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Exponentially slow motion for a one-dimensional Allen–Cahn equation with memory

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Dedicated to Maria Assunta Pozio with esteem and affection

Abstract. *A reaction-diffusion equation with memory kernel of Jeffreys type and with a balanced bistable reaction term is considered in a bounded interval of the real line. Taking advantage of the fact that in this case the integro-differential equation can be transformed into a local partial differential equation, it is proved that there exist solutions which evolve very slowly in time and maintain a transition layer structure for an exponentially long time $T_\varepsilon \geq c_1 \exp(c_2/\varepsilon)$ as $\varepsilon \to 0^+$, where ε^2 is the diffusion coefficient. Hence, we extend to reaction-diffusion equations with memory kernel of Jeffreys type the well-known results valid for the classic Allen–Cahn equation.*

1. Introduction

1.1. Description of the model

In this paper, we consider the following *reaction-diffusion equation with memory*

$$u_t = \int_{-\infty}^{t} k_\varepsilon(t-s) u_{xx}(x,s)\, ds + f(u), \tag{1.1}$$

where $u := u(x,t) \colon [a,b] \times \mathbb{R} \to \mathbb{R}$ and f is a balanced bistable reaction term. More precisely, regarding f, in all the paper it is assumed that $-f$ is the derivative of a double well potential with wells of equal depth, that is $f = -F'$ with $F \in C^3(\mathbb{R})$ satisfying

$$F(\pm 1) = F'(\pm 1) = 0, \quad F''(\pm 1) > 0, \quad F(u) > 0 \ \text{ for } u \neq \pm 1. \tag{1.2}$$

The main example of F satisfying (1.2) is $F(u) = \frac{1}{4}(u^2-1)^2$, which yields the reaction term $f(u) = u - u^3$; however, we stress that F could be any non–negative function, which vanishes only at ± 1, with $F''(\pm 1) > 0$, and in particular it is not excluded that F has other critical points in the interval $(-1,1)$. The memory kernel $k_\varepsilon \colon \mathbb{R} \to \mathbb{R}$ in (1.1) is assumed to be of Jeffreys type [25], namely

$$k_\varepsilon(s) := \frac{(1-\sigma)\varepsilon^2}{\tau} \exp\left(-\frac{s}{\tau}\right) + \sigma\varepsilon^2\delta(s), \tag{1.3}$$

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where $\tau > 0$ is a relaxation time, $\sigma \in [0,1]$ is the ratio of a retardation time to the relaxation time, $\varepsilon > 0$ is a small parameter such that ε^2 is the diffusion coefficient and δ is the Dirac delta function.

The model (1.1) with $f(u) = Ru - u^3$ and memory kernel given by (1.3) (with $\varepsilon = 1$), subject to homogeneous Dirichlet boundary conditions and appropriate initial conditions was introduced in [25] as a model for the flow of a viscoelastic fluid. In this case, the double well potential is given by $F(u) = \frac{1}{4}(u^2 - R)^2$ and in [25] the Authors study the stability and bifurcation of stationary solutions when the Rayleigh number R increases, for different values of the parameter τ. Since its introduction in [25], the reaction-diffusion equation with memory (1.1) has been studied in many papers, which sometimes include different reaction terms and memory kernels. Without claiming to be complete, we list some of the contributions. In [7], the Authors consider the n-dimensional versional of the model proposed in [25] and, after proving existence of a global attractor, they study the stability and bifurcation of stationary solutions in order to compare the results on the n-dimensional case with the ones of [25]. For other works on (1.1), we quote [6, 17, 18, 22] and references therein.

The goal of this paper is to study the limiting behavior as $\varepsilon \to 0^+$ of solutions to (1.1) with memory kernel given by (1.3) and appropriate boundary and initial conditions, with the aim of proving that solutions starting with an unstable N-*transition layer structure* (see Definition 4.1), maintain such a structure for an exponentially long time as $\varepsilon \to 0^+$, that is for a time $T_\varepsilon \geq c_1 \exp(c_2/\varepsilon)$, for some $c_1, c_2 > 0$. To do this, we use the fact that the integro-differential equation (1.1) with kernel given by (1.3) can be transformed into different local partial differential equations as the parameter σ changes in the interval $[0,1]$. Indeed, if we choose the kernel (1.3) with $\sigma = 1$ in (1.1), we recover the *Allen–Cahn* equation

$$u_t = \varepsilon^2 u_{xx} + f(u), \tag{1.4}$$

which is a classic (parabolic) reaction-diffusion, originally proposed in [1] to describe the motion of antiphase boundaries in iron alloys. Conversely, if $\sigma = 0$ in (1.3), let us introduce the function

$$S(x,t) := \frac{\varepsilon^2}{\tau} \int_{-\infty}^{t} \exp\left(-\frac{t-s}{\tau}\right) u(x,s)\, ds,$$

and rewrite (1.1) as the relaxation system

$$\begin{cases} u_t = S_{xx} + f(u), \\ \tau S_t = \varepsilon^2 u - S. \end{cases} \tag{1.5}$$

From the system of partial differential equation (1.5) we can obtain a single equation for u. Indeed, multiplying by τ and differentiating with respect to time the first equation in (1.5) and differentiating with respect to x the second one, we eliminate the function S and obtain the hyperbolic reaction-diffusion equation

$$\tau u_{tt} + \{1 - \tau f'(u)\} u_t = \varepsilon^2 u_{xx} + f(u), \tag{1.6}$$

also known as *Allen–Cahn equation with relaxation*, see [12, 20]. More generally, equation (1.6) with $f = 0$ has been introduced in [5] as a hyperbolic variation to the classic heat equation in order to avoid some unphysical properties, the most famous being the infinite speed of propagation of disturbances, see among others [12] and references therein. Notice that, by (formally) taking the limit as $\tau \to 0^+$ in (1.6), one obtains (1.4).

