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Nonlinear degenerate parabolic equations with irregular initial data

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This paper is dedicated to my colleague and friend Maria Assunta Pozio who I want to thank for all the nice time we spent together.

Abstract. *Existence and regularity results for a class of degenerate nonlinear parabolic equations are proved for irregular initial data like the Dirac mass. Indeed the diffusion operator may degenerate as the solution diverges and may depend on space and time variables in a non–regular way, too.*

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a sufficiently regular boundary $\partial\Omega$ (i.e. such that the Sobolev embeddings hold true) and let Ω_T denote the cylindrical domain $\Omega \times (0,T)$, for $0 < T < +\infty$. Set $S_T \equiv \partial\Omega \times (0,T)$. We will consider the following nonlinear parabolic problem

$$\begin{cases} u_t - \operatorname{div}(A(x,t,u)\nabla u) = 0 & \text{in} \quad \Omega_T, \\ u(x,t) = 0, & \text{on} \quad S_T, \\ u(x,0) = u_0, & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where u_0 can be a Dirac mass or in general an element of $M_b(\Omega)$, i.e. a bounded Radon measure. We assume that A is a bounded symmetric Caratheodory matrix function satisfying one of the following structural assumptions (for any $\xi \in \mathbb{R}^N$, $\sigma \in \mathbb{R}$ and a.e. $(x,t) \in \Omega_T$)

$$\frac{\alpha|\xi|^2}{(1+|\sigma|)^\gamma} \leq \langle A(x,t,\sigma)\xi, \xi\rangle \leq \beta|\xi|^2, \tag{1.2}$$

or

$$\frac{\alpha|\xi|^2}{(1+|\sigma|)^\gamma} \leq \langle A(x,t,\sigma)\xi, \xi\rangle \leq \frac{\beta|\xi|^2}{(1+|\sigma|)^\gamma}, \tag{1.3}$$

where α, β and γ are positive constants.

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This problem was investigated by the author with M. Assunta Pozio in [29] when the initial datum u_0 belongs to a Lebesgue space $L^m(\Omega)$, $m \geq 1$. See also [9] where the particular case $\gamma = m = 2$ is investigated. Hence, main aim of this paper is to complete these results understanding what happens in the more general case of bounded Radon measure as initial data. We point out that the main difficulty of this class of problems is that it degenerates when the solution is unbounded; in addition, the presence of irregular data, like measure Radon, increases the difficulties.

There is a wide literature on these problems due to the various applications of this class of nonlinear parabolic equations (see [12, 36, 4, 5, 6, 10, 17, 27] and the references therein). Also the stationary case was investigated by many authors (see [1, 7, 8, 15, 16, 30] and the references therein).

When the initial datum u_0 belongs to a Lebesgue space (hence also in case of L^1-data) problem (1.1) admits a solution if (1.2) is satisfied with γ not too large, i.e. if

$$\gamma < \frac{1}{N}, \tag{1.4}$$

(see Theorem 2.13 in [29]). Moreover, if the stronger assumption (1.3) is satisfied, together with a further structural assumption on A, then problem (1.1) admits solutions for every choice of u_0 in $L^m(\Omega)$ $(m \geq 1)$ and hence there is no need to assume any restriction on γ or on m (see Theorem 2.5 in [29]).

In addition, if (1.3) holds true and m satisfies the following condition

$$m > \frac{\gamma N}{2} \tag{1.5}$$

then, without assuming any further structural assumption on A, it is possible to prove the existence of a solution of (1.1) that becomes "immediately bounded" and satisfies the following decay estimate

$$\|u(t)\|_{L^\infty(\Omega)} \leq c\,\frac{\|u_0\|_{L^m(\Omega)}}{t^\eta e^{\sigma t}} \qquad t \in (0,T) \tag{1.6}$$

where η and σ are positive constants depending only on the data in the structure conditions (see Theorem 2.15 in [29]). Notice that condition (1.5) is a sharp condition to have the boundedness of a solution for every $t > 0$ when the initial datum is not a bounded function (see section 6 in [29]).

We observe that the bound (1.6) is satisfied also by the solutions of non-degenerate problems like, for example, the heat equation

$$\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega_T, \\ u(x,t) = 0, & \text{on} \quad S_T, \\ u(x,0) = u_0, & \text{in} \quad \Omega, \end{cases} \tag{1.7}$$

with $\eta = N/2$.

