Rend. Mat. Appl. (7).
Volume 43 (2022), 1 – 22

RENDICONTI DI MATEMATICA
E DELLE SUE APPLICAZIONI

Entropy solutions of some nonlinear elliptic problems with measure data in Musielak-Orlicz spaces

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Abstract. *In this work, we prove an existence theorem of entropy solutions for nonlinear elliptic problem of the type* $-\operatorname{div}(a(x,u,\nabla u)+\Phi(x,u))=\mu$ *in* Ω, *in the setting of Musielak-Orlicz spaces. The lower order term* Φ *verifies the natural growth condition, no* Δ_2*-condition is assumed on the Musielak function, and the datum* μ *is assumed to belong to* $L^1(\Omega)+W^{-1}E_\psi(\Omega)$.

1. Introduction and Basic Assumptions

In this note, we prove an existence theorem of entropy solutions for nonlinear elliptic problem whose model is :

$$\begin{cases} A(u)-\operatorname{div}\Phi(x,u)=f-\operatorname{div}F & in \quad \Omega \\ u\equiv 0 & on \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N, N\geq 2,\ A(u)=-\operatorname{div}(a(x,u,\nabla u))$ is a Leray-Lions operator defined from the space $W_0^1L_\varphi(\Omega)$ into its dual $W^{-1}L_{\bar\varphi}(\Omega)$, with φ and $\bar\varphi$ are two complementary Musielak-Orlicz functions and where a is a function satisfying the following conditions:

$$a:\Omega\times\mathbb{R}\times\mathbb{R}^N\longrightarrow\mathbb{R}^N \text{ is a Carathéodory function.} \tag{1.2}$$

There exist two Musielak-Orlicz functions φ and P such that $P\prec\prec\varphi$, a positive function $d(x)\in E_{\bar\varphi}(\Omega),\ \alpha>0$ and $k_i>0$ for $i=1,\ldots,4$, such that for a.e. $x\in\Omega$ and all $s\in\mathbb{R}^N$ and all $\xi,\ \xi'\in\mathbb{R}^N,\ \xi\neq\xi'$:

$$|a(x,s,\xi)|\leq k_1\left(d(x)+\bar\varphi_x^{-1}\left(P\left(x,k_2|s|\right)\right)+\bar\varphi_x^{-1}\left(\varphi\left(x,k_3|\xi|\right)\right)\right), \tag{1.3}$$

$$\left(a(x,s,\xi)-a\left(x,s,\xi'\right)\right)\left(\xi-\xi'\right)>0, \tag{1.4}$$

$$a(x,s,\xi).\xi\geq\alpha\varphi(x,|\xi|). \tag{1.5}$$

The lower order term Φ is a Carathéodory function satisfying, for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, the following condition:

$$|\Phi(x,s)|\leq\overline{P_x}^{-1}P_x(|s|). \tag{1.6}$$

2020 Mathematics Subject Classification: 35J25, 35J60, 46E30.
Keywords: Musielak-Orlicz-Sobolev spaces, elliptic equation, entropy solutions, truncations.

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The right-hand side of (1.1) is assumed to satisfy

$$\begin{array}{c}\mu \in L^1(\Omega) + W^{-1}E_{\bar{\varphi}}(\Omega) \text{ such that },\\ \mu = f - \operatorname{div}(F),\\ \text{with } f \in L^1(\Omega) \text{ and } F \in (E_{\bar{\varphi}}(\Omega))^N.\end{array} \tag{1.7}$$

The notion of entropy solution, used in [19], allows us to give a meaning to a possible solution of (1.1)

Boccardo proved in [19], for p such that $2 - 1/N < p < N$, the existence and regularity of an entropy solution u of problem (1.1) . For the case $\phi = 0$ and f is a bounded measure, Bnilan et al. proved in [10] the existence and uniqueness of entropy solutions,the same problem is treated using the notion of entropy solution introduced in [31] where $f \in L^1(\Omega)$, and $F \in L^{p'}(\Omega)^N$. We mention as a parallel development, the work of Lions and Murat [32] who consider similar problems in the context of the renormalized solutions introduced by Diperna and Lions [28] for the study of the Boltzmann equations,they can prove existence and uniqueness of renormalized solution.

For the case of Orlicz spaces, Gossez and Mustonen have studied in [29] the following strongly nonlinear elliptic problem

$$A(u) + g(x,u) = f \quad \text{in} \quad \Omega \tag{1.8}$$

they have proved the existence of solutions for the unilateral elliptic problem (1.8).

Several researches deals with the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts (see [2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 17, 18, 21, 22, 20, 23, 24, 27, 28, 34, 36, 37, 38] for more details).

In this work, we will prove the existence of solutions for the elliptic problem (1.1) in Musielak-Orlicz-Sobolev spaces, where the lower order term Φ verifies the natural growth condition, no Δ_2-condition is assumed on the Musielak function, and the datum μ is assumed to belong to $L^1(\Omega) + W^{-1}E_{\psi}(\Omega)$. Where $\Phi \equiv 0$ one of the motivations for studying the Eq. (1.1) in the generalized Orlicz-Sobolev spaces come from the fact that these spaces are more adequate for studying the behavior of some physical phenomenon like the flow electro-rheological fluids that is characterized by their ability to drastically change the mechanical properties under the influence of an extremal electromagnetic field. A mathematical model of electro-rheological fluids was proposed by Ruzicka in [40] .

The paper is organized as follows: In Section 2, we give some preliminaries and background. Section 3 is devoted to some auxiliary lemmas which can be used to our result. In Section 4, we state our main result and finally give the prove of an existence solution in Section 5 .