Finally, for a generic $\sigma \in (0, 1)$, introduce the function

$$S(x,t) := \frac{(1-\sigma)\varepsilon^2}{\tau} \int_{-\infty}^{t} \exp\left(-\frac{t-s}{\tau}\right) u(x,s)\, ds,$$

and rewrite (1.1) as

$$\begin{cases} u_t = S_{xx} + \sigma\varepsilon^2 u_{xx} + f(u), \\ \tau S_t = (1-\sigma)\varepsilon^2 u - S. \end{cases}$$

Reasoning as in the derivation of equation (1.6) from system (1.5), we get the partial differential equation

$$\tau u_{tt} + \{1 - \tau f'(u)\} u_t = \varepsilon^2 u_{xx} + f(u) + \sigma\varepsilon^2 \tau u_{xxt}. \tag{1.7}$$

Motivated by the classical results on the exponentially slow motion of the solutions to the Allen–Cahn equation (1.4), cfr. [2, 3, 4, 15] and their extension to the Allen–Cahn equation with relaxation (1.6) in the recent papers [9, 10, 12], in this paper we are interested in the exponentially slow motion of solutions to (1.7). The main questions are: the well-known metastable dynamics of the solutions to (1.4) and (1.6) is preserved in the case (1.7)? What is the effect of the additional term $\sigma\varepsilon^2\tau u_{xxt}$ in the exponentially slow motion of the solutions? This paper furnishes detailed answers to these questions and in particular, thanks to the study of equation (1.7), we can state that solutions to equation (1.1) with memory kernel (1.3) exhibit the phenomenon of *metastability* for any values of the parameter $\sigma \in [0, 1]$, provided that $\tau > 0$ is sufficiently small if $\sigma \neq 1$.

1.2. Comparison with hyperbolic Allen–Cahn models

Let us consider the partial differential equation (1.7) with a generic damping coefficient g, namely

$$\tau u_{tt} + g(u) u_t = \varepsilon^2 u_{xx} + f(u) + \sigma\varepsilon^2 \tau u_{xxt}, \qquad x \in (a,b),\ t > 0, \tag{1.8}$$

complemented with Dirichlet boundary conditions

$$u(a,t) = \alpha, \qquad u(b,t) = \beta, \qquad t \geq 0, \qquad \alpha, \beta = \pm 1, \tag{1.9}$$

and initial conditions

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \tag{1.10}$$

The main example we have in mind is $g(u) = 1 - \tau f'(u)$, but the results of this paper hold true for a non-negative sufficiently smooth function g, that is we only require that

$$g(u) \geq 0, \qquad \forall\, u \in \mathbb{R}. \tag{1.11}$$

The condition (1.11) in the case (1.7) imposes a restriction on the relaxation time τ, given by

$$\tau f'(u) \leq 1, \qquad \forall\, u \in \mathbb{R}.$$

For instance, in the case $f(u) = u - u^3$, the latter condition becomes $\tau \leq 1$. Therefore, the main novelty with respect to the case $\sigma = 0$ is that g can vanish; indeed, in the hyperbolic reaction-diffusion equation (1.6), the crucial assumption on the damping coefficient, which ensures the exponentially slow motion of the solutions when $\varepsilon \to 0^+$, is $g(u) \geq \kappa > 0$, see the previous works [10, 12]. The additional term $\sigma\varepsilon^2\tau u_{xxt}$ allows g to vanish in some points or, in general, g can entirely vanish. In fact, even if the goal of the paper is to study the model (1.1) with memory kernel (1.3), meaning that $g(u) = 1 - \tau f'(u)$ and $\sigma \in (0,1)$ in (1.8), it is also interesting to study (1.8) for a general damping coefficient g and for $\sigma > 1$. Let us briefly analyze the differences between equation (1.8) and the hyperbolic reaction-diffusion equation

$$\tau u_{tt} + g(u)u_t = \varepsilon^2 u_{xx} + f(u), \qquad x \in (a,b),\ t > 0, \tag{1.12}$$

in order to highlight the role played by the additional term $\sigma\varepsilon^2\tau u_{xxt}$. If g is strictly positive there is no difference between the results of this paper and the previous ones about (1.12) in [10, 12]. As mentioned above, the main novelty is that g can vanish. In particular, it is very interesting to consider the case $g \equiv 0$, corresponding to a nonlinear wave equation with the additional term $\sigma\varepsilon^2\tau u_{xxt}$:

$$\tau u_{tt} = \varepsilon^2 u_{xx} + f(u) + \sigma\varepsilon^2\tau u_{xxt}. \tag{1.13}$$

In this case the term $\sigma\varepsilon^2\tau u_{xxt}$ plays a crucial role and it entirely determines metastability of the solutions. Indeed, it is well-known that there is no metastable dynamics in the case of the nonlinear wave equation

$$\tau u_{tt} = \varepsilon^2 u_{xx} + f(u),$$

which exhibits completely different dynamics. In other words, adding the term $\sigma\varepsilon^2\tau u_{xxt}$ has the same effect of adding a damping term in the nonlinear wave equation.

Another interesting feature of equation (1.7) is that if the damping coefficient g is bounded from below, then it is possible to choose the parameters σ, τ large enough that solutions exhibit exponentially slow motion. Thus, the assumption (1.11) can be removed and g can be also negative. To be more precise, we have exponentially slow motion if

$$\min_u g(u) + \frac{\pi^2}{(b-a)^2}\sigma\varepsilon^2\tau > 0, \qquad \text{for any } \varepsilon > 0, \tag{1.14}$$

for details see (3.3)-(3.4) and Remark 4.5. Nevertheless, in the rest of the paper it is assumed that g satisfies (1.11), because the reaction-diffusion equation (1.1) with memory kernel (1.3), where $\sigma \in (0,1)$ becomes (1.8) with $g(u) = 1 - \tau f'(u)$ and in this case condition (1.14) is satisfied if and only if (1.11) holds true when $\varepsilon \to 0^+$ (because $\sigma \in (0,1)$).

1.3. Plan of the paper

We conclude the Introduction with a short plan of the paper. In Section 2 we consider some special solutions to (1.8) in the whole real line and we prove that there exist traveling waves solutions $\phi(x-ct)$ connecting the two minima of the potential F if and only if $c=0$. In Section 3 we introduce a Lyapunov functional for (1.8)-(1.9) and show some energy estimates, which are crucial to prove the exponentially slow motion of solutions. Finally, Section 4 contains the main results of the paper: we prove that if the initial profile u_0 has a transition layer structure, i.e. $u_0 \approx \pm 1$ except to a finite number of transition points, and the initial velocity u_1 is sufficiently small, then the solution to (1.8)-(1.9)-(1.10) maintains the same transition layer structure of the initial profile for a time $T_\varepsilon \geq c_1 \exp(c_2/\varepsilon)$ as $\varepsilon \to 0^+$, see Theorems 4.2-4.4, and the transition points move with an exponentially small velocity, see Theorem 4.6.