Estimates like (1.6) or, more in general, of the following type

$$\|u(t)\|_{L^\infty(\Omega)} \leq c\,\frac{\|u_0\|^{r_0}_{L^m(\Omega)}}{t^{r_1}} \qquad t \in (0,T) \tag{1.8}$$

(with r_0 and r_1 positive constants) are known in literature as decay or ultracontractive estimates and are very important because not only allow to describe the behavior in time of the solutions (both for t large or t small) but because generally allow to prove many other important properties of solutions like continuity or uniqueness (see [10, 26, 28] and the references therein). These estimates appear also for numerous other degenerate parabolic problems like the porous medium equation and the degenerate p-Laplacian equation ($p > 2$) (see [23, 24, 36] and the references therein). Indeed, this strong regularization appears also for some solutions of singular nonlinear parabolic problems like the fast diffusion equation (see [36, 23] and the references therein) together with problems which are not singular and not degenerate like, as recalled above, the heat equation.

Notice that if (1.3) holds true and m does not satisfy (1.5), as recalled above, we know that the solutions of (1.1) do not become bounded and hence cannot decay in the L^∞-norm. Anyway, if it results

$$1 < m \leq \frac{\gamma N}{2}, \quad \gamma \leq m,$$

then, by means of the techniques in [25] it is possible to prove the existence of a solution u of (1.1) that decays in every Lebesgue space $L^r(\Omega)$ with $r \in (1, m)$. Moreover, the following estimate holds true

$$\|u(t)\|_{L^r(\Omega)} \leq c\,\frac{\|u_0\|_{L^m(\Omega)}}{t^{\frac{m-r}{r\gamma}}} \quad t \in (0, T), \tag{1.9}$$

(see Theorem 3.3 in [27]). Finally, when u_0 is a bounded Radon measure existence results of properly defined solutions (measure valued solutions) which are not required to belong to any Lebesgue space can be found in [21, 22, 31, 32, 33].

In this paper we want to investigate on the existence of regular solutions of (1.1) when the initial datum u_0 is not in a Lebesgue space. In particular, we want to understand if in presence of irregular data, like bounded Radon measures, there still exist solutions of (1.1) that immediately become bounded or not and which influence have in this improvement of regularity the structure assumptions (1.2) and (1.3).

Our first result is that if $u_0 \in M_b(\Omega)$ and the structural assumption (1.3) is retained, if we assume

$$0 < \gamma < \frac{2}{N}, \tag{1.10}$$

then there exists a solution of (1.1) that becomes immediately bounded and satisfies the decay estimate (1.6). Notice that the condition (1.10) is equal to the bound (1.5) when $m = 1$. Hence, we have a sort of "continuity" in the requirement we need on γ to have solutions that become "immediately bounded" passing from the case of Lebesgue data to the case of Radon measures data (see Theorem 2.4 below).

Indeed, we can prove that the existence of regular solutions holds true also under the weaker structure assumption (1.2). In details, if $u_0 \in M_b(\Omega)$ and (1.2) holds true with

$$0 < \gamma < \frac{1}{N},$$

(i.e. under the same assumptions that guarantee the existence of a solution for L^1-data) then there exists a solution that becomes immediately bounded and satisfies the decay estimate (1.6) (see Theorem 2.5 below).

Notice that this last result substantially says that "what jokes a fundamental role" in the improvement of regularity is the following bound from below

$$\frac{\alpha|\xi|^2}{(1+|\sigma|)^\gamma} \leq \langle A(x,t,\sigma)\xi,\xi\rangle\,.$$

The plan of the paper is the following: in next section we state our results in all the details. In Section 3 we give some known results we need in the proofs. Finally, Section 4 is devoted to the proofs of all the results set out in Section 2.

2. Statement of the results

Before stating our results in all the details we recall what we mean here by weak solution of (1.1).

Definition 2.1. We will say that a measurable function u is a weak solution of (1.1) if

1. $u \in L^1(\Omega_T)$;
2. $G(u) \in L^1(0,T;W_0^{1,1}(\Omega))$;
3. $A(x,t,u)(1+|u|)^\gamma \nabla G(u) \in L^1(\Omega_T)^N$;

and if it results

$$\iint_{\Omega_T} \{-u\varphi_t + \langle A(x,t,u)(1+|u|)^\gamma \nabla G(u), \nabla\varphi\rangle\}dxdt = \langle u_0, \varphi(x,0)\rangle, \qquad (2.1)$$

for every $\varphi \in W^{1,1}(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;W_0^{1,\infty}(\Omega))$ with compact support in $[0,T)\times\overline{\Omega}$ and such that $\varphi(x,0) \in C(\Omega)$.

Here we denote

$$G(s) := \int_0^s \frac{1}{(1+|\sigma|)^\gamma}d\sigma, \qquad s \in \mathbb{R}\,. \qquad (2.2)$$

Remark 2.2. Observe that since φ belongs to $W^{1,1}(0,T;L^\infty(\Omega))$ it is continuous from $[0,T]$ to $L^\infty(\Omega)$: hence the requirement on $\varphi(x,0)$ is well posed. Moreover it results $\varphi(x,0) \in C_b(\Omega)$, i.e. $\varphi(x,0)$ is continuous and bounded on Ω so that $< u_0, \varphi(x,0) >$ is well defined for every $u_0 \in M_b(\Omega)$.

