta_b \Delta B + \mu(x) B = \nu w b (1 - b) & \text{in } \Omega, \\ -\delta_w \Delta W + \gamma b W + \mathcal{E}_b W = p(x) & \text{in } \Omega, \\ B = W = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

and where W^* is the unique solution of

$$\begin{cases} -\delta_w \Delta W^* = p(x) & \text{in } \Omega, \\ W^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

It is clear that $0 \leq B \leq 1$ and we have

$$\begin{cases} -\delta_w \Delta (W^* - W) = \gamma b W + \mathcal{E}_b W > 0 & \text{in } \Omega, \\ W - W^* = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.16}$$

Hence, $W \leq W^*$ by the Maximum Principle. We do not prove that T is compact. The reason is that a similar result will be proved later in a more involved framework.

2.5. Solutions with $\delta_w \equiv 0$

In this case $\gamma b w + \mathcal{E}_b w = \mathcal{I}_b \sqrt{h}$, which gives

$$w = \frac{\mathcal{I}_b \sqrt{h}}{\gamma b + \mathcal{E}_b}$$

and the corresponding problem is

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = \dfrac{\nu \mathcal{I}_b \sqrt{h}}{\gamma b + \mathcal{E}_b} b(1-b) & \text{in } \Omega, \\ -\delta_h \Delta h + \mathcal{I}_b \sqrt{h} = p(x) & \text{in } \Omega, \\ b = h = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.17}$$

The system is not cooperative since

$$\frac{\partial}{\partial b}\{p(x) - \mathcal{I}_b \sqrt{h}\} = -\mathcal{I}_b' \sqrt{h} < 0.$$

Again we should use Schauder's fixed point theorem. We will take $K = [0,1] \times [0, h^*]$ where h^* is given by

$$\begin{cases} -\delta_h \Delta h^* = p(x) & \text{in } \Omega, \\ h^* = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.18}$$

and define $T(b,h) = (B,H)$ from the system

$$\begin{cases} -\delta_b \Delta B + \mu(x) B = \dfrac{\nu \mathcal{I}_b \sqrt{h}}{\gamma b + \mathcal{E}_b} b(1-b) & \text{in } \Omega, \\ -\delta_h \Delta H + \mathcal{I}_b \sqrt{H} = p(x) & \text{in } \Omega, \\ B = H = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.19}$$

As above, $0 \le B \le 1$ and

$$\begin{cases} -\delta_h \Delta (h^* - H) = \mathcal{I}_b \sqrt{H} > 0 & \text{in } \Omega, \\ H - h^* = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.20}$$

gives $H \le h^*$.

2.6. Solutions with h fixed

In this case the system (2.1) is reduced to

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = \nu w b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w = -\gamma b w - \mathcal{E}_b w + \mathcal{I}_b \sqrt{h} & \text{in } \Omega, \\ -\delta_h \Delta h + \mathcal{I}_b \sqrt{h} = p(x) & \text{in } \Omega, \\ b = w = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.21}$$

Once again, we check if this system is cooperative. First we have that

$$\frac{\partial}{\partial w}(\nu w b(1-b)) \ge 0 \qquad \forall b \in [0,1].$$

However

$$\frac{\partial}{\partial b}(-\gamma b w - \mathcal{E}_b w + \mathcal{I}_b \sqrt{h}) = -\gamma w - \mathcal{E}_b' w + \mathcal{I}_b' \sqrt{h}$$

which is positive if $-\mathcal{E}_b' w > \gamma$ for any $b \in [0,1]$, which holds as above under condition (2.12).

Now we can see that $(0,0)$ is a subsolution and $(1, w^*)$, where w^* is the unique solution of

$$\begin{cases} -\delta_w \Delta w^* = \mathcal{I}_1 \sqrt{h} & \text{in } \Omega, \\ \qquad w^* = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

is a supersolution. Indeed, $\forall b \in [0,1]$

$$-\delta_w \Delta w^* + \gamma b w^* + \mathcal{E}_b w^* - \mathcal{I}_1 \sqrt{h} = (\mathcal{I}_1 - \mathcal{I}_b)\sqrt{h} + \gamma b w^* + \mathcal{E}_b w^* > 0,$$

since $\mathcal{I}_b$ is increasing. Thus we have proved

Proposition 2.9. *Under condition (2.12) there exists at least a solution of system (2.21) for $h > 0$ given.*

3. Existence when $min(\delta_b, \delta_w, \delta_h) > 0$: the local system

We study now the full version of the system

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = \nu w b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w = -\gamma b w - \mathcal{E}_b w + \mathcal{I}_b \sqrt{h} & \text{in } \Omega, \\ -\delta_h \Delta h = -\mathcal{I}_b \sqrt{h} + p(x) & \text{in } \Omega, \\ b = w = h = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Since $\mathcal{I}_b' > 0$, the right hand side of the last equation is not increasing in b, the system is not cooperative. We apply Schauder's fixed point theorem. For this, we define the nonlinear operator $T : E \to E$, where $E = C(\overline{\Omega})^3$ by

$$T(b, w, h) = (B, W, H),$$

where (B, W, H) is the solution of the system

$$\begin{cases} -\delta_b \Delta B + \mu(x) B = \nu w b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta W + \gamma b W + \mathcal{E}_b W = \mathcal{I}_b \sqrt{h} & \text{in } \Omega, \\ -\delta_h \Delta H + \mathcal{I}_b \sqrt{H} = p(x) & \text{in } \Omega, \\ B = W = H = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

It is clear that this system has a unique solution from the linear theory and Lemma 2.2. Now, we look for $K \subset E$, K convex, bounded, closed and such that $T(K) \subset K$ with T continuous and compact. To define K we take H^0 as the unique solution $H^0 > 0$ of

$$\begin{cases} -\delta_h \Delta H^0 = p(x) & \text{in } \Omega, \\ H^0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

(recall that $\mathcal{I}_1 > 0$) and with $W^0 > 0$ the unique solution to

$$\begin{cases} -\delta_w \Delta W^0 = \mathcal{I}_1 \sqrt{h} & \text{in } \Omega, \\ W^0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

We define $\mathcal{K}$ =$[0,1]\times[0,W^0]\times[0,H^0]$, which is convex, bounded and closed in E. Let us check that $T(K)\subset K$. We have already seen that $0\leq B\leq 1$ if $0\leq b\leq 1$. Next we have, from (2.21),

$$\begin{cases} -\delta_h\Delta(H^0-H)=\mathcal{I}_b\sqrt{H} & \text{in } \Omega,\\ H^0-H=0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

and by the Maximum Principle $H^0\geq H$. Finally, we also have

$$\begin{cases} -\delta_w\Delta(W^0-W)=+\gamma bW+(\mathcal{I}_1\sqrt{H^0}-\mathcal{I}_b\sqrt{H})>0 & \text{in } \Omega,\\ W^0-W=0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

and again, from the Maximum Principle $W^0\geq W$ follows.