2. Standing waves

Travelling fronts form an important class of global in time solutions of reaction-diffusion equations and, in many situations, they describe the transition between two different states. In general, existence and stability of travelling wave solutions for classic reaction-diffusion equations (1.4) is a well-know fact, see among others the landmark articles [19] and [8]. Concerning the existence and stability of travelling waves for the Allen–Cahn equation with relaxation (1.6), we quote the recent works [20, 21]. Here, we are interested in a travelling wave solution to equation (1.8) connecting the two states $u=-1$ and $u=1$, the global minimal points of the potential F satisfying (1.2). Thus, we look for a solution to (1.8) in the whole real line of the form $\phi(x-ct)$ where $c \in \mathbb{R}$ is the *wave speed* and the *wave profile* ϕ is monotone increasing and connects the states $u=-1$ and $u=1$. Introducing the variable $\xi = x - ct$, denoting by $\phi = \phi(\xi)$, $\phi' = \partial_\xi \phi(\xi)$, and inserting the travelling wave form in (1.8), we get the following boundary value problem

$$c^2\tau\phi'' - cg(\phi)\phi' = \varepsilon^2\phi'' + f(\phi) - c\sigma\varepsilon^2\tau\phi''', \qquad \text{in } \mathbb{R}, \qquad \lim_{\xi \to \pm\infty} \phi(\xi) = \pm 1.$$

Multiplying by ϕ' and integrating in $\mathbb{R}$, we deduce

$$\frac{c^2\tau - \varepsilon^2}{2} \int_{\mathbb{R}} \frac{d}{d\xi}(\phi')^2 d\xi - c\int_{\mathbb{R}} g(\phi)(\phi')^2 d\xi = -\int_{\mathbb{R}} \frac{d}{d\xi} F(\phi)\, d\xi - c\sigma\varepsilon^2\tau \int_{\mathbb{R}} \phi'''\phi'\, d\xi,$$

where we used $F'(s) = -f(s)$. Taking advantage of the fact that $\phi(\pm\infty) = \pm 1$, $\phi'(\pm\infty) = 0$ and using integration by parts, we end up with the relation

$$c\int_{\mathbb{R}} g(\phi)(\phi')^2\,d\xi + c\sigma\varepsilon^2\tau\int_{\mathbb{R}}(\phi'')^2\,d\xi = F(+1) - F(-1) = 0,$$

being $F(\pm 1) = 0$. Since $g \geq 0$ and all the parameters $\sigma, \varepsilon, \tau$ are strictly positive, we deduce that a travelling wave with speed $c \neq 0$ can not exist. Hence, exactly like the classical reaction-diffusion equation (1.4), if the potential F satisfies (1.2), then the only travelling wave solutions to (1.8) connecting -1 and $+1$ are stationary for any $\sigma \geq 0$. It is worth pointing out that such a result holds true also when $g \equiv 0$ in (1.8), i.e. for equation (1.13), where the term $\sigma\varepsilon^2\tau u_{xxt}$ plays a crucial role and allows us to exclude travelling waves with $c \neq 0$.

In the case $c = 0$, we obtain an *increasing standing wave* connecting -1 and $+1$, given by the unique (up to translation) solution to the problem

$$\varepsilon^2\Phi_\varepsilon'' + f(\Phi_\varepsilon) = 0, \quad \text{in } \mathbb{R}, \qquad \lim_{x\to\pm\infty}\Phi_\varepsilon(x) = \pm 1. \tag{2.1}$$

Notice that Φ_ε is the same standing wave of equation (1.4), and so, we extended the well-known results on the existence of travelling/standing waves when F satisfies (1.2) to the case of the reaction-diffusion equation (1.1) with memory kernel given by (1.3). We recall that, in the simplest example $F(u) = \frac{1}{4}(u^2-1)^2$, the solution to (2.1) is given by $\Phi_\varepsilon(x) = \tanh\left(\dfrac{x-h}{\sqrt{2}\,\varepsilon}\right)$, $h \in \mathbb{R}$, and that $\Phi_\varepsilon(-x)$ is a *decreasing standing wave* connecting $+1$ and -1.

The existence of standing waves solutions (or, the non-existence of travelling wave solutions with $c \neq 0$) suggests the existence of *metastable patterns* for (1.8)-(1.9): by using the functions (2.1), we can construct initial profiles such that the corresponding solutions to (1.8)-(1.9)-(1.10) evolve very slowly in time and maintain an unstable structure for a time $T_\varepsilon \geq c_1\exp(c_2/\varepsilon)$, see Figure 1. Let us fix an integer $N > 0$, N *transition layers* $a < h_1 < h_2 < \cdots < h_N < b$ and define the function

$$U_\varepsilon^N(x) := \Phi_\varepsilon\big((-1)^{i+1}(x-h_i)\big) \qquad \text{for } x \in [h_{i-1/2}, h_{i+1/2}], \tag{2.2}$$

and $i = 1,\dots,N$, where Φ_ε is the solution of (2.1) with $\Phi_\varepsilon(0) = 0$ and

$$h_{i+1/2} := \frac{h_i + h_{i+1}}{2}, \qquad i = 1,\dots,N-1, \qquad h_{1/2} = a, \quad h_{N+1/2} = b.$$

We will show in Section 4 that solutions to (1.8)-(1.9), starting with an initial profile like the one in (2.2) and depicted in Figure 1, and a sufficiently small initial velocity u_1 (see assumptions (4.1) and (4.3)), evolve exponentially slowly in time, that is they maintain such an unstable structure for an exponentially long time and the layers move with an exponentially small velocity as $\varepsilon \to 0^+$.