2. Some Preliminary Results

Here we give some definitions and properties that concern Musielak-Orlicz spaces (see [33]). Let Ω be an open subset of $\mathbb{R}^n$, a Musielak-Orlicz function φ is a real-

valued function defined in $\Omega \times \mathbb{R}^+$ such that:
a) $\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, nondecreasing, continuous, $\varphi(x,0)=0$, $\varphi(x,t)>0$ for all $t>0$ and $\lim\limits_{t\to 0}\sup\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t}=0$ and $\lim\limits_{t\to\infty}\inf\limits_{x\in\Omega}\dfrac{\varphi(x,t)}{t}=\infty$).
b) $\varphi(.,t)$ is a measurable function for all $t \geq 0$.

For a Musielak-Orlicz function φ, let $\varphi_x(t)=\varphi(x,t)$ and let φ_x^{-1} be the non-negative reciprocal function with respect to t, i.e. the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t$$

The Musielak-Orlicz function φ is said to satisfy the Δ_2 -condition if for some $k>0$, and a nonnegative function h, integrable in Ω, we have

$$\varphi(x,2t) \leq k\varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2 -condition near infinity. Let φ and γ be two Musielak-Orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$:

$$\gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0, \text{ (resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0, \quad \text{(resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x,ct)}{\varphi(x,t)}\right)=0).$$

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x,|u(x)|)\,dx.$$

The set $K_\varphi(\Omega) = \{u\colon \Omega \to \mathbb{R} \text{ measurable}/\rho_{\varphi,\Omega}(u) < \infty\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \left\{u\colon \Omega \longrightarrow \mathbb{R} \text{ measurable}/\ \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}$$

For a Musielak-Orlicz function φ we put:

$$\psi(x,s) = \sup_{t>0}\{st - \varphi(x,t)\}.$$

Note that ψ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0/ \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_\psi \leq 1} \int_\Omega |u(x)v(x)|\, dx$$

where ψ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent (see [33]). The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$, It is a separable space (see [33, Theorem 7.10]).

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\left(\frac{u_n - u}{\lambda}\right) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega)/\ \forall|\alpha| \leq m, D^\alpha u \in L_\varphi(\Omega)\}$$

and

$$W^m E_\varphi(\Omega) = \{u \in E_\varphi(\Omega)/\ \forall|\alpha| \leq m, D^\alpha u \in E_\varphi(\Omega)\}$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ with nonnegative integers $\alpha_i, |\alpha| = |\alpha_1| + \cdots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz Sobolev space. Let for $u \in W^m L_\varphi(\Omega)$:

$$\bar{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}\left(D^\alpha u\right) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\left\{\lambda > 0/\ \bar{\rho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

these functionals are a convex modular and a norm on $W^m L_M(\Omega)$, respectively, and the pair $\left(W^m L_\varphi(\Omega), \|\cdot\|^m_{\varphi,\Omega}\right)$ is a Banach space if φ satisfies the following condition (see [33]):

$$\text{There exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c_0. \tag{2.2}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product

$$\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi,$$

this subspace is $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closed. The space $W_0^m L_\varphi(\Omega)$ is defined as the $\sigma\left(\Pi L_\varphi, \Pi E_\psi\right)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$, and the space $W_0^m E_\varphi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. The following spaces of distributions will also be used:

$$W^{-m}L_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) / \ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \Big\}$$

and

$$W^{-m}E_\psi(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) / \ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \Big\}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \bar{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

We recall that

$$\varphi(x,t) \leq t\psi^{-1}(\varphi(x,t)) \leq 2\varphi(x,t) \quad \text{ for all } t \geq 0. \tag{2.3}$$

For φ and her complementary function ψ, the following inequality is called the Young inequality (see [33]):

$$ts \leq \varphi(x,t) + \psi(x,s), \quad \forall t,s \geq 0, \text{ a.e. } x \in \Omega. \tag{2.4}$$

This inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1\,. \tag{2.5}$$

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \quad \text{ if } \|u\|_{\varphi,\Omega} > 1 \tag{2.6}$$

and

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \quad \text{ if } \|u\|_{\varphi,\Omega} \leq 1. \tag{2.7}$$

For two complementary Musielak-Orlicz functions φ and ψ, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Hölder inequality (see [33]):

$$\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{\varphi,\Omega} \| |v| \|_{\psi,\Omega}. \tag{2.8}$$

Definition 2.1. A Musielak function φ is called locally integrable on Ω if

$$\int_E \varphi(x,t)dx = \int_\Omega \varphi\left(x, t\chi_E(x)\right) dx < +\infty$$

for all $t \geq 0$ and all measurable set $E \subset \Omega$ with $\text{mes}(E) < +\infty$.

Remark 2.2. If $P \prec\prec \varphi$ near infinity such that P is locally integrable on Ω, then $\forall c > 0$ there exists a nonnegative integrable function h such that

$$P(x,t) \leq \varphi(x,ct) + h(x), \text{ for all } t \geq 0 \text{ and for a.e. } x \in \Omega.$$

Definition 2.3. A Musielak function φ satisfies the log-Hölder continuity condition on Ω if there exists a constant $A > 0$ such that

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t^{\left(\frac{A}{\log\left(\frac{1}{x-y|}\right)}\right)}$$

for all $t \geq 1$ and for all $x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$.

3. Some Auxiliary Lemmas

We will use the following technical lemmas.

Lemma 3.1 ([1])**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the log-Hölder continuity such that*

$$\bar{\varphi}(x,1) \leq c_1 \qquad a.e \text{ in } \Omega \text{ for some } c_1 > 0 \tag{3.1}$$

Then $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W_0^1 L_\varphi(\Omega)$ for the modular convergence.