Remark 2.3. Let's notice that in definition (2.1) we do not require ∇u to exist. However, if $\nabla u \in L^1_{loc}(\Omega_T)^N$ then $A(x,t,u)(1+|u|)^\gamma \nabla G(u) = A(x,t,u)\nabla u$, hence the requirement $A(x,t,u)(1+|u|)^\gamma \nabla G(u) \in L^1(\Omega_T)^N$ becomes $A(x,t,u)\nabla u \in L^1(\Omega_T)^N$ and (2.1) turns to

$$\iint_{\Omega_T} \{-u\varphi_t + \langle A(x,t,u)\nabla u, \nabla\varphi\rangle\} dxdt = \langle u_0, \varphi(x,0)\rangle. \tag{2.3}$$

Moreover if (1.3) is assumed, the requirement $A(x,t,u)(1+|u|)^\gamma \nabla G(u) \in L^1(\Omega_T)^N$ is a consequence of $G(u) \in L^1(0,T;W_0^{1,1}(\Omega))$.

Lastly, if $u \in L^1(0,T;W_0^{1,1}(\Omega))$ then $A(x,t,u)\nabla u$ is in $L^1(\Omega_T)^N$ under the weaker hypothesis (1.2).

Theorem 2.4. *Assume* (1.3), $u_0 \in M_b(\Omega)$ *and* γ *satisfying*

$$0 < \gamma < \frac{2}{N}\,. \tag{2.4}$$

Then there exists a weak solution u *of* (1.1) *in* $L^\infty_{loc}(0,T;L^\infty(\Omega)) \cap L^2_{loc}(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^1(\Omega))$. *Moreover* u *satisfies* (2.3), *the following regularity property*

$$A(x,t,u)\nabla u \in (L^q(\Omega_T))^N, \quad \forall\, q \in [1, \frac{2}{2-\gamma}), \tag{2.5}$$

and the decay estimate

$$\|u(t)\|_{L^\infty(\Omega)} \le \frac{K_1}{(\alpha t)^\eta} e^{-K\alpha t}, \quad \text{a.e. } t \in (0,T), \tag{2.6}$$

where $\eta > \frac{N}{2-N\gamma}$ *can be fixed arbitrarily and* K *and* K_1 *only depend on* γ, N, Ω, u_0 *and* η.

We point out that the solution u described in the previous Theorem not only becomes immediately bounded but has the property that also its gradient becomes immediately more regular since it results $\nabla u \in (L^2(\Omega \times (\varepsilon, T))^N$ for every $\varepsilon \in (0,T)$.

As observed in the introduction it is still possible to prove the existence of a solution assuming the weaker hypothesis (1.2), if γ is suitable small. In details, we have the following result where, differently from the previous one, the regularity of the gradient of the solution is controlled up to $t = 0$.

Theorem 2.5. *Let* (1.2) *hold true. If* $u_0 \in M_b(\Omega)$ *and* γ *satisfies*

$$0 < \gamma < \frac{1}{N}, \tag{2.7}$$

then there exists a weak solution $u \in L^\infty(0,T;L^1(\Omega))$ *of* (1.1). *Moreover we have*

$$\nabla u \in M^h(\Omega_T)\,, \qquad h = 1 + \frac{1-\gamma N}{N+1}\,.$$

Hence it results $u \in L^r(0,T;W_0^{1,r}(\Omega))$ *for every* $1 \le r < h$ *and* (2.3) *holds true. Finally* u *belongs to* $L^\infty_{loc}(0,T;L^\infty(\Omega))$, *satisfies the decay estimate* (2.6) *and it results*

$$u \in M^s(\Omega_T), \quad \forall\, s = \frac{2}{N} + 1 - \gamma\,. \tag{2.8}$$

Remark 2.6. Observe that the hypothesis (2.7) is equivalent to require $h > 1$. Moreover, by slight modifications of the analogous result in [29], we prove here also the regularity (2.8). Notice that it results

$$\frac{2}{N} + 1 - \gamma = h\frac{N+1}{N}\,.$$

3. Preliminary results

This section is devoted to some preliminary results which will be used in the proofs of our results. We start from a consequence of the Gagliardo-Niremberg embedding theorem.