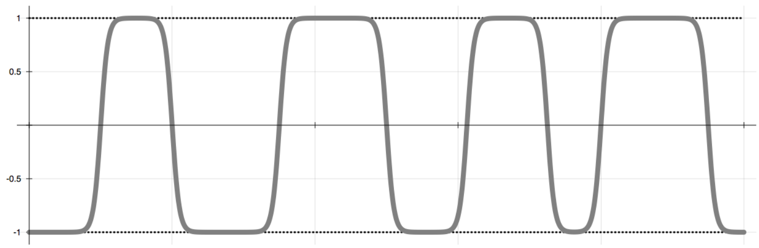

Figure 1: Example of function U_ε^N in the case $f(u) = u - u^3$: $N = 8, \varepsilon = 0.1$.

3. Energy estimates

In this section, we introduce the energy functional for the initial boundary value problem (1.8)-(1.9), which plays the key role in the proof of the main result of this paper.

3.1. Lyapunov functional

First of all, let us prove that the assumption (1.11) on g implies that there exists a Lyapunov functional for the boundary value problem (1.8)-(1.9), for any $\sigma, \varepsilon, \tau > 0$. Indeed, we shall prove that the functional

$$E[u,u_t](t) := \int_a^b \left[\frac{\tau}{2}u_t^2(x,t) + \frac{\varepsilon^2}{2}u_x^2(x,t) + F(u(x,t))\right] dx, \tag{3.1}$$

where $F(u) = -\displaystyle\int_0^u f(s)\,ds$, satisfies the following energy estimates.

Proposition 3.1. *If $u \in C^1([0,T], H^2(a,b)) \cap C^2([0,T], L^2(a,b))$ is a solution to (1.8)-(1.9), then the energy (3.1) satisfies the following equality:*

$$E[u,u_t](0) - E[u,u_t](T) = \int_0^T\int_a^b g(u)u_t^2\,dxdt + \sigma\varepsilon^2\tau\int_0^T\int_a^b u_{xt}^2\,dxdt. \tag{3.2}$$

Proof. Multiplying by u_t equation (1.8) and integrating in $[a,b]$, we infer

$$\frac{d}{dt}\int_a^b \frac{\tau}{2}u_t^2\,dx + \int_a^b g(u)u_t^2\,dx = \int_a^b \left(\varepsilon^2 u_{xx}u_t + \sigma\varepsilon^2\tau u_{xxt}u_t\right)dx - \frac{d}{dt}\int_a^b F(u)dx,$$

where we used $F' = -f$. Using integration by parts and the boundary conditions (1.9), we obtain

$$\frac{d}{dt}\int_a^b \frac{\tau}{2}u_t^2\,dx + \int_a^b g(u)u_t^2\,dx = -\int_a^b \left(\varepsilon^2 u_x u_{xt} + \sigma\varepsilon^2\tau u_{xt}^2\right)dx - \frac{d}{dt}\int_a^b F(u)dx,$$

and, as a consequence, we have

$$\int_a^b g(u)u_t^2\,dx + \sigma\varepsilon^2\tau\int_a^b u_{xt}^2\,dx = -\frac{d}{dt}\int_a^b\left[\frac{\tau}{2}u_t^2 + \frac{\varepsilon^2}{2}u_x^2 + F(u)\right]dx$$
$$= -\frac{d}{dt}E[u,u_t](t),$$

where E is defined in (3.1). Integrating in $[0,T]$ the latter equality, we end up with (3.2) and the proof is complete. □

Thanks to equality (3.2) and the fact that g is a non–negative function, we can state that (1.8)-(1.9) possesses a Lyapunov functional for any $\sigma,\varepsilon,\tau>0$, or, in other words, that the energy defined in (3.1) is a non-increasing function of time t along the solutions to (1.8)-(1.9). In general, if g could be negative, the equality (3.2) does not ensure directly the dissipative character of equation (1.8) with Dirichlet boundary conditions (1.9), because we need to compare the two integrals in the right hand side of (3.2). To do this, we can use Poincaré–Wirtinger inequality to obtain the following result.

Proposition 3.2. *If $u\in C^1([0,T],H^2(a,b))\cap C^2([0,T],L^2(a,b))$ is a solution to (1.8)-(1.9), then the energy (3.1) satisfies the following equality:*

$$E[u,u_t](0) - E[u,u_t](T) \geq \left(\min_u g(u) + c_p^2\sigma\varepsilon^2\tau\right)\int_0^T\int_a^b u_t^2\,dxdt, \tag{3.3}$$

where $c_p := \dfrac{\pi}{b-a}$.

Proof. The Dirichlet boundary conditions (1.9) imply that $u_t(a,t)=u_t(b,t)=0$ for any $t\geq 0$ and we can apply the Poincaré–Wirtinger inequality to the function u_{xt}, namely we have

$$\int_a^b u_{xt}^2\,dx \geq \frac{\pi^2}{(b-a)^2}\int_a^b u_t^2\,dx,$$

and inequality (3.3) becomes a trivial consequence of the equality (3.2). □

Remark 3.3. In all the rest of the paper, we deal with sufficiently regular solutions to (1.8)-(1.9)-(1.10) such that the equality (3.3) holds true. The well-posedness of the latter initial boundary value problem is beyond the scope of this paper, but we mention the fundamental contribution [23], where the Author applies the semigroup theory for solutions of differential equations on Banach spaces to study well-posedness of a general abstract equation, which includes (1.8).

The energy estimate (3.3) guarantees the dissipative character of (1.8)-(1.9) if

$$\min_u g(u) + \frac{\pi^2}{(b-a)^2}\sigma\varepsilon^2\tau > 0. \tag{3.4}$$

However, in this paper we are interested in the behavior of the solutions to (1.8)-(1.9), when the diffusion coefficient $\varepsilon\to 0^+$, with $\sigma\in[0,1]$ and $\tau>0$ independent from ε. Therefore, the condition (3.4) is not satisfied if the minimum of g is strictly negative and ε is sufficiently small and this is the reason why we need to assume that g is non-negative. Hence, substituting (1.11) in (3.3), we conclude that

$$E[u,u_t](0)-E[u,u_t](T)\geq\frac{\pi^2\sigma\varepsilon^2\tau}{(b-a)^2}\int_0^T\int_a^b u_t^2\,dxdt. \tag{3.5}$$

Finally, notice that if g is strictly positive, that is $g(u)\geq\kappa$, for any $u\in\mathbb{R}$, for some $\kappa>0$ independent on ε, then there is no need to take advantage of the positive term $\sigma\varepsilon^2\tau\displaystyle\int_0^T\int_a^b u_{xt}^2\,dxdt$ in (3.2) and we can simply state that

$$E[u,u_t](0)-E[u,u_t](T)\geq\kappa\int_0^T\int_a^b u_t^2\,dxdt. \tag{3.6}$$

It is worth to observe that the difference between (3.5) and (3.6) is that the constant in front of the L^2–norm of u_t in the right hand side is of $\mathcal{O}(\varepsilon^2)$ in (3.5), while it does not depend on ε in (3.6). We will see in Section 4 how this affects the study of the limiting behavior as $\varepsilon\to0^+$ of the solutions to (1.8)-(1.9).