Remark 3.2. Note that if $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$, then (3.1) holds:

Example 3.3. Let $p \in P(\Omega)$ a bounded variable exponent on Ω, such that there exists a constant $A > 0$ such that for all points $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$, we have the inequality

$$|p(x) - p(y)| \leq \frac{A}{\log\left(\frac{1}{|x-y|}\right)}$$

We can verify that the Musielak function defined by $\varphi(x,t) = t^{p(x)} \log(1+t)$ satisfies the conditions of Lemma 3.1

Lemma 3.4 ([1])**.** *(Poincare's inequality: Integral form) Let Ω be a bounded Lipschitz domain of $R^N (N \geq 2)$ and let φ be a Musielak function satisfying the conditions of Lemma 3.1. Then there exists positive constants β, η and λ depending only on Ω and φ such that*

$$\int_\Omega \varphi(x,|v|)dx \leq \beta + \eta \int_\Omega \varphi(x, \lambda|\nabla v|)dx \text{ for all } v \in W_0^1 L_\varphi(\Omega). \tag{3.2}$$

Lemma 3.5 ([1] Poincaré's inequality)**.** *Let Ω be a bounded Lipchitz domain of $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the same conditions of Lemma 3.4. Then there exists a constant $C > 0$ such that*

$$\|v\|_\varphi \leq C\|\nabla v\|_\varphi \quad \forall v \in W_0^1 L_\varphi(\Omega).$$

Lemma 3.6 ([35]). *Let $F\colon \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1L_\varphi(\Omega)$. Then $F(u) \in W_0^1L_\varphi(\Omega)$.*

Moreover, if the set D of discontinuity points of F' is finite, we have

$$\frac{\partial}{\partial x_i}F(u) = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e \text{ in } \{x \in \Omega : u(x) \in D\} \\ 0 & a.e \text{ in } \{x \in \Omega : u(x) \notin D\} \end{cases}$$

Lemma 3.7 ([17]). *Suppose that Ω satisfies the segment property and let $u \in W_0^1L_\varphi(\Omega)$. Then, there exists a sequence $(u_n) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1L_\varphi(\Omega).$$

Furthermore, if $u \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \leq (N+1)\|u\|_\infty$.

Lemma 3.8 ([30]). *Let (f_n), $f \in L^1(\Omega)$ such that*
i) $f_n \geq 0$ a.e in Ω,
ii) $f_n \longrightarrow f$ a.e in Ω,
iii) $\displaystyle\int_\Omega f_n(x)\,dx \longrightarrow \int_\Omega f(x)\,dx$.
Then $f_n \longrightarrow f$ strongly in $L^1(\Omega)$.

Lemma 3.9 ([18]). *If a sequence $g_n \in L_\varphi(\Omega)$ converges in measure to a measurable function g and if g_n remains bounded in $L_\varphi(\Omega)$, then $g \in L_\varphi(\Omega)$ and $g_n \rightharpoonup g$ for $\sigma(\Pi L_\varphi, \Pi E_\psi)$.*

Lemma 3.10 ([18]). *Let u_n, $u \in L_\varphi(\Omega)$. If $u_n \to u$ with respect to the modular convergence, then $u_n \to u$ for $\sigma(L_\varphi(\Omega), L_\psi(\Omega))$.*

Lemma 3.11 ([26]). *If $P \prec \varphi$ and $u_n \to u$ for the modular convergence in $L_\varphi(\Omega)$ then $u_n \to u$ strongly in $E_P(\Omega)$.*

Lemma 3.12 ([39] Jensen inequality). *Let $\varphi\colon \mathbb{R} \longrightarrow \mathbb{R}$ a convex function and $g\colon \Omega \longrightarrow \mathbb{R}$ is function measurable, then*

$$\varphi\left(\int_\Omega g d\mu\right) \leq \int_\Omega \varphi \circ g \, d\mu.$$

Lemma 3.13 ([25] The Nemytskii Operator). *Let Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions.*

Let $f\colon \Omega \times \mathbb{R}^p \to \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:

$$|f(x,s)| \leq c(x) + k_1\psi_x^{-1}\varphi(x, k_2|s|)$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$. Then the Nemytskii Operator N_f defined by $N_f(u)(x) = f(x,u(x))$ is continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p = \prod\left\{u \in L_M(\Omega) : d\left(u, E_M(\Omega)\right) < \frac{1}{k_2}\right\}$$

into $(L_\psi(\Omega))^q$ for the modular convergence. Furthermore if $c(\cdot) \in E_\gamma(\Omega)$ and $\gamma \prec\prec \psi$ then N_f is strongly continuous from $\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p$ to $(E_\gamma(\Omega))^q$.

Throughout the paper, T_k denotes the truncation function at height $k \geq 0$:

$$T_k(s) = \max(-k, \min(k, s)).$$

4. Main Result

Theorem 4.1. *Under the assumptions* (1.2)-(1.7)*; there exists an entropy solution* u *of the problem* (1.1) *in the following sense:*

$$\begin{cases} T_k(u) \in W_0^1 L_\varphi(\Omega) \\ \displaystyle\int_\Omega a(x,u,\nabla u)\nabla T_k(u-v)dx + \int_\Omega \Phi(x,u)\nabla T_k(u-v)dx \\ \displaystyle\leq \int_\Omega f T_k(u-v)dx + \int_\Omega F\nabla T_k(u-v)dx, \end{cases}$$

for every $v \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$ *and for every* $k \geq 0$.