Lemma 3.1 (cf. Prop. 3.1 of [14])**.** *Let* $v \in L^\infty(0,T;L^m(\Omega)) \cap L^p(0,T;W_0^{1,p}(\Omega))$, $m,\, p \ge 1$. *Then* v *belongs to* $L^q(\Omega_T)$, *where*

$$q = p\ \frac{(N+m)}{N},$$

and there exists a constant c *that depends only on* N *and* p *such that*

$$\iint_{\Omega_t} |v|^q \le c \left(\sup_{t\in(0,T)} \int_\Omega |v|^m \right)^{\frac{p}{N}} \iint_{\Omega_T} |\nabla v|^p. \tag{3.1}$$

We recall a very useful compactness result.

Lemma 3.2 (cf. [34], Corollary 4, page 85)**.** *Let* $X, B,$ *and* Y *be Banach spaces such that*

$$X \subset B \subset Y,$$

with compact embedding $X \to B$. *Let* F *be bounded in* $L^q(0,T;X)$ *where* $1 \le q < \infty$, *and* $\dfrac{\partial F}{\partial t} = \{\partial f/\partial t : f \in F\}$ *be bounded in* $L^1(0,T;Y)$. *Then* F *is relatively compact in* $L^q(0,T;B)$.

We conclude this section with a result proved in [29, Proposition 4.1] that we will use to prove the local L^∞-regularity together with the decay estimates. It deals with bounded initial data.

Proposition 3.3 (L^∞–decay estimates)**.** *Assume* (1.2). *Then all the solutions of* (1.1) *belonging to* $C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ *and corresponding to "initial data"* $u_0 \in L^\infty(\Omega)$ *satisfy the following decay estimate*

$$\|u(t)\|_\infty \le K_0 \frac{\|u_0\|_m}{(\alpha t)^\eta} \mathrm{e}^{-K\alpha t}\,, \quad a.e.\ t \in (0,T), \quad \forall\, m > \gamma\frac{N}{2}\,, \tag{3.2}$$

where $\eta > \dfrac{mN}{2m - N\gamma}$ *can be fixed arbitrarily and the constants* K_0 *and* K *only depend on* $m, \gamma, N, |\Omega|, \|u_0\|_m$*, and* η*. Hence, they are uniform for a set of bounded initial data having a uniform bound on their* $L^m(\Omega)$ *norm. In the estimate above the constants are independent from* T *, i.e. the case* $T = +\infty$ *is allowed.*

4. Proof of the existence results

The proofs of our results make use of many tools used by the author with M. Assunta Pozio in [29] which here are suitable adapted to treat our case of more irregular initial data.

Let us consider the following approximating problems

$$\begin{cases} (u_n)_t - \operatorname{div}(A(x,t,u_n)\nabla u_n) = 0 & \text{in} \quad \Omega_T, \\ u_n(x,t) = 0, & \text{on} \quad S_T, \\ u_n(x,0) = f_n(x), & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where

$$f_n \in L^\infty(\Omega), \quad \|f_n\|_{L^1(\Omega)} \le C_0, \quad f_n \to u_0 \quad \text{in } M_b(\Omega). \tag{4.2}$$

By Theorem 2.6 in [29], for every fixed $n \in \mathbb{N}$ there exists a solution $u_n \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^\infty(\Omega_T) \cap C^{\delta,\delta/2}(\Omega \times (0,T))$ (for a suitable $\delta \in (0,1)$) of (4.1) satisfying

$$\|u_n\|_{L^\infty(\Omega_T)} \le \|f_n\|_{L^\infty(\Omega)}. \tag{4.3}$$

In the proofs of our existence results we need the following estimates.

Lemma 4.1. *Assume* (1.2) *with* $0 < \gamma < \frac{2}{N}$ *and let*

$$q \in \left[1, \frac{2}{2-\gamma}\right). \tag{4.4}$$

Then there exists a constant C*, independent on* n*, such that the following estimate holds true*

$$\iint_{\Omega_T} \frac{|\nabla u_n|^q}{(1+|u_n|)^{\gamma q}} \le C. \tag{4.5}$$

Moreover the constant C *depends only on* α*,* $|\Omega|$*,* γ*,* T*,* C_0 *and on* q*, where* C_0 *is as in* (4.2)*. Finally it results*

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C_0. \tag{4.6}$$

Proof. Let δ be a positive constant and take

$$v_n = \left(1 - \frac{1}{(1+|u_n|)^\delta}\right) \operatorname{sign}(u_n)$$

as a test function in (4.1) (the use of such a test function can be made rigorous by means of Steklov averaging process). By the structure assumption (1.2), for any $t \in (0,T]$, we obtain

$$\int_\Omega \left[\int_0^{u_n}\left(1-\frac{1}{(1+|s|)^\delta}\right)\operatorname{sign}(s)ds\right]_0^t + \alpha\delta\iint_{\Omega_t}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\delta+1}} \le 0.$$