3.2. A lower bound on the energy

In the study of the exponentially slow motion of solutions to (1.8)-(1.9) it is crucial to use the following renormalized version of the energy functional defined in (3.1)

$$E_\varepsilon[u,u_t](t):=\frac{\tau}{2\varepsilon}\|u_t(\cdot,t)\|^2_{L^2}+P_\varepsilon[u](t), \tag{3.7}$$

where

$$P_\varepsilon[u]:=\int_a^b\left[\frac{\varepsilon}{2}u_x^2+\frac{F(u)}{\varepsilon}\right]dx. \tag{3.8}$$

Thus, the functional E_ε has simply been obtained by multiplying by ε^{-1} the energy defined in (3.1). We introduced the classical Ginzburg–Landau functional (3.8), because a very important variational result on P_ε together with the energy estimates (3.5) (or (3.6)) allows us to prove exponentially slow motion of solutions to (1.8)-(1.9). Here, we present the aforementioned variational result.

There is a vast literature of works about the Γ-convergence of the Ginzburg–Landau functional (3.8) both in the one space dimensional and in the multidimensional case. A comprehensive list of all these contributions goes beyond the scope of our presentation; here we only recall [24, 26] and that in the one space dimensional case, the minimum energy of a transition between -1 and $+1$ for (3.8) is given by a strictly positive constant, which depends only on F (independent on ε). Such a constant can be derived in the following way: consider a monotone function u connecting -1 and $+1$, then Young inequality gives

$$P_\varepsilon[u]\geq\int_a^b u_x\sqrt{2F(u)}\,dx\geq\int_{-1}^{+1}\sqrt{2F(s)}\,ds=:c_0. \tag{3.9}$$

Notice that the constant c_0 obtained in (3.9) is strictly positive and independent on ε, because of the renormalization of the energy functional. It represents the minimum energy of a transition between -1 and $+1$ in the following sense: here and for the rest of the paper, fix $N \in \mathbb{N}$, a *piecewise constant function* v and a constant $r > 0$ such that

$$\begin{aligned} &v : [a, b] \to \{-1, +1\} \text{ has } N \text{ jumps located at } a < h_1 < h_2 < \cdots < h_N < b, \\ &r < \frac{h_{i+1} - h_i}{2} \qquad \text{for} \quad i = 1, \dots, N-1 \qquad \text{and} \quad a \le h_1 - r, \quad h_N + r \le b. \end{aligned} \tag{3.10}$$

If $\{u^\varepsilon\}$ is a sequence converging to v in L^1, then

$$\liminf_{\varepsilon \to 0^+} P_\varepsilon[u^\varepsilon] \ge N c_0, \tag{3.11}$$

with equality if the sequence $\{u^\varepsilon\}$ is chosen properly (for instance, the sequence U_ε^N constructed in (2.2); for details see [9, Section 3] or [10, Section 3]).

The crucial variational result to prove exponentially slow motion of solutions to (1.8)-(1.9) is an improvement of (3.11), stating that if u is sufficiently close to a piecewise constant function v as in (3.10), then $P_\varepsilon[u^\varepsilon] \ge N c_0 - C \exp(-A/\varepsilon)$ for some constants $A, C > 0$.

Proposition 3.4. *Assume that the potential $F \in C^3(\mathbb{R})$ satisfies (1.2) and define $\lambda := \min\{F''(\pm 1)\}$. Let v, r as in (3.10) and $A \in (0, r\sqrt{2\lambda})$. Then, there exist constants $C, \delta > 0$ (depending only on F, v and A) such that if $u \in H^1$ satisfies*

$$\|u - v\|_{L^1} \le \delta, \tag{3.12}$$

then

$$P_\varepsilon[u] \ge N c_0 - C \exp(-A/\varepsilon),$$

where P_ε and c_0 are defined in (3.8) and (3.9), respectively.

A more general version of Proposition 3.4 was first proved by Grant in [16], where the Author use this variational result to prove exponentially slow motion of solutions to the Cahn–Morral system. In particular, in [16] $u \in \mathbb{R}^m$ is a vector, the potential $F \colon \mathbb{R}^m \to \mathbb{R}$ vanishes in a finite number of points $K \ge 2$ and the constant $N c_0$ is replaced by an appropriate asymptotic energy. Moreover, Grant imposes a weaker assumption on u, which is guaranteed by (3.12). On the other hand, the proof of Proposition 3.4 in the scalar case can be found also in [13].

4. Exponentially slow motion

This section contains the main results of the paper, which are obtained by using the energy approach introduced by Bronsard and Kohn in [2] in the study of the classic Allen–Cahn equation and then improved by Grant in [16] to study exponentially slow motion for the Cahn–Morral system. This energy approach is quite elementary yet very powerful and it can be applied to different evolution

PDEs exhibiting slow motion of solutions: just to mention some contributions, it has been used to prove slow motion of solutions for hyperbolic variations of the Allen–Cahn equation [9, 10], hyperbolic Cahn–Hilliard equation [13], and parabolic reaction-diffusion equations with different nonlinear diffusions [11, 14].