5. Proof of Theorem 4.1

5.1. Approximate problem

For $n \in \mathbb{N}^*$, let define the following approximations of f, and Φ

Let f_n be a sequence of $L^\infty(\Omega)$ functions that converge strongly to f in $L^1(\Omega)$, and $\|f_n\|_{L^1} \leq \|f\|_{L^1}$. Let $\Phi_n(x,s) = \Phi(x, T_n(s))$.
Then we consider the approximate equation (1.1) for $n \geq 1$: $u_n \in W_0^1 L_\varphi(\Omega)$

$$-\operatorname{div}\Big(a(x,u_n,\nabla u_n)\Big) + \operatorname{div}\Big(\Phi_n(x,u_n)\Big) = f_n - \operatorname{div}(F) \quad \text{in } \mathcal{D}'(\Omega). \tag{5.1}$$

there exists at last one solution $u_n \in W_0^1 L_\varphi(\Omega)$ of (5.1) (see [26]).

5.2. A Priori Estimates

Taking $v = T_k(u_n), k > 0$, as test function in (1.1) we get

$$\begin{aligned} &\int_\Omega a(x,u_n,\nabla u_n).\nabla T_k(u_n)\,dx + \int_\Omega \Phi_n(x,u_n)\nabla T_k(u_n)\,dx \\ &\qquad = \int_\Omega f_n T_k(u_n)\,dx + \int_\Omega F_n \nabla T_k(u_n)\,dx. \end{aligned} \tag{5.2}$$

Thanks to (1.6) and applying Young inequality one has

$$\begin{aligned} \int_\Omega \Phi_n(x,u_n)\nabla T_k(u_n)\,dx &\leq \int_\Omega \overline{P_x}^{-1} P_x(|T_k(u_n)|)\nabla T_k(u_n)\,dx \\ &\leq \int_\Omega P(x,|T_k(u_n)|)\,dx + \int_\Omega P(x,|\nabla T_k(u_n)|)\,dx. \end{aligned}$$

By choosing $\varepsilon > 0$ such that $\varepsilon = \frac{\alpha}{(\alpha+2)(\lambda\eta+1)}$, and thanks to Remark 2.2, we can have

$$\int_\Omega \Phi_n(x,u_n)\nabla T_k(u_n)\,dx \leq \int_\Omega \varphi\left(x, \frac{\varepsilon\lambda}{\lambda}|T_k(u_n)|\right)dx + \varepsilon\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx + 2\int_\Omega h(x)dx$$

Thanks to Lemma3.4 and choosing $v = \frac{|T_k(u_n)|}{\lambda}$ with $\varepsilon\lambda \leq 1$, we get

$$\int_\Omega \Phi_n(x,u_n)\nabla T_k(u_n)\,dx \leq \beta + \varepsilon(\lambda\eta+1)\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx + 2\int_\Omega h(x)dx \tag{5.3}$$

On the other hand we have

$$\int_\Omega F_n\nabla T_k(u_n)\,dx \leq \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla T_k(u_n)\,u_n|)\,dx \tag{5.4}$$

from (5.2), (5.3) and (5.4) we get

$$\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n dx \leq \varepsilon(\lambda\eta+1)\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx + \beta + c_1k + 2\int_\Omega h(x)dx + \frac{\alpha}{2}\int_\Omega \varphi(x,|\nabla T_k(u_n)\,u_n|)\,dx$$

which give

$$\int_{\{|u_n|\leq k\}} a(x,u_n,\nabla u_n)\nabla u_n dx \leq \left(\varepsilon(\lambda\eta+1)+\frac{\alpha}{2}\right)\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx + \beta + c_1k + 2\int_\Omega h(x)dx\,.$$

Thus, by virtue of (1.5) and since $(\frac{\alpha}{2} - \varepsilon(\lambda\eta+1)) > 0$, we get

$$\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx \leq c_1 + c_2k. \tag{5.5}$$

Now, choosing $v = \frac{1}{\lambda}|T_k(u_n)|$ in (3.2) we obtain

$$\int_\Omega \varphi\left(x,\frac{1}{\lambda}|T_k(u_n)|\right)dx \leq \beta + \eta\int_\Omega \varphi(x,|\nabla T_k(u_n)|)\,dx \leq c_3 + c_4k \tag{5.6}$$

then

$$\begin{aligned}
\operatorname{meas}\{|u_n|>k\} &\leq \frac{1}{\inf_{x\in\Omega}\varphi\left(x,\frac{k}{\lambda}\right)}\int_{\{|u_n|>k\}}\varphi\left(x,\frac{k}{\lambda}\right)dx \\
&\leq \frac{1}{\inf_{x\in\Omega}\varphi\left(x,\frac{k}{\lambda}\right)}\int_{\Omega}\varphi\left(x,\frac{1}{\lambda}\left|T_k\left(u_n\right)\right|\right)dx \qquad (5.7)\\
&\leq \frac{c_3+c_4k}{\inf_{x\in\Omega}\varphi\left(x,\frac{k}{\lambda}\right)} \qquad \forall n, \forall k>0\,.
\end{aligned}$$

For any $\delta>0$, we have

$$\begin{aligned}
&\operatorname{meas}\{|u_n-u_m|>\delta\}\\
&\qquad \leq \operatorname{meas}\{|u_n|>k\}+\operatorname{meas}\{|u_m|>k\}+\operatorname{meas}\{|T_k\left(u_n\right)-T_k\left(u_m\right)|>\delta\}
\end{aligned}$$

and so that

$$\operatorname{meas}\{|u_n-u_m|>\delta\} \leq \frac{2\left(c_3+c_4k\right)}{\inf_{x\in\Omega}\varphi\left(x,\frac{k}{\lambda}\right)}+\operatorname{meas}\{|T_k\left(u_n\right)-T_k\left(u_m\right)|>\delta\}\,. \qquad (5.8)$$

From (5.5), we deduce that $T_k\left(u_n\right)$ is bounded in $W_0^1L_\varphi(\Omega)$ and we can assume that $T_k\left(u_n\right)$ is a Cauchy sequence in measure in Ω.