Notice that it results

$$\int_0^{u_n}\left(1-\frac{1}{(1+|s|)^\delta}\right)\operatorname{sign}(s)ds = |u_n| - \frac{(1+|u_n|)^{1-\delta}}{1-\delta} + \frac{1}{1-\delta},$$

and hence we deduce that

$$\begin{aligned}&\int_\Omega \left[|u_n(t)| - \frac{(1+|u_n|)^{1-\delta}(t)}{1-\delta} - |f_n| + \frac{(1+|f_n|)^{1-\delta}}{1-\delta}\right]dx \\ &\quad + \alpha\delta\iint_{\Omega_t}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\delta+1}} \le 0.\end{aligned} \tag{4.7}$$

If $\delta > 1$ from (4.7) it follows that

$$\int_\Omega |u_n(t)| + \alpha\delta\iint_{\Omega_t}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\delta+1}} \le \int_\Omega |f_n| + \frac{|\Omega|}{\delta-1},$$

from which passing to the sup on $(0,T)$ we deduce

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} + \alpha\delta\iint_{\Omega_T}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\delta+1}} \le C_0 + \frac{|\Omega|}{\delta-1}, \tag{4.8}$$

and hence ($\delta > 1$ is arbitrary)

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \le C_0. \tag{4.9}$$

Thus (4.6) is proved and to conclude the proof it remains to show that (4.5) holds true too.

If $0 < \delta < 1$ from (4.7) and (4.9) we deduce that

$$\begin{aligned}&\int_\Omega |u_n(t)|dx + \alpha\delta\iint_{\Omega_t}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\delta+1}} \\ &\le \int_\Omega \frac{(1+|u_n|)^{1-\delta}(t)}{1-\delta} + \|f_n\|_{L^1(\Omega)} \\ &\le \int_\Omega \frac{1+|u_n(t)|}{1-\delta}dx + C_0 \le \frac{c_4}{1-\delta} + C_0,\end{aligned} \tag{4.10}$$

where $c_4 = |\Omega| + C_0$. Thus for every $0 < \delta < 1$ it results

$$\iint_{\Omega_t}\frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\delta+1}} \le \frac{c_4}{\alpha\delta(1-\delta)} + \frac{C_0}{\alpha\delta}. \tag{4.11}$$

Since we are assuming that $\gamma \leq 1$, for $q \in (1, \frac{2}{2-\gamma})$, let $\delta = \frac{2-q(2-\gamma)}{q}$. Then it results $\delta > 0$ since $q < \frac{2}{2-\gamma}$ and we have also that $\delta < 1$ since $q > 1$, hence we get

$$\delta = \frac{2(1-q)+\gamma q}{q} < \gamma \leq 1\,.$$

Using the Hölder inequality we deduce

$$\begin{aligned}\iint_{\Omega_T} \frac{|\nabla u_n|^q}{(1+|u_n|)^{\gamma q}} &= \iint_{\Omega_T} \frac{|\nabla u_n|^q}{(1+|u_n|)^{(\gamma+\delta+1)\frac{q}{2}}}(1+|u_n|)^{(\gamma+\delta+1)\frac{q}{2}-\gamma q} \\ &\leq \left(\iint_{\Omega_T} \frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma+\delta+1}}\right)^{\frac{q}{2}} \left(\iint_{\Omega_T} (1+|u_n|)^{[(\gamma+\delta+1)\frac{q}{2}-\gamma q]\frac{2}{2-q}}\right)^{1-\frac{q}{2}}. \qquad (4.12)\end{aligned}$$

The definition of δ entails

$$[(\gamma+\delta+1)\frac{q}{2}-\gamma q]\frac{2}{2-q} = \frac{(\gamma+1)q+2-q(2-\gamma)-2\gamma q}{2-q} = 1\,,$$

hence (4.9), (4.11) and (4.12) imply (4.5) for every $q \in (1, \frac{2}{2-\gamma})$, where the constant C also depends on q. □

4.1. Proof of Theorem 2.4

Using u_n as test function in (4.1), using assumption (1.3) (but it would be sufficient to assume (1.2)) and noticing that

$$\int_{t_0}^{T}\int_{\Omega} \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\gamma}} \leq \int_{t_0}^{T}\int_{\Omega} \frac{|\nabla u_n|^2}{(1+|u_n|)^{\gamma}},$$

we deduce that

$$\int_{t_0}^{T}\int_{\Omega} \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\gamma}} \leq c_1, \qquad (4.13)$$

for every fixed $0 \leq t_0 < T$, where $c_1 = \dfrac{\|u_n(t_0)\|_{L^2(\Omega)}}{\alpha}$.