The key points to apply the energy approach proposed in [2] are the energy estimates (3.5) or (3.6) (depending whether g is strictly positive or it vanishes) and Proposition 3.4, which establishes a lower bound on the energy functional. In particular, Proposition 3.4 is a variational result concerning the Ginzburg–Landau functional, in which equation (1.8) plays no role, and the exponentially slow motion is consequence of the fact that solutions to the boundary value problem (1.8)-(1.9) satisfy the energy estimates (3.5) or (3.6) with the particular energy functional (3.7). Moreover, if g is strictly positive, we can use (3.6) and, by proceeding exactly as in [10], we obtain the same results of the hyperbolic reaction-diffusion equation (1.12) on the exponentially slow motion of the solutions, see Theorem 4.4 below. Here, since we consider the case when g vanishes and satisfies (1.11), we need to slightly modify the standard procedure, in particular in the proof of Proposition 4.3, and we shall obtain a weaker result. We still obtain exponentially slow motion of the solutions, but, when g vanishes, the evolution is faster than the one of the case $g(u) \geq \kappa > 0$, for any $u \in \mathbb{R}$; compare Theorem 4.2 and Theorem 4.4 below.

Let us start with the following definition.

Definition 4.1. Let us fix v, r as in (3.10). We say that a function $u^\varepsilon \in H^1(a,b)$ has an *N-transition layer structure* if

$$\lim_{\varepsilon \to 0} \|u^\varepsilon - v\|_{L^1} = 0, \tag{4.1}$$

and there exist $C > 0$, $A \in (0, r\sqrt{2\lambda})$ and $\lambda = \min\{F''(\pm 1)\}$ (independent on ε) such that

$$P_\varepsilon[u^\varepsilon] \leq Nc_0 + C\exp(-A/\varepsilon), \tag{4.2}$$

for any $\varepsilon \ll 1$, where the energy P_ε and the positive constant c_0 are defined in (3.8) and (3.9), respectively.

An example of function with a N-transition layer structure can be found in [13, Section 2.4] or, alternatively, one can check that the function defined in (2.2) satisfies (4.1) and (4.2).

Roughly speaking, the main result of this paper states that if the initial profile u_0 has a N-transition layer structure and the initial velocity u_1 is sufficiently small, then the solution to (1.8)-(1.9)-(1.10) maintains the N-transition layer structure for an exponentially long time as $\varepsilon \to 0^+$, provided that g satisfies (1.11) and $f = -F'$ with F satisfying (1.2). More precisely, we assume that the initial data (u_0, u_1) in (1.10) depend on ε, with u_0^ε satisfying (4.1) and there exist $C > 0$, $A \in (0, r\sqrt{2\lambda})$ independent from ε, with $\lambda := \min\{F''(\pm 1)\}$, such that

$$E_\varepsilon[u_0^\varepsilon, u_1^\varepsilon] \leq Nc_0 + C\exp(-A/\varepsilon), \tag{4.3}$$

for any $\varepsilon \in (0, \varepsilon_0)$, where the energy E_ε and the positive constant c_0 are defined in (3.7) and (3.9), respectively. Clearly, the assumptions (4.1) and (4.3) imply that u_0^ε has a N-transition layer structure. Moreover, if u_0^ε satisfies (4.1), by using (4.3) and Proposition 3.4, we infer

$$\frac{\tau}{2\varepsilon}\|u_1\|_{L^2}^2 \leq Nc_0 + C\exp(-A/\varepsilon) - P_\varepsilon[u_0^\varepsilon] \leq C\exp(-A/\varepsilon), \tag{4.4}$$

for some $C > 0$. Hence, the assumptions on the initial data (4.1), (4.3) are equivalent to (4.1), (4.2) and (4.4): the initial profile u_0^ε has an N-transition layer structure and the L^2-norm of the initial velocity u_1^ε is exponentially small as $\varepsilon \to 0^+$.

The main result of this paper is the following.

Theorem 4.2. *Assume that g satisfies (1.11), $F \in C^3(\mathbb{R})$ satisfies (1.2) and define $\lambda := \min\{F''(\pm 1)\} > 0$. Let v, r be as in (3.10) and let $A \in (0, r\sqrt{2\lambda})$. If u^ε is the solution to (1.8)-(1.9)-(1.10) with initial data u_0^ε, u_1^ε satisfying (4.1) and (4.3), then*

$$\sup_{0\leq t\leq m\varepsilon^{1+\gamma}\exp(A/\varepsilon)} \|u^\varepsilon(\cdot,t) - v\|_{L^1} \xrightarrow[\varepsilon\to 0]{} 0, \tag{4.5}$$

for any $m, \gamma > 0$.

The proof of Theorem 4.2 is a consequence of the following proposition, which is proved using only (3.5) and Proposition 3.4.

Proposition 4.3. *Assume that g satisfies (1.11) and that $F \in C^3(\mathbb{R})$ satisfies (1.2). Let u^ε be the solution of (1.8)-(1.9)-(1.10) with initial data u_0^ε, u_1^ε satisfying (4.1) and (4.3). Then, there exist positive constants $\varepsilon_0, C_1, C_2 > 0$ (independent on ε) such that*

$$\int_0^{C_1\varepsilon\exp(A/\varepsilon)} \|u_t^\varepsilon\|_{L^2}^2\,dt \leq C_2\varepsilon^{-1}\exp(-A/\varepsilon), \tag{4.6}$$

for all $\varepsilon \in (0, \varepsilon_0)$.

Proof. Let $\varepsilon_0 > 0$ so small that for all $\varepsilon \in (0, \varepsilon_0)$, (4.3) holds and

$$\|u_0^\varepsilon - v\|_{L^1} \leq \frac{1}{2}\delta, \tag{4.7}$$

where δ is the constant of Proposition 3.4. Let $\hat{T} > 0$; we claim that if

$$\int_0^{\hat{T}} \|u_t^\varepsilon\|_{L^1}\,dt \leq \frac{1}{2}\delta, \tag{4.8}$$

then there exists $C > 0$ such that

$$E_\varepsilon[u^\varepsilon, u_t^\varepsilon](\hat{T}) \geq Nc_0 - C\exp(-A/\varepsilon). \tag{4.9}$$