Let $\varepsilon>0$, by using (5.8) and the fact that $\frac{2(c_3+c_4k)}{\inf_{x\in\Omega}\varphi\left(x,\frac{k}{\lambda}\right)}\to 0$ as $k\to+\infty$ there exists $k(\varepsilon)>0$ such that

$$\operatorname{meas}\{|u_n-u_m|>\delta\}\leq\varepsilon, \qquad \text{for all } n,m\geq n_0(k(\varepsilon),\delta)\,.$$

This proves that (u_n) is a Cauchy sequence in measure in Ω; thus, u_n converges almost everywhere to some measurable function u. Finally, for all $k>0$, we have for a subsequence

$$\begin{cases} T_k\left(u_n\right)\rightharpoonup T_k(u) & \text{weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi,\Pi E_{\bar\varphi}\right)\\ T_k\left(u_n\right)\to T_k(u) & \text{strongly in } E_\varphi(\Omega) \text{ and a.e. in } \Omega. \end{cases} \qquad (5.9)$$

5.3. Boundedness of $\left(a\left(x,u_n,\nabla u_n\right)\right)_n$ in $\left(L_{\bar\varphi}(\Omega)\right)^N$.

Let $\vartheta\in\left(E_\varphi(\Omega)\right)^N$ such that $\|\vartheta\|_{\varphi,\Omega}=1$. We have

$$\int_\Omega\left[a\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)-a\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\right]\left[\nabla T_k\left(u_n\right)-\frac{\vartheta}{k_3}\right]dx\geq 0\,.$$

This implies that

$$\begin{aligned}
&\int_\Omega \frac{1}{k_3} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta dx \\
&\leq \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx \\
&\quad -\int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\left(\nabla T_k\left(u_n\right)-\frac{\vartheta}{k_3}\right) dx \\
&\leq k C_1+C_2-\int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right) \nabla T_k\left(u_n\right) dx \\
&\quad +\frac{1}{k_3} \int_\Omega a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right) \vartheta dx .
\end{aligned}$$

By using Young's inequality in the last two terms of the last side and (5.5) we get

$$\begin{aligned}
&\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta dx \leq (kC_1+C_2)k_3 \\
&\quad +3 k_1\left(1+k_3\right) \int_\Omega \bar{\varphi}\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) dx \\
&\quad +3 k_1 k_3 \int_\Omega \varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right) dx+3 k_1 \int_\Omega \varphi(x,|\vartheta|)\, dx \\
&\leq (kC_1+C_2)k_3+3k_1k_3(kC_1+C_2)+3k_1 \\
&\quad +3 k_1\left(1+k_3\right) \int_\Omega \bar{\varphi}\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) dx .
\end{aligned}$$

Now, by using (1.3) and the convexity of $\bar{\varphi}$ we get

$$\bar{\varphi}\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{\vartheta}{k_3}\right)\right|}{3 k_1}\right) \leq \frac{1}{3}\left(\bar{\varphi}(x, d(x))+P\left(x, k_2\left|T_k\left(u_n\right)\right|\right)+\varphi(x,|\vartheta|)\right).$$

Thanks to Remark 2.2 there exists $h \in L^1(\Omega)$ such that

$$P\left(x, k_2\left|T_k\left(u_n\right)\right|\right) \leq P\left(x, k_2 k\right) \leq \varphi(x, 1)+h(x);$$

then by integrating over Ω we deduce that

$$\begin{aligned}
&\int_\Omega \bar{\varphi}\left(x, \frac{\left|a\left(x, T_k\left(u_n\right), \frac{v}{k_3}\right)\right|}{3 k_1}\right) dx \leq \frac{1}{3}\left(\int_\Omega \bar{\varphi}(x, c(x)) dx+\int_\Omega h(x) dx\right. \\
&\quad \left.+\int_\Omega \varphi(x, 1) dx+\int_\Omega \varphi(x,|\vartheta|) dx\right) \leq c_k^{\prime}
\end{aligned}$$

where $c_k^{\prime}$ is a constant depending on k. So,

$$\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \vartheta dx \leq c_k^{\prime}, \quad \forall \vartheta \in\left(E_{\varphi}(\Omega)\right)^N \quad \text{ with } \|\vartheta\|_{\varphi, \Omega}=1$$

and thus $\|a(x, T_k(u_n), \nabla T_k(u_n))\|_{\bar{\varphi},\Omega} \leq c'_k$, which implies that,

$$(a(x, T_k(u_n), \nabla T_k(u_n)))_n \text{ is bounded in } L_{\bar{\varphi}}(\Omega)^N. \tag{5.10}$$

5.4. Almost everywhere convergence of the gradients

Let $v_j \in \mathfrak{D}(\Omega)$ be a sequence which converges to $T_k(u)$ for the modular convergence in $W_0^1 L_\varphi(\Omega)$. Let $h > 2k > 0$ and define the functions

$$\begin{aligned} \omega_{n,j}^h &= T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(v_j)), \\ \omega_j^h &= T_{2k}(u - T_h(u) + T_k(u) - T_k(v_j)), \\ \omega^h &= T_{2k}(u - T_h(u)). \end{aligned}$$

Choosing $\omega_{n,j}^h$, as test function in (5.1) we obtain

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n) \cdot \nabla \omega_{n,j}^h dx + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(x, u_n) \nabla \omega_{n,j}^h dx \\ &= \int_\Omega f_n \omega_{n,j}^h dx + \int_\Omega F \cdot \nabla \omega_{n,j}^h dx. \end{aligned} \tag{5.11}$$