The proof of the compactness of the operator K follows from the fact that when the right hand side data (b,w,h) are uniformly bounded in $E=C(\overline{\Omega})^3$ then we know that the solutions are also uniformly bounded in E which implies the same property for $(\Delta B,\Delta W,\Delta H)$. Then by the linear theory (see, e.g., the exposition made in [5]) (B,W,H) are uniformly bounded in $W^{2,p}(\Omega)$ for any $p\geq 1$ which implies the equicontinuity and the Ascoli-Arzelà result leads to the compactness of K. In order to prove the continuity of the operator T it suffices to apply Proposition 6 of [6]) and then, if $f(b)=b(1-b)$,

$$\|B_1-B_2\|_{L^\infty(\Omega)}\leq C_1\|f(b_1)-f(b_2)\|_{L^\infty(\Omega)}\leq C_1\omega\|b_1-b_2\|_{L^\infty(\Omega)},$$

where $C_1=\nu\max_{i=1,2}\|w_i\|_{L^\infty(\Omega)}$ and ω is the Lipschitz constant of the nonlinear function $f(b)$. Analogously,

$$\|W_1-W_2\|_{L^\infty(\Omega)}\leq C_2\left\|\sqrt{h_1}-\sqrt{h_2}\right\|_{L^\infty(\Omega)}\leq C_2\sqrt{\|h_1-h_2\|_{L^\infty(\Omega)}}$$

where $C_2=\max_{i=1,2}\|\mathcal{I}_{b_i}\|_{L^\infty(\Omega)}$. Finally, since $\mathcal{I}_b$ is a Lipschitz function of b we have

$$-\delta_h\Delta(H_1-H_2)+\mathcal{I}_{b_1}(\sqrt{H_1}-\sqrt{H_2})=(\mathcal{I}_{b_2}-\mathcal{I}_{b_1})\sqrt{H_2}\leq C_3C_4\|b_1-b_2\|_{L^\infty(\Omega)},$$

where C_3 is the Lipschitz constant of function $\mathcal{I}_b$ and $C_4\geq\|H_2\|_{L^\infty(\Omega)}^{1/2}$. Then, arguing as in the proof of Proposition 6 of [6] we get

$$\|H_1-H_2\|_{L^\infty(\Omega)}\leq C_3C_4\|b_1-b_2\|_{L^\infty(\Omega)}$$

(notice that the above inequalities are connected with the accretiveness in $L^\infty(\Omega)$ of each one of the scalar operators associated to the equations of W and H, and with the $\omega-$accretiveness in $L^\infty(\Omega)$ of the operator associated to B: see, e.g., [4]). Thus, since, in fact, $(b_i,w_i,h_i),(B,W,H)\in E$, for $i=1,2$, we get the continuity of T.

4. Existence: the non-local system when $min(\delta_b, \delta_w, \delta_h) > 0$

We study now the original non-local system

$$\begin{cases} -\delta_b \Delta b + \mu(x) b = G_b b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta w + G_w w + \mathcal{E}_b w = \mathcal{I}_b \sqrt{h} & \text{in } \Omega, \\ -\delta_h \Delta h + \mathcal{I}_b \sqrt{h} = p & \text{in } \Omega, \\ b = w = h = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

We have already seen that the system (4.1) is not cooperative. We proceed in a very similar way as in the non-local case. The nonlinear operator $T : E \to E$, where $E = C(\overline{\Omega})^3$, is defined by

$$T(b, w, h) = (B, W, H),$$

where (B, W, H) is the solution of the system

$$\begin{cases} -\delta_b \Delta B + \mu(x) B = G_b(x) b(1-b) & \text{in } \Omega, \\ -\delta_w \Delta W + G_w(x) W + \mathcal{E}_b W = \mathcal{I}_b \sqrt{h} & \text{in } \Omega, \\ -\delta_h \Delta H + \mathcal{I}_b \sqrt{H} = p(x) & \text{in } \Omega, \\ B = W = H = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

The system (4.2) has a unique solution: this follows from linear theory (coefficients $G_b(x)$ and $G_w(x)$ are smooth enough) and Lemma 2.2. Again, the Maximum Principle yields $0 \le B \le 1$ if $0 \le b \le 1$. We define $H^0 > 0$ as above, the same for $W^0 > 0$. Both are positive by the Maximum Principle (notice that $G_b(x)$, $G_w(x) \ge 0$).

We consider $\mathcal{K} =[0, 1] \times [0, W^0] \times [0, H^0]$ again. The proof that $H^0 \ge H$ is the same. It is only slightly different for $W^0 \ge W$. We have

$$\begin{cases} -\delta_w \Delta(W^0 - W) = G_w(x) W + \mathcal{E}_b W + (\mathcal{I}_1 \sqrt{H^0} - \mathcal{I}_b \sqrt{h}) > 0 & \text{in } \Omega, \\ W^0 - W = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

and the comparison follows. The proof of the compactness and continuity of the operator T is exactly the same than the one given in the above sections since only a priori estimates in the coefficients of the system were used in the arguments of the proof.

5. Multiplicity of solutions when $\delta_b > 0$ and $\delta_w = 0$

We study in this Section the existence of positive solutions to the system when $\delta_b > 0$ and $\delta_w = 0$.