Indeed, from the definition (3.7) we have $E_\varepsilon[u^\varepsilon, u_t^\varepsilon](\hat{T}) \geq P_\varepsilon[u^\varepsilon](\hat{T})$ and inequality (4.9) follows from Proposition 3.4 if the condition $\|u^\varepsilon(\cdot,\hat{T}) - v\|_{L^1} \leq \delta$ holds true. By using triangle inequality, (4.7) and (4.8), we obtain

$$\|u^\varepsilon(\cdot,\hat{T}) - v\|_{L^1} \leq \|u^\varepsilon(\cdot,\hat{T}) - u_0^\varepsilon\|_{L^1} + \|u_0^\varepsilon - v\|_{L^1} \leq \int_0^{\hat{T}} \|u_t^\varepsilon\|_{L^1} + \frac{1}{2}\delta \leq \delta,$$

and the claim (4.9) is proved. By multiplying by ε^{-1} the inequality (3.5), we deduce

$$E_\varepsilon[u_0^\varepsilon, u_1^\varepsilon] - E_\varepsilon[u^\varepsilon, u_t^\varepsilon](\hat{T}) \geq \frac{\pi^2\sigma\varepsilon\tau}{(b-a)^2}\int_0^{\hat{T}}\int_a^b u_t(x,t)^2\,dxdt. \tag{4.10}$$

Substituting (4.3) and (4.9) into (4.10), one has

$$\int_0^{\hat{T}} \|u_t^\varepsilon\|_{L^2}^2\,dt \leq C_2\varepsilon^{-1}\exp(-A/\varepsilon). \tag{4.11}$$

It remains to prove that inequality (4.8) holds for $\hat{T} \geq C_1\varepsilon\exp(A/\varepsilon)$. If

$$\int_0^{+\infty} \|u_t^\varepsilon\|_{L^1}\,dt \leq \frac{1}{2}\delta,$$

there is nothing to prove. Otherwise, choose $\hat{T}$ such that

$$\int_0^{\hat{T}} \|u_t^\varepsilon\|_{L^1}\,dt = \frac{1}{2}\delta.$$

Using Hölder's inequality and (4.11), we infer

$$\frac{1}{2}\delta \leq [\hat{T}(b-a)]^{1/2}\left(\int_0^{\hat{T}} \|u_t^\varepsilon\|_{L^2}^2\,dt\right)^{1/2} \leq \left[\hat{T}(b-a)C_2\varepsilon^{-1}\exp(-A/\varepsilon)\right]^{1/2}.$$

It follows that there exists $C_1 > 0$ such that

$$\hat{T} \geq C_1\varepsilon\exp(A/\varepsilon),$$

and the proof is complete. □

Now, we have all the tools to prove (4.5).

Proof of Theorem 4.2. Fix $m, \gamma > 0$. Triangle inequality gives

$$\|u^\varepsilon(\cdot,t) - v\|_{L^1} \leq \|u^\varepsilon(\cdot,t) - u_0^\varepsilon\|_{L^1} + \|u_0^\varepsilon - v\|_{L^1}, \tag{4.12}$$

for all $t \in [0, m\varepsilon^{1+\gamma}\exp(A/\varepsilon)]$. The last term of inequality (4.12) tends to 0 by assumption (4.1). Regarding the first term, we have

$$\sup_{0\leq t\leq m\varepsilon^{1+\gamma}\exp(A/\varepsilon)} \|u^\varepsilon(\cdot,t) - u_0^\varepsilon\|_{L^1} \leq \int_0^{m\varepsilon^{1+\gamma}\exp(A/\varepsilon)} \|u_t^\varepsilon(\cdot,t)\|_{L^1}\,dt.$$

By using Cauchy–Schwarz inequality and by taking ε so small that $C_1 \geq m\varepsilon^\gamma$ in order to apply Proposition 4.3, we deduce

$$\int_0^{m\varepsilon^{1+\gamma}\exp(A/\varepsilon)} \|u_t^\varepsilon\|_{L^1}\,dt \leq \left\{(b-a)m\varepsilon^{1+\gamma}\exp(A/\varepsilon)C_2\varepsilon^{-1}\exp(-A/\varepsilon)\right\}^{1/2} \leq C\varepsilon^{\frac{\gamma}{2}}.$$

Hence (4.5) follows. □

Thanks to Theorem 4.2 and the fact the energy (3.7) is a non-increasing function of t along the solutions to (1.8)-(1.9), we can state that if the initial data $u_0^\varepsilon, u_1^\varepsilon$ in (1.10) satisfy the conditions (4.1) and (4.3), then the solution u^ε maintains the N-transition layer structure for a time $T_\varepsilon \geq m\varepsilon^{1+\gamma}\exp(A/\varepsilon)$ for any $m, \gamma > 0$ (see (4.5)) and the velocity u_t satisfies

$$\frac{\tau}{2\varepsilon}\|u_t(\cdot,t)\|_{L^2}^2 \leq C\exp(-A/\varepsilon),$$

for any $t \in [0, T_\varepsilon]$ (see (4.4)).

By using the energy estimate (3.6) instead of (3.5), we can improve the estimate (4.6) and, by proceeding as in the proof of Proposition 4.3 (in particular, by using (3.6) instead of (3.5) in (4.10)), we can prove that there exist positive constants $\varepsilon_0, C_1, C_2 > 0$ (independent on ε) such that

$$\int_0^{C_1\varepsilon^{-1}\exp(A/\varepsilon)} \|u_t^\varepsilon\|_{L^2}^2\,dt \leq C_2\varepsilon\exp(-A/\varepsilon),$$

for all $\varepsilon \in (0, \varepsilon_0)$. As a consequence, we have the following result.

Theorem 4.4. *Assume that $g(u) \geq \kappa > 0$ for any $u \in \mathbb{R}$, that $F \in C^3(\mathbb{R})$ satisfies (1.2) and define $\lambda := \min\{F''(\pm 1)\} > 0$. Let v, r be as in (3.10) and let $A \in (0, r\sqrt{2\lambda})$. If u^ε is the solution to (1.8)-(1.9)-(1.10) with initial data u_0^ε, u_1^ε satisfying (4.1) and (4.3), then*

$$\sup_{0\leq t\leq m\exp(A/\varepsilon)} \|u^\varepsilon(\cdot,t) - v\|_{L^1} \xrightarrow[\varepsilon\to 0]{} 0,$$

for any $m > 0$.

Therefore, when g is a strictly positive function, we obtain the same result of the hyperbolic Allen–Cahn equation (1.12), cfr. [10], where a vectorial version of (1.12) is considered.