A consequence of (4.6) of Lemma 4.1 and the hypothesis (2.4) is that we can apply Proposition 3.3 with $m = 1$ and deduce that for every fixed $k \in \mathbb{N}$ with $k > \frac{1}{T}$ it results

$$\|u_n\|_{L^\infty(\Omega\times(\frac{1}{k},T))} \leq K_0 \frac{\|u_n(\frac{1}{k})\|_1}{(\alpha\frac{1}{k})^\eta} \mathrm{e}^{-K\alpha\frac{1}{k}} \leq c_0, \qquad (4.14)$$

where c_0 is a constant that depends only on k, N, $|\Omega|$, C_0, α and γ and hence independent on n. Moreover (1.3) implies that for all entries a_{ij} of the symmetric matrix $A(x,t,\sigma)$ we have

$$|a_{ij}(x,t,\sigma)| \leq \frac{\beta}{(1+|\sigma|)^\gamma}\,, \qquad \text{a.e. } (x,t) \in \Omega_T,\ \forall \sigma \in \mathbb{R}.$$

A further consequence of Lemma 4.1 is that there exists a subsequence of u_n, that we call again u_n, such that

$$A(x,t,u_n)\nabla u_n \rightharpoonup v \quad \text{weakly in} \quad L^q(\Omega_T)^N. \tag{4.15}$$

Applying (4.13) we deduce that

$$\|\nabla u_n\|_{L^2(\Omega\times(\frac{1}{k},T))} \leq c_1, \tag{4.16}$$

where c_1 is a constant that does not depend on n. As a matter of fact by estimate (4.14), $\|u_n(\frac{1}{k})\|_{L^\infty(\Omega)}$ is bounded by c_0 independently from n, hence this is true also for $\|u_n(\frac{1}{k})\|_{L^2(\Omega)}$. Thus applying (4.13) in the time interval $(\frac{1}{k},T)$, i.e. taking $u_n(\frac{1}{k})$ as initial data, we get

$$\int_{\frac{1}{k}}^T \int_\Omega \frac{|\nabla u_n|^2}{(1+|u_n|)^{2\gamma}} \leq c_2, \tag{4.17}$$

where c_2 is independent on n too. This last estimate and (4.14)) imply (4.16). Hence working in the set $\Omega\times(\frac{1}{k},T)$, by standard compactness arguments (see Lemma 3.2 and [29] for further details) we obtain a subsequence of u_n, that we call again u_n, converging strongly in $L^2(\Omega\times(\frac{1}{k},T))$ and a.e. in $\Omega\times(\frac{1}{k},T)$ to a function u. Moreover by (4.16), eventually passing to a subsequence, we also get that

$$\nabla u_n \rightharpoonup \nabla u \quad \text{weakly in} \quad L^2(\Omega\times(\frac{1}{k},T)). \tag{4.18}$$

By a diagonal argument we determine a subsequence that we denote again by u_n, such that

$$\begin{cases} u_n \to u \quad \text{a.e. in} \quad \Omega_T, \\ \nabla u_n \rightharpoonup \nabla u \quad \text{weakly in} \quad L^2_{loc}(0,T;L^2(\Omega)). \end{cases} \tag{4.19}$$

We need to prove that u is a weak solution of (1.1). Remark that u belongs to $L^\infty_{loc}(0,T;L^\infty(\Omega))\cap L^2_{loc}(0,T;H^1_0(\Omega))$.

By (4.19) and (4.15) we deduce that $v = A(x,t,u)\nabla u$.

To conclude that u solves the problem (1.1) it is sufficient to show that

$$\exists \lim_{n\to+\infty} \iint_{\Omega_T} (u_n-u)\varphi_t = 0, \quad \forall\varphi\in C_0^\infty(\Omega_T), \tag{4.20}$$

as the general case follows by density arguments. Hence let ε be arbitrarily fixed. We notice that

$$\left|\iint_{\Omega_T}(u_n-u)\varphi_t\right| \leq \left|\int_0^{\frac{1}{k}}\int_\Omega(u_n-u)\varphi_t\right| + \left|\int_{\frac{1}{k}}^T\int_\Omega(u_n-u)\varphi_t\right|. \tag{4.21}$$

Moreover by (4.6) and (4.19) we deduce that

$$\|u\|_{L^\infty(0,T;L^1(\Omega))} \leq c_3,$$

and for $\varphi \in W^{1,\infty}(0,T;L^\infty(\Omega))$ we get

$$\left|\int_0^{\frac{1}{k}}\int_\Omega (u_n-u)\varphi_t\right| \le 2c_3\|\varphi_t\|_{L^1(0,\frac{1}{k};L^\infty(\Omega))} \le \frac{2c_3\|\varphi_t\|_{L^\infty(\Omega_T)}}{k}. \tag{4.22}$$

Thus we can fix $k=k_0$ (sufficiently large) such that

$$\left|\int_0^{\frac{1}{k_0}}\int_\Omega (u_n-u)\varphi_t\right| \le \frac{\varepsilon}{2}. \tag{4.23}$$

Now thanks to (4.14) and to (4.19) we can apply the dominate convergence Theorem and conclude that if n tends to $+\infty$

$$\int_{\frac{1}{k_0}}^T\int_\Omega (u_n-u)\varphi_t \to 0,$$

and hence (4.20) follows.