Let us start by assuming also that $\delta_h = 0$ and we will consider the case $\delta_h > 0$ to the end of the Section. As above, μ and p are smooth functions such that $0 \le \mu(x) \le \bar{\mu}$ and $p(x) > 0$ on Ω. By simplicity we assume $\delta_b = 1$. We introduce a real parameter λ in the equation

$$\begin{cases} -\Delta b + \mu(x) b = \lambda p(x) f(b) & \text{in } \Omega, \\ b = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where

$$f(b) = \frac{\nu b(1-b)(1+\rho b)}{\nu + \gamma b(1+\rho b)},$$

with $\nu, \gamma, \rho > 0$. It is clear that $f(0) = f(1) = 0$, $f'(0) = 1$ and $f(b) < 0$ for $b > 1$. As above solutions satisfy $0 \le b \le 1$ and, by the Strong Maximum Principle, if $b \ge 0$ is a solution $b > 0$ in Ω and $\dfrac{\partial b}{\partial n} < 0$ on $\partial\Omega$.

We have a first existence result by using sub and supersolutions. The linear eigenvalue problem

$$\begin{cases} -\Delta\varphi + \mu(x)\varphi = \lambda p(x)\varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

has a first positive eigenvalue $\lambda_1 > 0$ with eigenfunction $\varphi_1 > 0$ such that $\|\varphi_1\|_{L^\infty} = 1$. We point out that this holds even if $p(x)$ vanishes on $\partial\Omega$ (a case which will be considered in Section 6 below): see, e.g., [17] and its references.

Theorem 5.1. *For any $\lambda > \lambda_1$ there exists at least a positive solution of* (5.1).

Proof. It is easy to see that $b^0 \equiv 1$ is a supersolution. We look for subsolutions of the form $b_0 = c\varphi_1$, $c > 0$. We have

$$\begin{aligned} -\Delta b_0 + \mu(x)b_0 - \lambda p(x)f(b_0) &= c\lambda_1 p(x)\varphi_1 - \lambda p(x)f(b_0) \\ &= c\lambda_1 p(x)\varphi_1 - \lambda p(x)c\varphi_1 - \lambda p(x)(f(c\varphi_1) - c\varphi_1) \\ &= (\lambda_1 - \lambda)p(x)c\varphi_1 + o(c\varphi_1) < 0 \end{aligned}$$

for $c > 0$ small. ☐

We study the uniqueness of the positive solutions. For this we try to check if the well-known "concavity" condition holds, [7, 8]. This condition reads

$$\left(\frac{f(b)}{b}\right)' = \frac{-\gamma\rho^2 b^2 - 2\rho(\nu+\gamma)b + \nu(\rho - 1) - \gamma}{(\nu + \gamma b + \rho\gamma b^2)^2} < 0 \ \forall b \in [0,1],$$

or equivalently

$$\gamma\rho^2 b^2 + 2\rho(\nu+\gamma)b + \nu(1-\rho) + \gamma > 0, \ \forall b \in [0,1]. \tag{5.3}$$

We consider two cases, the first is when

$$\nu(1-\rho) + \gamma > 0. \tag{5.4}$$

This means that both real roots of (5.3) have the same sign, which is actually negative. Hence $\left(\dfrac{f(b)}{b}\right)' < 0$ for any $b \in [0,1]$ and we get uniqueness. The second case is when

$$\nu(1-\rho) + \gamma < 0. \tag{5.5}$$

Now both roots of (5.3) have opposite sign and the uniqueness condition is not satisfied. We have thus proved the first part of the

Theorem 5.2. *If (5.4) holds there exists a unique positive solution for any $\lambda > \lambda_1$ and there is no solution if $\lambda \leq \lambda_1$. Moreover, this unique solution is linearly stable.*

Proof. First, it remains to show that there is no solution if $\lambda < \lambda_1$. Indeed, assume that $b > 0$ is a solution. Since $\left(\frac{f(b)}{b}\right)' < 0$, $f(b) \leq b$ for $0 \leq b \leq 1$. If we multiply (5.1) by b and integrate by parts on Ω we get

$$\int_\Omega (|\nabla b|^2 + \mu(x)b^2) = \lambda \int_\Omega p(x)f(b)b \leq \lambda \int_\Omega p(x)b^2$$

and hence

$$\lambda_1 \leq \frac{\int_\Omega (|\nabla b|^2 + \mu(x)b^2)}{\int_\Omega p(x)b^2} = \lambda.$$

If $\bar{b}$ is a solution to (5.1), the corresponding linearized eigenvalue problem is

$$\begin{cases} -\Delta z + \mu(x)z - \lambda p(x)f'(\bar{b})z = \upsilon z & \text{in } \Omega, \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

If υ_1 is the first eigenvalue to (5.6) with eigenfunction $\psi_1 > 0$, $\|\psi_1\|_{L^\infty} = 1$, we have

$$\begin{cases} -\Delta\psi_1 + \mu(x)\psi_1 - \lambda p(x)f'(\bar{b})\psi_1 = \upsilon_1\psi_1 & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.7}$$

Multiplying (5.1) by ψ_1 and (5.7) by $\bar{b}$ and integrating on Ω with Green's formula we obtain

$$\int_\Omega \nabla\bar{b}\cdot\nabla\psi_1 + \int_\Omega \mu(x)\bar{b}\psi_1 - \int_\Omega \lambda p(x)f(\bar{b})\psi_1 = 0$$

$$= \int_\Omega \nabla\bar{b}\cdot\nabla\psi_1 + \int_\Omega \mu(x)\bar{b}\psi_1 - \lambda\int_\Omega p(x)\bar{b}f'(\bar{b})\psi_1 - \upsilon_1\int_\Omega \bar{b}\psi_1$$

and hence

$$\upsilon_1 = \frac{\lambda\int_\Omega p(x)\left[f(\bar{b}) - \bar{b}f'(\bar{b})\right]\psi_1}{\int_\Omega \bar{b}\psi_1}. \tag{5.8}$$

Since $\left[f(\bar{b}) - \bar{b}f'(\bar{b})\right] > 0$ from the uniqueness condition, $\upsilon_1 > 0$, which ends the proof. □

Remark 5.3. Actually the above computation in the proof of Theorem 5.2 shows that 0 is not an eigenvalue of the linearization along a solution $\bar{b}$ of (5.1). This, together with an application of the well-known Crandall-Rabinowitz local inversion theorem ([9]) at the simple eigenvalue λ_1 and the Implicit Function Theorem shows that the branch of solutions $\lambda \to \bar{b}(\lambda)$ is a smooth mapping in some function space. We skip the details.