Remark 4.5. As it was mentioned at the beginning of this section and as we saw in the proofs of Proposition 4.3 and Theorem 4.2, the key ingredients of the energy approach are the energy estimates (3.5) or (3.6) and Proposition 3.4, which establishes a lower bound on the energy functional. Notice that the energy estimate (3.6) can be also obtained from (3.3) for a generic damping coefficient g

which does not satisfy (1.11), if it is possible to choose $\sigma\tau = C\varepsilon^{-2}$ for some $C > 0$ so large that

$$\inf_u g(u) + \frac{\pi^2}{(b-a)^2} C = \kappa > 0.$$

Hence, as we mentioned in the Introduction, solutions to (1.8) exhibit exponentially slow motion for any bounded from below function g, provided that the product $\sigma\tau$ is sufficiently large (and depending on ε).

Similarly, an energy estimate like (3.5) can be also obtained in the case $\sigma = 0$ (equation (1.12)) if the damping coefficient satisfies $g(u) \geq C\varepsilon^2$, for any $u \in \mathbb{R}$. Hence, proceeding as in the proof of Theorem 4.2, we can extend the results of [10] valid for (1.12) to the case of a positive damping coefficient g depending on ε and vanishing (in some points or entirely) as $\varepsilon \to 0^+$.

4.1. Layer dynamics

We have proved that solutions to (1.8)-(1.9) with appropriate initial conditions maintain a transition layer structure for an exponentially long time as $\varepsilon \to 0^+$. Now, we conclude this paper with an estimate of the velocity of the layers. Fix a piecewise constant function v as in (3.10) and define its *interface* by

$$I[v] := \{h_1, h_2, \ldots, h_N\}.$$

Moreover, for an arbitrary function $u\colon [a,b] \to \mathbb{R}$ and an arbitrary closed subset $K \subset \mathbb{R}\backslash\{\pm 1\}$, the *interface* $I_K[u]$ is defined by

$$I_K[u] := u^{-1}(K).$$

Finally, we recall that for any $X, Y \subset \mathbb{R}$ the *Hausdorff distance* $d(X,Y)$ between X and Y is defined by

$$d(X,Y) := \max\left\{\sup_{x\in X} d(x,Y),\ \sup_{y\in Y} d(y,X)\right\},$$

where $d(y,X) := \inf\{|y-x| : x \in X\}$. Then, we can prove that the velocity of the layers is exponentially small as $\varepsilon \to 0^+$.

Theorem 4.6. *Assume that g satisfies (1.11), $F \in C^3(\mathbb{R})$ satisfies (1.2) and define $\lambda := \min\{F''(\pm 1)\} > 0$. Let v, r be as in (3.10), $A \in (0, r\sqrt{2\lambda})$ and let u^ε be the solution to (1.8)-(1.9)-(1.10) with initial data u_0^ε, u_1^ε satisfying (4.1) and (4.3). Given $m, \gamma > 0$, $\delta_1 \in (0, r)$ and a closed subset $K \subset \mathbb{R}\backslash\{\pm 1\}$, set*

$$t_\varepsilon(\delta_1) = \inf\{t :\ d(I_K[u^\varepsilon(\cdot,t)], I_K[u_0^\varepsilon]) > \delta_1\}.$$

There exists $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then

$$t_\varepsilon(\delta_1) > m\varepsilon^{1+\gamma} \exp(A/\varepsilon).$$

Proof. The proof is consequence of Theorem 4.2 and the following variational result on the Ginzburg–Landau functional (3.8). Given $\delta_1 \in (0, r)$ and a closed subset $K \subset \mathbb{R}\backslash\{\pm 1\}$, there exist constants $\hat{\delta}, \varepsilon_0, L > 0$ (independent on ε) such that for any $u \in H^1([a,b])$ satisfying

$$\|u - v\|_{L^1} < \hat{\delta} \qquad \text{and} \qquad P_\varepsilon[u] \leq Nc_0 + L, \tag{4.13}$$

for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$d(I_K[u], I[v]) < \frac{1}{2}\delta_1. \tag{4.14}$$

For the proof of this result see [10, Lemma 4.2] or [13, Lemma 2.9].

Fix $\varepsilon_0 > 0$ so small that the assumptions on the initial data (4.1), (4.3) imply that u_0^ε satisfies (4.13) for all $\varepsilon \in (0, \varepsilon_0)$; thus, (4.14) for u_0^ε reads as

$$d(I_K[u_0^\varepsilon], I[v]) < \frac{1}{2}\delta_1. \tag{4.15}$$

On the other hand, the solution $u^\varepsilon(\cdot, t)$ satisfies (4.13) for all $t \in (0, m\varepsilon^{1+\gamma}\exp(A/\varepsilon)]$, for any fixed $m, \gamma > 0$. Indeed, the first condition in (4.13) holds true for (4.5) in Theorem 4.2, while the second one is valid because

$$P_\varepsilon[u^\varepsilon(\cdot, t)] \leq E_\varepsilon[u^\varepsilon, u_t^\varepsilon](t) \leq E_\varepsilon[u_0^\varepsilon, u_1^\varepsilon] \leq Nc_0 + C\exp(A/\varepsilon),$$

for any $t \in [0, m\varepsilon^{1+\gamma}\exp(A/\varepsilon)]$, where we used the fact that $E_\varepsilon[u^\varepsilon, u_t^\varepsilon](t)$ is a non-increasing function of t and assumption (4.3). Then, (4.14) yields

$$d(I_K[u^\varepsilon(t)], I[v]) < \frac{1}{2}\delta_1, \tag{4.16}$$

for all $t \in [0, m\varepsilon^{1+\gamma}\exp(A/\varepsilon)]$. Combining (4.15) and (4.16), we obtain

$$d(I_K[u^\varepsilon(t)], I_K[u_0^\varepsilon]) < \delta_1,$$

for all $t \in [0, m\varepsilon^{1+\gamma}\exp(A/\varepsilon)]$ and the proof is complete. □

In conclusion, according to Theorem 4.6 one must wait a time of order $\varepsilon\exp(A/\varepsilon)$ to see an appreciable change in the position of the zeros of u^ε.

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