Put $m = h + 5k$, and denote by $\varepsilon(n, j, h)$ any quantity such that

$$\lim_{h\to\infty} \lim_{j\to\infty} \lim_{n\to\infty} \varepsilon(n, j, h) = 0,$$

and by $\varepsilon_h(n, j)$ any quantity such that

$$\lim_{j\to\infty} \lim_{n\to\infty} \varepsilon_h(n, j) = 0, \text{ for } h \text{ fixed}.$$

Remark that $\nabla \omega_{n,j}^h = 0$ on the set $\{x \in \Omega : |u_n| > m\}$, then by tanking to (5.11) we have

$$\begin{aligned} &\int_\Omega a(x, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla \omega_{n,j}^h dx \\ &\quad + \int_{\{m \leq |u_n| \leq m+1\}} \Phi(x, T_m(u_n)) \nabla \omega_{n,j}^h dx \\ &= \int_\Omega f_n \omega_{n,j}^h dx + \int_\Omega F \cdot \nabla \omega_{n,j}^h dx. \end{aligned} \tag{5.12}$$

Thanks to (5.9), we have $\omega_{n,j}^h \to \omega_j^h$ weakly * in $L^\infty(\Omega)$ as $n \to \infty$ and then

$$\int_\Omega f_n \omega_{n,j}^h dx \to \int_\Omega f \omega_j^h dx \text{ as } n \to \infty,$$

$$\int_\Omega F \cdot \nabla \omega_{n,j}^h dx \to \int_\Omega F \cdot \nabla \omega_j^h dx \text{ as } n \to \infty$$

letting j and h to infinity and by applying the Lebesgue's Theorem we obtain

$$\int_\Omega f_n \omega_{n,j}^h dx = \varepsilon(n, j, h),$$

$$\int_{\Omega} F \cdot \nabla \omega_{n,j}^h dx = \varepsilon(n,j,h)\,.$$

Concerning the first term of (5.12), we have

$$\begin{aligned}\int_{\Omega} a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \nabla \omega_{n,j}^h dx \\ = \int_{\{|u_n|\le k\}} a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\right) dx \\ + \int_{\{|u_n|>k\}} a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \nabla \omega_{n,j}^h dx\end{aligned} \tag{5.13}$$

For the second term of the right-hand side of (5.13) we have

$$\begin{aligned}\int_{\{|u_n|>k\}} a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \nabla \omega_{n,j}^h dx \\ \ge -\int_{\{|u_n|>k\}} \left|a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right)\right| \left|\nabla v_j\right| dx\,.\end{aligned} \tag{5.14}$$

since $\left|a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right)\right|$ is bounded in $L_{\bar{\varphi}}(\Omega)$, we have for a subse- quence

$$\left|a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right)\right| \rightharpoonup l_m$$

weakly in $L_{\bar{\varphi}}(\Omega)$, for $\sigma\left(\Pi L_{\varphi}, \Pi E_{\varphi}\right)$ as $n \to \infty$ and since $\nabla v_j \chi_{\{u_n>k\}} \to \nabla v_j \chi_{\{u>k\}}$ strongly in $E_{\varphi}(\Omega)$ as $n \to \infty$, we have

$$-\int_{\{|u_n|>k\}} \left|a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right)\right| \left|\nabla v_j\right| dx \to -\int_{\{|u|>k\}} l_m \left|\nabla v_j\right| dx$$

as $n \to \infty$. By using the modular convergence of v_j, we get

$$-\int_{\{|u|>k\}} l_m \left|\nabla v_j\right| dx \to -\int_{\{|u|>k\}} l_m |\nabla u| dx \quad \text{as } j \to \infty\,.$$

Finally,

$$-\int_{\{|u_n|>k\}} \left|a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right)\right| \left|\nabla v_j\right| dx = \varepsilon_h(n,j).$$

By virtue of (5.12), (5.13) and (5.14) we have

$$\begin{aligned}\int_{\Omega} a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \nabla \omega_{n,j}^h dx \\ \ge \int_{\Omega} a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right)\left(\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right)\right) dx \\ + \varepsilon_h(n,j).\end{aligned} \tag{5.15}$$

It follows that,

$$\begin{aligned}\int_\Omega a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \nabla \omega_{n,j}^h dx \\ \geq \int_\Omega \left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), T_k\left(v_j\right) \chi_j^s\right)\right] \\ \times\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right] dx \\ +\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right] dx \\ -\int_{\Omega \backslash \Omega_j^a} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) dx+\varepsilon_h(n, j),\end{aligned} \tag{5.16}$$

where χ_j^s denotes the characteristic function of the subset

$$\Omega_j^s=\left\{x \in \Omega: \quad\left|\nabla T_k\left(v_j\right)\right| \leq s\right\} .$$

We can easily show that

$$-\int_{\Omega \backslash \Omega_j^s} a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) . \nabla T_k\left(v_j\right) dx=\varepsilon(n, j, s). \tag{5.17}$$

Concerning the second term of (5.16), remark that by using Lemma 3.13 and the fact that $\nabla T_k\left(u_n\right) \rightarrow \nabla T_k(u)$ weakly in $\left(L_{\varphi}(\Omega)\right)^N$, by (5.9) we obtain

$$a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \rightarrow a\left(x, T_k(u), \nabla T_k\left(v_j\right) \chi_j^s\right)$$

strongly in $E_{\bar{\varphi}}(\Omega)^N$ as $n \rightarrow \infty$, then

$$\begin{aligned}\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right] dx \\ \rightarrow \int_\Omega a\left(x, T_k(u), \nabla T_k\left(v_j\right) \chi_j^s\right)\left[\nabla T_k(u)-\nabla T_k\left(v_j\right) \chi_j^s\right] dx \text{ as } n \rightarrow \infty .\end{aligned}$$