It remains to study the case (5.5). We start by showing the existence of an unbounded continuum of positive solutions bifurcating from λ_1 in both cases (5.4) and (5.5).

Since solutions satisfy $0 \leq b \leq 1$, we replace f by the continuous function $\overline{f}(b) = f(b)$ if $0 \leq b \leq 1$, $\overline{f}(b) = 0$ if $b > 1$. It is clear that the associate Nemitskii operator $\overline{F} : C(\overline{\Omega}) \to C(\overline{\Omega})$, $\overline{F}(u)(x) = \overline{f}(u(x))$ is well-defined and continuous. If we denote by K the solution operator of

$$\begin{cases} -\Delta u + \mu(x)u = \lambda p(x)h(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.9}$$

for any $h \in C(\overline{\Omega})$, i.e., $K = (-\Delta + \mu(x)I)^{-1} : C(\overline{\Omega}) \to C(\overline{\Omega})$ is well-defined and continuous as follows easily from the classical linear theory and (5.1) can be written equivalently as $b = \lambda K\overline{F}(b)$ in $C(\overline{\Omega})$. Since $K\overline{F}$ is a compact and positive operator and right-differentiable at $b = 0$, we can apply the global bifurcation result (Theorem 18.3 in [2], see also [11]) and get the following

Theorem 5.4. *There exists an unbounded continuum of positive solutions of* (5.1) *bifurcating from the line of trivial solutions at the point* $(\lambda_1, 0)$.

Now we see that if (5.4) holds this continuum coincides with the branch of positive solutions for $\lambda > \lambda_1$, previously obtained.

Next we study the case where (5.5) holds. Let us consider firstly the associate heuristics. The McLaurin expansion of the function f is actually

$$f(b) = b + ((\rho - 1) - \frac{\gamma}{\nu})b^2 + \cdots$$

By the way,

$$f''(0) = 2((\rho - 1) - \frac{\gamma}{\nu}).$$

We see immediately that (5.4) (resp. (5.5)) holds if $(\rho - 1) - \frac{\gamma}{\nu} < 0$ (resp. $(\rho - 1) - \frac{\gamma}{\nu} > 0$). Heuristics tells us that for "small" solutions the equation (5.1) can be "approximated" by

$$\begin{cases} -\Delta b + \mu(x)b = \lambda p(x)\left[b + ((\rho - 1) - \frac{\gamma}{\nu})b^2\right] & \text{in } \Omega, \\ b = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.10}$$

If (5.4) holds, (5.10) is the logistic equation and we have obtained the corresponding results in Theorems 5.1 and 5.2.

If (5.5) holds, (5.10) is the well-known semilinear subcritical equation with a branch of positive solutions "bifurcating" to the left at $\lambda = \lambda_1$. Since the continuum in Theorem 5.4 should go to infinity as $\lambda \to +\infty$ (see Lemma 5.6 below), it seems that there should be at least two solutions in a left-neighborhood of λ_1.

Moreover, if (5.5) holds it is easy to see that, if $b_2 > 0$ is the only positive solution of

$$\gamma\rho^2 b^2 + 2\rho(\nu + \gamma)b + \nu(1 - \rho) + \gamma = 0, \tag{5.11}$$

we have $0 < b_2 < 1$. Indeed, if not

$$1 < \frac{-2\rho(\nu + \gamma) + \sqrt{4\rho^2(\nu + \gamma)^2 - 4\gamma\rho^2(\nu(1 - \rho) + \gamma)}}{2\gamma\rho^2}$$

is equivalent to

$$\gamma\rho^2 + 2(\nu+\gamma) < \nu(1-\rho) + \gamma < 0,$$

a contradiction since all coefficients are positive. Hence the integral in (5.8) is negative if $\|b\|_{L^\infty} < b_2$, i.e., such "small" solutions (if they exist!) should be linearly unstable, something which fits well with the left bifurcation above argument.

Next, we give a rigorous proof of the heuristic results.

Theorem 5.5. *If* (5.5) *holds, then there exists a smooth curve of positive solutions of* (5.1) *bifurcating to the left from* λ_1 *and there exists a positive value* $\lambda^* < \lambda_1$ *such that no positive solution* b *is possible if* $\lambda \in [0, \lambda^*)$. *Solutions on this branch of small norm are linearly unstable.*

Let us start by proving that the possible bifurcation branch does not touch the axis $\lambda = 0$ since no positive solution b is possible if λ is small enough. That was already shown when (5.4) holds. Let us prove it for the case in which (5.5) is satisfied

Lemma 5.6. *Assume* (5.5). *Then there exists a positive value* $\lambda^* < \lambda_1$ *such that no positive solution* b *is possible if* $\lambda \in [0, \lambda^*)$.