Finally, the proof that u satisfies the decay estimate (2.6) is an immediate consequence of the construction of u as the limit a.e. in Ω_T of the approximating solutions u_n and on the following estimate (that can be obtained applying Proposition 3.3 and using (4.2))

$$\|u_n(t)\|_\infty \le K_0\frac{\|f_n\|_1}{(\alpha t)^\eta}\mathrm{e}^{-K\alpha t} \le \frac{K_1}{(\alpha t)^\eta}\mathrm{e}^{-K\alpha t}, \quad \text{a.e. } t\in(0,T),$$

where $K_1 = K_0C_0$, $\eta > \frac{N}{2-N\gamma}$ can be fixed arbitrarily and the constants K_0 and K only depend on $\gamma, N, |\Omega|, u_0$, and η.

q.e.d.

4.2. Proof of Theorem 2.5

Let u_n be as before a solution of the approximating problem (4.1). Since the hypotheses of Lemma 4.1 are satisfied, estimate (4.6) holds true. We estimate now ∇u_n. Let

$$\psi_k(s) = \int_0^s T_k(\sigma)\,d\sigma, \quad k>0, \tag{4.24}$$

where

$$T_k(\sigma) = \operatorname{sign}(\sigma)\min\{|\sigma|,k\}. \tag{4.25}$$

It results

$$\frac{1}{2}|T_k(s)|^2 \le \psi_k(s) \le k|s|, \quad \forall k>0, \quad \forall s\in\mathbb{R}. \tag{4.26}$$

Taking $T_k(u_n)$ as test function (again such a choice can be made rigorous by means of Steklov averaging process) we obtain

$$\iint_{\Omega_t}(u_n)_tT_k(u_n) + \alpha\iint_{\Omega_t}\frac{|\nabla T_k(u_n)|^2}{(1+|u_n|)^\gamma} \le 0. \tag{4.27}$$

Notice that by (4.26) we have

$$\begin{aligned}\langle (u_n)_t, T_k(u_n)\rangle = \iint_{\Omega_t} \frac{\partial \psi_k(u_n)}{\partial t} &= \int_\Omega \psi_k(u_n(t)) - \psi_k(f_n) \\ &\geq \frac{1}{2}\int_\Omega |T_k(u_n)|^2(t) - k\|f_n\|_{L^1(\Omega)},\end{aligned} \tag{4.28}$$

that together with (4.27) gives

$$\frac{1}{2}\|T_k(u_n)\|^2_{L^\infty(0,T;L^2(\Omega))} + \alpha \iint_{\Omega_T} \frac{|\nabla T_k(u_n)|^2}{(1+|u_n|)^\gamma} \leq kC_0, \tag{4.29}$$

where C_0 is, as before, the constant that appears in (4.2).

Let $\sigma \in [1,2)$ be a constant to be determined. By (4.29) it results

$$\begin{aligned}\iint_{\Omega_T} |\nabla T_k(u_n)|^\sigma &= \iint_{\Omega_T} \frac{|\nabla T_k(u_n)|^\sigma}{(1+|T_k(u_n)|)^{\gamma\frac{\sigma}{2}}}(1+|T_k(u_n)|)^{\gamma\frac{\sigma}{2}} \\ &\leq \left(\iint_{\Omega_T} \frac{|\nabla T_k(u_n)|^2}{(1+|u_n|)^\gamma}\right)^{\frac{\sigma}{2}} \left(\iint_{\Omega_T} (1+|T_k(u_n)|)^{\frac{\gamma\sigma}{2-\sigma}}\right)^{1-\frac{\sigma}{2}} \\ &\leq \left(\frac{k}{\alpha}\right)^{\frac{\sigma}{2}} C_0^{\frac{\sigma}{2}} \left(\iint_{\Omega_T} (1+|T_k(u_n)|)^{\frac{\gamma\sigma}{2-\sigma}}\right)^{1-\frac{\sigma}{2}}.\end{aligned} \tag{4.30}$$