On the other hand, since $\nabla T_k\left(v_j\right) \chi_j^s \rightarrow \nabla T_k(u) \chi^s$ strongly in $E_{\varphi}(\Omega)^N$ as $j \rightarrow \infty$, we have

$$\int_\Omega a\left(x, T_k(u), \nabla T_k\left(v_j\right) \chi_j^s\right)\left[\nabla T_k(u)-\nabla T_k\left(v_j\right) \chi_j^s\right] dx \rightarrow 0 \text{ as } j \rightarrow \infty$$

and thus

$$\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right] dx=\varepsilon(n, j, s). \tag{5.18}$$

By (5.16), (5.17) and (5.18) one has

$$\begin{aligned}&\int_{\Omega} a\left(x, T_m\left(u_n\right), \nabla T_m\left(u_n\right)\right) \nabla \omega_{n, j}^h d x \\ &\quad \geq \int_{\Omega}\left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), T_k\left(v_j\right) \chi_j^s\right)\right] \\ &\qquad \times\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right] d x+\varepsilon_h(n, j)+\varepsilon(n, j, s) .\end{aligned} \tag{5.19}$$

Now, for the second term on the left-hand side of (5.12), we will show that the sequence $\Phi\left(x, T_m\left(u_n\right)\right)$ converges strongly in $E_{\varphi}(\Omega)^N$ By using the pointwise convergence of u_n to u as $n \rightarrow \infty$, we obtain

$$\bar{P}\left(x, \frac{\left|\Phi\left(x, T_m\left(u_n\right)\right)-\Phi\left(x, T_m(u)\right)\right|}{\mu}\right) \rightarrow 0 \text{ a.e. in}$$

and for μ and n large enough, we have

$$\begin{aligned}\bar{P}\left(x, \frac{\left|\Phi\left(x, T_m\left(u_n\right)\right)-\Phi\left(x, T_m(u)\right)\right|}{\mu}\right) &\leq \frac{1}{\mu} P\left(x,\left|T_m\left(u_n\right)\right|\right) \\ &\quad+\frac{1}{\mu} P\left(x,\left|T_m(u)\right|\right) \\ &\leq \frac{2}{\mu} P(x, m)=g_m(x)\end{aligned}$$

where $g_m \in L^1(\Omega)$. By applying Lebesgue's dominated convergence theorem, we obtain

$$\Phi\left(x, T_m\left(u_n\right)\right) \rightarrow \Phi\left(x, T_m(u)\right)$$

with respect to modular convergence in $L_{\bar{P}}(\Omega)$ as $n \rightarrow \infty$ since $\bar{\varphi} \prec\prec \bar{P}$, then, thanks to Lemma 3.11 we obtain $\Phi\left(x, T_m\left(u_n\right)\right) \rightarrow \Phi\left(x, T_m(u)\right)$ in $E_{\bar{\varphi}}(\Omega)$, and by virtue of $\nabla T_{2k}\left(u_n\right) \rightarrow \nabla T_{2k}(u)$ weakly in $\left(L_{\bar{\varphi}}(\Omega)\right)^N$, as $n \rightarrow \infty$ and then $h \rightarrow \infty$ we get,

$$\begin{aligned}\int_{\Omega} \Phi\left(x, T_m\left(u_n\right)\right) \nabla \omega_{n, j}^h d x &= \int_{\Omega} \Phi\left(x, T_m(u)\right) \nabla T_{2k}\left(u-T_h(u)\right) d x+\varepsilon_h(n, j) \\ &= \varepsilon(n, j, s) .\end{aligned} \tag{5.20}$$

Thanks to (5.12), (5.19) and (5.20) we obtain

$$\begin{aligned}&\int_{\Omega}\left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), T_k\left(v_j\right) \chi_j^s\right)\right] \\ &\quad \cdot\left[\nabla T_k\left(u_n\right) \quad-\nabla T_k\left(v_j\right) \chi_j^s\right] d x \leq \varepsilon(n, j, h, s).\end{aligned} \tag{5.21}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega \left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right)\right] \cdot\left[\nabla T_k\left(u_n\right)\right. \\
&\left.-\nabla T_k(u) \chi^s\right] d x \\
&=\int_\Omega\left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right. \\
&\quad \left.-a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\right] \cdot\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right] d x \\
&\quad +\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot\left[\nabla T_k\left(v_j\right) \chi_j^s-\nabla T_k(u) \chi^s\right] d x \\
&\quad -\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right) \cdot\left[\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right] d x \\
&\quad +\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right] d x .
\end{aligned}$$

By letting n to infinity and using the modular convergence of v_j in the last three terms of the right-hand side of the above equality, it is easy to get

$$\begin{aligned}
&\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\left[\nabla T_k\left(v_j\right) \chi_j^s-\nabla T_k(u) \chi^s\right] d x=\varepsilon(n, j) \\
&\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right)\left[\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right] d x=\varepsilon(n, j)
\end{aligned}$$

and

$$\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\left[\nabla T_k\left(u_n\right)-\nabla T_k\left(v_j\right) \chi_j^s\right] d x=\varepsilon(n, j). \tag{5.22}$$

It follows taht

$$\begin{aligned}
&\int_\Omega \left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right)\right.\right. \\
&\qquad \left.\left.\nabla T_k(u) \chi^s\right)\right] \cdot\left[\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right] d x \\
&=\int_\Omega\left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right)\right] \cdot\left[\nabla T_k\left(u_n\right)\right. \\
&\quad \left.-\nabla T_k\left(v_j\right) \chi_j^s\right] d x+\varepsilon(n, j).
\end{aligned}$$