Proof. Multiplying the equation by Φ_1 with

$$\begin{cases} -\Delta\Phi_1 = \widetilde{\lambda}_1\Phi_1 & \text{in } \Omega, \\ \Phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.12}$$

and integrating on Ω we get

$$\widetilde{\lambda}_1 \int_\Omega b\Phi_1 + \int_\Omega \mu(x) b\Phi_1 = \lambda \int_\Omega p(x) f(b)\Phi_1 \,.$$

Thus, necessarily

$$\int_\Omega \Phi_1 \left[(\widetilde{\lambda}_1 + \mu(x))b - \lambda p(x) f(b) \right] = 0 \,. \tag{5.13}$$

But from (6.3)

$$0 \le p(x) \le \|p\|_{L^\infty(\overline{\Omega})} := \overline{p},$$

for any $x \in \Omega$. Then, (5.13) is clearly impossible if

$$\lambda\overline{p}\frac{f(b)}{b} \le \widetilde{\lambda}_1 \text{ for } b \in (0,1).$$

Moreover, since we can assume (thanks to (5.5)) that $\dfrac{f(b)}{b} \le M$ for $b \in (0,1)$, for some $M > 1$, we get that no positive solution may exists if

$$0 \le \lambda \le \frac{\widetilde{\lambda}_1}{\overline{p}M} \,.$$

□

To prove the rest of conclusions of Theorem 5.5 we will apply a local inversion result by Crandall-Rabinowitz [10]:

Theorem 5.7. *[10]. Let X and Y be real Banach spaces, let $I \subset \mathbb{R}$ be a bounded interval and $F : I \times X \to Y$, $F \in C^2$, let $\lambda_0 \in I$ and assume that satisfies*
i) $F(\lambda, 0) = 0$ for every $\lambda \in I$;
ii) $\dim\ Ker\ F_x(\lambda_0, 0) = co\dim R(F_x(\lambda_0, 0)) = 1$;
iii) $F_{\lambda x}(\lambda_0, 0)x_0 \notin R(F_x(\lambda_0, 0))$, where $Ker\ F_x(\lambda_0, 0) = [x_0]$.
Let Z be a complementary subespace of $[x_0]$ in X. Then there exists an interval J such that $0 \in J$ and C^1 functions $\lambda : J \to \mathbb{R}$ and $\psi : J \to Z$ such that $\lambda(0) = \lambda_0$, $\psi(0) = 0$ and $x(s) = sx_0 + s\psi(s)$ implies $F(\lambda(s), x(s)) = 0$. Moreover $F^{-1}(0)$ is uniquely formed (in a neighborhood of $(\lambda_0, 0)$) by the curves $x = 0$ and $(\lambda(s), x(s))$, $s \in J$.

Proof of Theorem 5.5. To complete the proof we use Theorem 5.7 with $\lambda_0 = \lambda_1$, $X = C_0^{2,\alpha}(\overline{\Omega}) = \{u \in C^{2,\alpha}(\overline{\Omega}) \mid u = 0 \text{ on } \partial\Omega\}$, $Y = C^\alpha(\overline{\Omega})$, for some $\alpha \in (0, 1)$ and

$$F(\lambda, u) = -\Delta u + \mu(x)u - \lambda p(x)f(u).$$

It is easy to see that $F \in C^2$ (actually C^∞). We have

$$\begin{aligned} F_u(\lambda, u)v &= -\Delta v + \mu(x)v - \lambda p(x)f'(u)v \\ F_{\lambda u}(\lambda, u)v &= -p(x)f'(u)v \\ F_{u\ u}(\lambda, u)(v, w) &= -\lambda p(x)f''(u)vw. \end{aligned}$$

Also, we can see that $Ker\ F_u(\lambda_1, 0) = [\varphi_1]$ and

$$R(F_u(\lambda_1, 0)) = \left\{ u \in C^\alpha(\overline{\Omega}) \;\middle|\; \int_\Omega u\varphi_1 = 0 \right\}.$$

Moreover $F_{\lambda u}(\lambda_1, 0)\varphi_1 \notin R(F_u(\lambda_1, 0))$ since $\int_\Omega p(x)\varphi_1^2 \neq 0$. Hence λ_1 is a bifurcating point with a smooth curve bifurcating from λ_1 with $u(s) > 0$ for $0 < s < s_0$, for some $s_0 > 0$.

That this curve bifurcates to the left follows, e.g., from the results in [1, pp. 96-97] since the curve is given by

$$\lambda = \lambda_1 - \frac{b}{a}s + o(s)$$

with

$$\begin{aligned} a &= \langle F_{\lambda u}(\lambda_1, 0)\varphi_1, \varphi_1 \rangle = -\textstyle\int_\Omega p(x)\varphi_1^2 < 0, \\ b &= \tfrac{1}{2}\langle F_{u\ u}(\lambda_1, 0)(\varphi_1, \varphi_1), \varphi_1 \rangle = -\tfrac{1}{2}\textstyle\int_\Omega p(x)f''(0)\varphi_1^3 < 0, \end{aligned}$$

since $f''(0) < 0$ by (5.5).

That these solutions are linearly unstable if $\|u\|_{L^\infty} \leq b_2$ (b_2 defined above: see (5.11) follows from (5.8). □

Remark 5.8. If $\nu(1-\rho)+\gamma = 0$ it is necessary to use the local bifurcation results in, e.g., [1, p. 97] involving $f'''(0)$.

Theorem 5.9. *If* (5.5) *holds there exists* $0 < \overline{\lambda} < \lambda_1$ *such that there are (at least) two positive solutions to* (5.1) *for* $\overline{\lambda} < \lambda < \lambda_1$*.*

Proof. The result follows from Theorems 5.5, 5.4, Lemma 5.6 and the fact that $0 \le b < 1$. □

We add another proof of existence of positive solutions to the left of λ_1. This is done under very special conditions on the coefficients $\mu(x)$ and $p(x)$. However we include it for the sake of completeness concerning the application of different methods. We shall built a family of subsolutions of the form $\underline{b}_\lambda(x) = k(\lambda)\psi_1(x)$, where $k(\lambda) \in (0,1)$ will be determined later and $\psi_1(x)$ is the normalized ($\psi_1 > 0$, $\|\psi_1\|_{L^\infty} = 1$) first eigenfunction associated to the first eigenvalue $\widehat{\lambda}_1$ of the auxiliary problem (without absorption term)

$$\begin{cases} -\Delta\psi_1 = \widehat{\lambda}_1 p(x)\psi_1 & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.14}$$