Let $q = \sigma\frac{N+1}{N}$. Using inequality (3.1) and estimates (4.6) and (4.30) we obtain

$$\begin{aligned}\iint_{\Omega_T} |T_k(u_n)|^q &\leq c\left(\sup_{[0,T]}\int_\Omega |T_k(u_n)|\right)^{\frac{\sigma}{N}} \iint_{\Omega_T} |\nabla T_k(u_n)|^\sigma \\ &\leq c_4 k^{\frac{\sigma}{2}} \left(\iint_{\Omega_T} (1+|T_k(u_n)|)^{\frac{\gamma\sigma}{2-\sigma}}\right)^{1-\frac{\sigma}{2}} \\ &\leq 2^{\frac{\gamma\sigma}{2}} c_4 k^{\frac{\sigma}{2}} \left[|\Omega_T|^{1-\frac{\sigma}{2}} + \left(\iint_{\Omega_T} |T_k(u_n)|^{\frac{\gamma\sigma}{2-\sigma}}\right)^{1-\frac{\sigma}{2}}\right],\end{aligned} \tag{4.31}$$

where $c_4 = c\frac{C_0^{\frac{\sigma}{N}+\frac{\sigma}{2}}}{\alpha^{\frac{\sigma}{2}}}$ is a constant depending only on C_0, σ, α and N. Choose $\sigma = 2 - \frac{\gamma N}{N+1}$. Notice that it is an admissible choice as $\sigma \geq 1$ is equivalent to the restriction $\gamma \leq 1 + \frac{1}{N}$. Moreover it results $q = \frac{\gamma\sigma}{2-\sigma}$ and hence the previous inequality gives (using Young inequality)

$$\iint_{\Omega_T} |T_k(u_n)|^q \leq c_5 k, \quad \forall k \in \mathbb{N} \tag{4.32}$$

where c_5 is a constant depending only on C_0, σ, α, N, γ and $|\Omega_T|$. Using (4.32) in (4.30) we have

$$\iint_{\Omega_T} |\nabla T_k(u_n)|^\sigma \leq c_6 k, \tag{4.33}$$

where c_6 is a positive constant independent on n and k. From (4.32) it follows also

$$|\{|u_n| > k\}| \leq \frac{c_5}{k^{q-1}}. \tag{4.34}$$

Hence the sequence $\{u_n\}$ is equibounded in $M^s(\Omega_T)$ where $s \equiv q-1 = \frac{2}{N}+1-\gamma$. Moreover, analogously as before, from (4.33) it follows

$$|\{|\nabla u_n| > \lambda\} \cap \{|u_n| \leq k\}| \leq \frac{c_6 k}{\lambda^\sigma}. \tag{4.35}$$

Thus by (4.34) and (4.35) we deduce for every k, $\lambda > 0$

$$|\{|\nabla u_n| > \lambda\}| \leq |\{|\nabla u_n| > \lambda\} \cap \{|u_n| \leq k\}| + |\{|u_n| > k\}| \leq \frac{c_6 k}{\lambda^\sigma} + \frac{c_5}{k^{q-1}}.$$

Choosing $k = \lambda^{\frac{\sigma}{q}}$ we obtain

$$|\{|\nabla u_n| > \lambda\}| \leq \frac{c_7}{\lambda^h}$$

where $c_7 = c_5 + c_6$ and $h = \sigma\left(1 - \frac{1}{q}\right) = 1 + \frac{1-\gamma N}{N+1}$, i.e. $\nabla u_n \in M^h$. As just noticed in Remark 2.6 assumption (2.7) assures that $h > 1$. Hence ∇u_n belongs also to $L^r(\Omega_T)$ for every $1 \leq r < h$ and its $L^r(\Omega_T)$-norm can be estimated by a constant depending only from the data. Hence there exists a subsequence, that we can call again u_n, such that

$$u_n \to u, \quad \text{weakly in} \quad L^r(0,T;W_0^{1,r}(\Omega)).$$

Moreover, since from the equation (4.1) we deduce that $\{(u_n)_t\}$ is bounded in $L^r(0,T;W_0^{-1,r}(\Omega))$, using compactness arguments (see Lemma 3.2), it follows that

$$u_n \to u, \quad \text{strongly in} \quad L^r(\Omega_T),$$

and hence, up to subsequences,

$$u_n \to u, \quad \text{a.e. in} \quad \Omega_T.$$

Thus we can pass to the limit on n in (4.1) and conclude that u solves (1.1). Moreover from the previous estimates we deduce that $u \in M^s(\Omega_T) \cap L^\infty(0,T;L^1(\Omega)) \cap L^r(0,T;W_0^{1,r}(\Omega))$ and $\nabla u \in M^h(\Omega_T)$. Finally, proceeding as in the proof of Theorem 2.4 we deduce that u satisfies the decay estimate (2.6).

q.e.d.

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