Notice that since $\underline{\mu} > 0$ and

$$\widehat{\lambda}_1 = \inf_{\psi\neq 0} \frac{\int_\Omega |\nabla\psi|^2}{\int_\Omega p(x)\psi^2} \text{ and } \lambda_1 = \inf_{\psi\neq 0} \frac{\int_\Omega (|\nabla\psi|^2 + \mu(x)\psi^2)}{\int_\Omega p(x)\psi^2}$$

we obviously know that $\widehat{\lambda}_1 < \lambda_1$. We shall assume now the stronger condition

$$\widehat{\lambda}_1 + C_{\mu,p} < \lambda_1, \tag{5.15}$$

where

$$C_{\mu,p} = \max_{x\in\overline{\Omega}} \frac{\mu(x)}{p(x)}.$$

Since, $\lambda_1 - \widehat{\lambda}_1$ is essentially related to the expression $\int_\Omega \mu(x)\psi^2 / \int_\Omega p(x)\psi^2$ and we have, for $\psi \in H_0^1(\Omega) \subset L^{2^*}(\Omega)$, $2^* = 2n/(n-2)$ if $n \ge 2$,

$$\frac{\int_\Omega \mu(x)\psi^2}{\int_\Omega p(x)\psi^2} \ge \frac{\underline{\mu}\int_\Omega \psi^2}{\left[\int_\Omega p(x)^q\right]^{\frac{1}{q}} \|\psi\|_{L^{2^*}}^2}, \text{ with } q = 2^*/(2^*-2),$$

then assumption (5.15) requires that $p(x)$ is essentially concentrated in Ω around the maximum of μ and with $\overline{p} >> \underline{p}$, where $\underline{p} = \min_{x\in\overline{\Omega}} p(x)$ (notice that if $p(x)$ and $\mu(x)$ are constant functions then (5.15) cannot be satisfied). Analogously, if we assume $p(x)$ almost a constant function then (5.15) requires that $\mu(x)$ is essentially concentrated on Ω around its minimum value $\underline{\mu}$ and with $\underline{\mu} << \overline{\mu}$ where $\overline{\mu} = \max_{x\in\overline{\Omega}} \mu(x)$.

Theorem 5.10. *Assume* (5.5) *and* (5.15)*. Then, for any* $\lambda \in (\widehat{\lambda}_1 + C_{\mu,p}, \lambda_1)$ *there exists at least one nontrivial solution* b *of problem* (5.1)*.*

Proof. Since we know that $\overline{b} \equiv 1$ is a supersolution of problem (5.1), by applying the method of super and subsolutions it is enough to built a branch of positive subsolutions for $\lambda \in (\widehat{\lambda}_1 + C_{\mu,p}, \lambda_1)$. We try with functions of the form $\underline{b}_\lambda(x) = k(\lambda)\psi_1(x)$. We have

$$-\Delta \underline{b}_\lambda + \mu(x)\underline{b}_\lambda = (\widehat{\lambda}_1 + \frac{\mu(x)}{p(x)})kp(x)\psi_1 \leq (\widehat{\lambda}_1 + C_{\mu,p})kp(x)\psi_1.$$

On the other hand, since $0 < \underline{b}_\lambda(x) \leq k(\lambda) < 1$,

$$\lambda p(x) f(\underline{b}_\lambda) = \lambda p(x)\left[\frac{\nu \underline{b}_\lambda(1-\underline{b}_\lambda)(1+\rho\underline{b}_\lambda)}{\nu + \gamma\underline{b}_\lambda(1+\rho\underline{b}_\lambda)}\right] \geq \lambda p(x)\psi_1(x)\left[\frac{\nu k(1-k)}{\nu+\gamma k(1+\rho k)}\right].$$

Thus, if we take $k(\lambda)$ such that

$$(\widehat{\lambda}_1 + C_{\mu,p})k = \lambda \frac{\nu k(1-k)}{\nu + \gamma k(1+\rho k)} \tag{5.16}$$

then $\underline{b}_\lambda(x)$ becomes a subsolution since

$$\begin{cases} -\Delta \underline{b}_\lambda + \mu(x)\underline{b}_\lambda \leq \lambda p(x) f(\underline{b}_\lambda) & \text{in } \Omega, \\ \underline{b}_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.17}$$

That equation (5.16) admits a positive solution $k(\lambda)$ for $\lambda \in (\widehat{\lambda}_1 + C_{\mu,p}, \lambda_1)$ is easy to check in the special case $\gamma = 0$, since then the equation leads simply to the expression

$$k(\lambda) = 1 - \frac{\widehat{\lambda}_1 + C_{\mu,p}}{\lambda}$$

and by assumption (5.15) $k(\lambda) > 0$ if $\lambda \in (\widehat{\lambda}_1 + C_{\mu,p}, \lambda_1]$ (and $k(\widehat{\lambda}_1 + C_{\mu,p}) = 0$). The case $\gamma > 0$ small (such that (5.5) holds) is similar. Indeed, now condition (5.16) can be rewritten as

$$\gamma\rho\left[\frac{\widehat{\lambda}_1 + C_{\mu,p}}{\lambda}\right]k^2 + \left(\gamma\left[\frac{\widehat{\lambda}_1 + C_{\mu,p}}{\lambda}\right] + \nu\right)k + \nu\left[\frac{\widehat{\lambda}_1 + C_{\mu,p}}{\lambda} - 1\right] = 0$$

which have a positive root $k(\lambda)$, for $\lambda \in (\widehat{\lambda}_1 + C_{\mu,p}, \lambda_1]$, given by

$$k(\lambda) = \frac{\sqrt{\left(\gamma\left[\frac{\widehat{\lambda}_1+C_{\mu,p}}{\lambda}\right]+\nu\right)^2 + 4\gamma\rho\left[\frac{\widehat{\lambda}_1+C_{\mu,p}}{\lambda}\right]\nu\left[1 - \frac{\widehat{\lambda}_1+C_{\mu,p}}{\lambda}\right]} - \left(\gamma\left[\frac{\widehat{\lambda}_1+C_{\mu,p}}{\lambda}\right]+\nu\right)^2}{2\gamma\rho\left[\frac{\widehat{\lambda}_1+C_{\mu,p}}{\lambda}\right]}.$$

Thus, we can apply the method of super and subsolutions and we get the result. □

Remark 5.11. It seems possible to get some similar results by taking other subsolutions as for instance $\underline{b}_\lambda(x) = k(\lambda)\psi_1(x)$ with $-\Delta\psi_1 + \overline{\mu}\psi_1 = \widehat{\lambda}\underline{p}\psi_1$ under some suitable additional assumption of the type (5.15).

Remark 5.12. In previous studies of the system, corresponding to Neumann boundary conditions (see [19, 20, 15] and the computational examples in [18, 23]), it was not indicated the starting value (from λ_1) of the bifurcation diagram, neither its characterization as the supercritical (resp. subcritical) nature of the bifurcation in terms of the positivity (resp. negativity) of the parameters balance expression $\nu(1-\rho)+\gamma$.

Remark 5.13. The above study can be extended to the case of other kinds of weak solutions, for instance when $p(x)$ is assumed to be merely in $h \in L^1(\Omega, d)$, but we will not enter into the details.

Remark 5.14. Most of the existence results of this paper can be extended to the case in which instead of Dirichlet boundary conditions we have Robin boundary conditions

$$\frac{\partial b}{\partial n} + \omega_b(x)b = 0,\ \frac{\partial w}{\partial n} + \omega_w(x)w = 0,\ \frac{\partial w}{\partial n} + \omega_h(x)w = 0 \text{ on } \partial\Omega,$$

with $\omega_b, \omega_w, \omega_h > 0$ given and smooth, and also when the boundary conditions are of mixed type (some equations with Dirichlet boundary conditions and the rest with Robin ones). The main reason is that the comparison of solutions remains valid and the rest of the arguments can be easily adapted to this case. As a matter of fact, nonlinear boundary conditions given in terms of maximal monotone graphs are also possible (see, e.g., [12] and its references) when dealing with other notions of weak solutions. The possible extension of the multiplicity results of this paper to the case of other boundary conditions is more delicate since the construction of subsolutions requires to be well adapted to the boundary conditions. In [15] this was made by means of constant subsolutions $\underline{b}_\lambda(x) = k(\lambda)$ but no so sharp information on the starting point of the bifurcation diagram was given there.

Remark 5.15. Variational methods may also be applied to our problem (see [15] for the case of Neumann boundary conditions). It is easy to see that the associated functional $E : H_0^1(\Omega) \to \mathbb{R}$ defined as

$$E(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \mu(x)u^2) - \lambda \int_\Omega p(x)F(u),$$

where $F(u) = \int_0^u f(s)ds$ is coercive and that its minimum is 0 for $\lambda < \lambda_1$ if (5.3) holds.

Finally, let us consider now the case $\delta_h > 0$ (always under the assumption that $\delta_b > 0$ and $\delta_w = 0$). As mentioned in Subsection 2.5, the equation satisfied by b is rather similar to the Equation (5.1) since we arrive now to problem

$$\begin{cases} -\Delta b + \mu(x)b = \lambda\sqrt{h(x)}f^*(b) & \text{in } \Omega, \\ b = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.18}$$

where

$$f^*(b) = f(b)I_b(b)$$

with $I_b(b)$ given by (2.2). Notice that $I_b(0) = \dfrac{1}{c} < 1$ and $\lim_{b\to+\infty} I_b(b) = 1$. In addition

$$I_b'(b) > 0$$

so that the qualitative behaviour of $f^*(b)$ is very similar to the one of $f(b)$ and thus the above treatment for the case $\delta_h = 0$ can be easily adapted to the case $\delta_h > 0$ (as we will explain in the following section $\sqrt{h(x)} > 0$ on Ω and thus this term behaves also entirely similar to the function $p(x)$ arising in the formulation of (5.1)).

6. Flat solutions for $p = p(x)$ vanishing on $\partial\Omega$

It seems interesting to consider the case in which the precipitation rate $p(x)$ is not completely constant in Ω, but in fact vanishes outside a closed subset ω of $\mathbb{R}^2$ (the study could be extended to $\mathbb{R}^n$ for any $n \geq 1$).

Notice that the interesting case corresponds now to the case $\delta_h > 0$ since if $\delta_h = 0$ we get that

$$\mathcal{I}_b h(x) = p(x) \text{ for any } x \in \overline{\Omega},$$

and since $\mathcal{I}_b > 0$ we conclude that $h(x) = 0$ if and only if $p(x) = 0$.

Thus, the rest of this Section concerns the case $\delta_h > 0$. The special case in which $p(x) = p\chi_\omega(x)$ on Ω, where χ_ω denotes the characteristic function of a subset $\omega \subset\subset \Omega$ (and with Neumann boundary conditions on $\partial\Omega$) was considered in [15]. In this paper we will extend the mentioned study to the case in which $\omega = \Omega$, i.e. in addition to (6.3) we will assume that

$$p = 0 \text{ on } \partial\Omega. \tag{6.1}$$

Let us consider the case in which $min(\delta_b, \delta_w, \delta_h) > 0$. Notice that by the previous existence theorem we can assume that $b(x) \in [0, 1]$ is a given positive solution of the corresponding equation of the local problem (2.1). Moreover, we know that there exists a positive constant c_b such that

$$c_b d(x) \leq b(x) \leq 1 \text{ for any } x \in \Omega,$$

where $d(x) = d(x, \partial\Omega)$. Indeed, it suffices to apply the strong maximum principle to the equation satisfied by b.

We set

$$\theta(x) := \alpha \frac{b(x) + q/c}{b(x) + q} \quad \text{in } \Omega.$$

Then

$$\alpha \frac{(c_b d(x) + q/c)}{1 + q} \leq \theta(x) \leq \alpha \frac{(1 + q/c)}{c_b d(x) + q} \text{ in } \Omega,$$

and the third equation of (2.1) can be written as

$$\begin{cases} -\Delta h + \dfrac{\theta(x)}{\delta_h}\sqrt{h} = \phi(x) & \text{in } \Omega, \\ h = 0 & \text{on } \partial\Omega, \end{cases} \tag{6.2}$$

with $\phi(x) := \dfrac{p(x)}{\delta_h}$. As mentioned before, for b fixed (i.e., for a given $\theta(x)$) there is a unique solution h of (6.2). The following result gives a sufficient condition on $p(x)$ in order to get that h is a *flat solution* (in the sense that also $\dfrac{\partial h}{\partial n} = 0$ on $\partial\Omega$).

Theorem 6.1. *Let $p(x)$ be satisfying (6.3). Then, $h(x) > 0$ a.e. $x \in \Omega$. If in addition $p(x)$ is such that*

$$0 \le p(x) \le \delta_h K d(x)^2 \text{ in } \Omega, \tag{6.3}$$

for some $K > 0$ small enough, then there exists a constant $C_\sigma^* > 0$ such that

$$0 \le h(x) \le C_\sigma^* d(x)^4 \text{ in } \Omega. \tag{6.4}$$

In particular, h is a flat solution.

Proof. The proof that $h(x) > 0$ a.e. $x \in \Omega$ is an easy consequence of a result due to G. Stampacchia (see, e.g. Lemma A.4 of [21]) since if $h(x) = 0$ on a positively measured subset $E \subset \Omega$ then $\Delta h = 0$ on E and thus $\mathcal{I}_b h(x) = p(x)$ for a a.e. $x \in E$, which implies a contradiction since $p(x) > 0$ on Ω. To complete the proof we will apply the *method of local supersolutions* such as presented in [12]. Let $x_0 \in \partial\Omega$ and define $\Omega_{x_0,R} = \Omega \cap B_R(x_0)$ for some $R > 0$ to be determined later. Observe that since $d(x) \le |x - x_0|$, we have

$$-\Delta h + \frac{\alpha q}{\delta_h c(1+q)}\sqrt{h} \le \phi(x) \le K\,|x - x_0|^2 \text{ in } \Omega_{x_0,R}.$$

Let $\overline{h}(x : x_0) = C\,|x - x_0|^4$. As a consequence of Theorem 1.15 of [12], if we denote $\sigma = \dfrac{\alpha q}{\delta_h c(1+q)}$ then we know that

$$-\Delta\overline{h} + \sigma\sqrt{\overline{h}} = [\sigma\sqrt{C} - (8+4N)C]\,|x - x_0|^2,$$

(in our model $N = 2$ but it is pedagogical to work with an arbitrary $N \ge 1$). The function

$$\theta(C) = \sigma\sqrt{C} - (8+4N)C$$

takes nonnegative values for $C \in [0, C_{N,\sigma}]$ with

$$C_{N,\sigma} = \frac{\sigma^2}{(8+4N)^2},$$

(notice that $\theta(C_{N,\sigma})=0$). Moreover $\theta(C)$ attains its maximum at

$$C^*_{N,\sigma}=\frac{\sigma^2}{4(8+4N)^2}=\frac{C_{N,\sigma}}{4}.$$

Then, a good choice of the constant K mentioned in (6.3) is

$$K=\frac{\theta(C^*_{N,\sigma})}{\delta_h}.$$

In that case we know that

$$-\Delta h+\sigma\sqrt{h}\leq-\Delta\overline{h}+\sigma\sqrt{\overline{h}}\text{ in }\Omega_{x_0,R}.$$

Moreover, clearly $h\leq\overline{h}$ on $\partial\Omega_{x_0,R}\cap\partial\Omega$ and we also have $h\leq\overline{h}$ on $\partial\Omega_{x_0,R}\setminus\partial\Omega$ if, for instance,

$$\|h\|_{L^\infty(\Omega)}\leq C^*_{N,\sigma}R^4. \tag{6.5}$$

Finally, we assume R "large enough" so that

$$R\geq\left[\frac{\|h\|_{L^\infty(\Omega)}}{C^*_{N,\sigma}}\right]^{1/4}$$

and then (6.5) holds. In conclusion, by the maximum principle

$$0\leq h(x)\leq C^*_{N,\sigma}\left|x-x_0\right|^4\ \text{ in }\Omega_{x_0,R},$$

and since $x_0\in\partial\Omega$ is arbitrary this implies (6.4). □

7. Conclusions and some open problems

The local and non-local systems (1.6) and (1.7) were considered for the biomass b, the soil-water content w and the surface-water height h when we assume Dirichlet boundary conditions. The main results of this paper concern the existence of positive solutions when $\delta_b>0$ and $\delta_w=0$ getting both uniqueness and multiplicity results. In contrast to previous results dealing with Neumann boundary condition, we obtain some bifurcation diagrams showing rigorously its starting value (from the first eigenvalue λ_1 of a linear operator with the corresponding weights and with Dirichlet boundary conditions) and characterizing the supercritical (resp. subcritical) nature of the branch (something unnoticed before in the literature) depending on (for instance when $\delta_h=0$) the positivity (resp. negativity) of the parameters balance expression $\nu(1-\rho)+\gamma$. Moreover, we study the case in which $p(x)$ vanishes on $\partial\Omega$ completing previous results in the literature. We show (for $\delta_h>0$) that if $p(x)$ grows near $\partial\Omega$ as $d(x,\partial\Omega)^2$ then $h(x)$ grows, at most, as $d(x,\partial\Omega)^4$. In particular h is a "flat solution", in the sense that $h=\frac{\partial h}{\partial n}=0$ on $\partial\Omega$, with $h>0$ on Ω if $p>0$ on Ω.

Notice that, although the results on the bifurcation diagrams requires the assumption $\delta_w = 0$, at least for small positive values of δ_w we expect to have some similar behaviours, but we live it as an open problem. Notice that the possibility of having a solution for $\lambda < \lambda_1$ is something which is not evident from the mere modeling arguments.

Several other open problems where mentioned in the paper, but we can also mention some related to the consideration of the associated parabolic system. For instance, it would be interesting to analyze the existence and behaviour of possible travelling waves, already in the one-dimensional framework, linking the stationary states (for the Cauchy problem on the whole space $(-\infty, +\infty)$), $b = 1$ (for instance when $x \to -\infty$) with $b = 0$ (when $x \to +\infty$). The presence of the slow diffusion for the surface-water height h presents some important technical difficulties.

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