Combining this with (5.21) we get

$$\begin{aligned}
&\int_\Omega\left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)\right. \\
&\quad \left.-a\left(x, T_k\left(u_n\right), T_k(u) \chi^s\right)\right] \cdot\left[\nabla T_k\left(u_n\right)-\nabla T_k(u) \chi^s\right] d x \\
&\quad \leq \varepsilon(n, j, h, s)
\end{aligned}$$

in which we pass to the limit as n, j, h and s tend to infinity to get

$$\begin{aligned}
&\int_\Omega \left[a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right)-a\left(x, T_k\left(u_n\right), \nabla T_k(u) \chi^s\right)\right] \cdot\left[\nabla T_k\left(u_n\right)\right. \\
&\left.-\nabla T_k(u) \chi^s\right] d x \rightarrow 0 \text { as } n, s \rightarrow \infty .
\end{aligned}$$

As in [11], we deduce that there exists a subsequence still denoted by u_n such that

$$\nabla u_n \to \nabla u \text{ a.e in } \Omega \tag{5.23}$$

which implies that, for all $k > 0$,

$$a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \rightharpoonup a\left(x, T_k(u), \nabla T_k(u)\right) \tag{5.24}$$

weakly in $L_{\bar{\varphi}}(\Omega)^N$ for $\sigma\left(\Pi L_{\bar{\varphi}}, \Pi E_{\varphi}\right)$.

From the estimate (5.21), we can read

$$\begin{aligned}
&\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) dx \\
&\quad \leq \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \chi_j^s dx \\
&\quad\quad + \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(v_j\right) \chi_j^s\right) \cdot \left[\nabla T_k\left(u_n\right) - \nabla T_k\left(v_j\right) \chi_j^s\right] dx \\
&\quad\quad + \int_{\Omega \setminus \Omega_s} l_k \cdot \nabla T_k(u) dx + \varepsilon_h(n, j).
\end{aligned}$$

by using (5.22), we have

$$\begin{aligned}
&\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) dx \\
&\quad \leq \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(v_j\right) \chi_j^s dx + \int_{\Omega \setminus \Omega_s} l_k \cdot \nabla T_k(u) dx + \varepsilon_h(n, j).
\end{aligned}$$

Passing to the limit sup over n and j in both sides of this inequality yields

$$\begin{aligned}
&\limsup_{n \to \infty} \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \cdot \nabla T_k\left(u_n\right) dx \\
&\quad \leq \int_\Omega a\left(x, T_k(u), \nabla T_k(u)\right) \cdot \nabla T_k(u) \chi^s dx + \int_{\Omega \setminus \Omega_a} l_k \cdot \nabla T_k(u) dx.
\end{aligned}$$

In which, we can pass to the limit in s to obtain

$$\limsup_{n \to \infty} \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx \leq \int_\Omega a\left(x, T_k(u), \nabla T_k(u)\right) \nabla T_k(u) dx.$$

Now,by applying Fatou's Lemma we have

$$\int_\Omega a\left(x, T_k(u), \nabla T_k(u)\right) \nabla T_k(u) dx \leq \liminf_{n \to \infty} \int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx.$$

Which implies that

$$\int_\Omega a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) dx \to \int_\Omega a\left(x, T_k(u), \nabla T_k(u)\right) \nabla T_k(u) dx,$$

as $n \to \infty$ and, using Lemma 3.8, we conclude that

$$a\left(x, T_k\left(u_n\right), \nabla T_k\left(u_n\right)\right) \nabla T_k\left(u_n\right) \to a\left(x, T_k(u), \nabla T_k(u)\right) \nabla T_k(u) \text{ in } L^1(\Omega).$$

By thanking to (1.5) we have

$$T_k\left(u_n\right) \to T_k(u) \text{ in } W_0^1 L_{\varphi}(\Omega) \text{ for the modular convergence, for all } k > 0.$$

5.5. Passing to the limit

Let $v \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega)$. By Lemma 3.7, there exists a sequence $(w_j) \subset \mathfrak{D}(\Omega)$ such that $w_j \to v$ in $W_0^1 L_\varphi(\Omega)$ for the modular convergence

$$\text{and } \|w_j\|_{\infty,\Omega} \leq (N+1)\|v\|_{\infty,\Omega}.$$

Using $T_k(u_n - w_j)$ as a test function in (5.1), we obtain

$$\begin{aligned} &\int_\Omega a\left(x, T_\rho(u_n), \nabla T_\rho(u_n)\right) \nabla T_k(u_n - w_j)\, dx \\ &\quad + \int_\Omega \Phi\left(x, T_\rho(u_n)\right) \nabla T_k(u_n - w_j)\, dx \\ &\quad \leq \int_\Omega f_n T_k(u_n - w_j)\, dx + \int_\Omega F \nabla T_k(u_n - w_j)\, dx, \end{aligned} \tag{5.25}$$

where $\rho = k + (N+1)\|v\|_\infty$.

The first term of the left-hand side of (5.25) reads as

$$\begin{aligned} &\int_\Omega a\left(x, T_\rho(u_n), \nabla T_\rho(u_n)\right) \nabla T_k(u_n - w_j)\, dx \\ &\quad = \int_{\{|u_n - w_j| \leq k\}} a\left(x, T_\rho(u_n), \nabla T_\rho(u_n)\right) \nabla u_n dx \\ &\qquad - \int_{\{|u_n - w_j| \leq k\}} a\left(x, T_\rho(u_n), \nabla T_\rho(u_n)\right) \nabla w_j dx. \end{aligned}$$

By Fatou's lemma, we have

$$\begin{aligned} &\int_{\{|u - w_j| \leq k\}} a\left(x, T_\rho(u), \nabla T_\rho(u)\right) \nabla u dx \\ &\quad \leq \liminf_{n\to\infty} \int_{\{|u_n - w_j| \leq k\}} a\left(x, T_\rho(u_n), \nabla T_\rho(u_n)\right) \nabla u_n dx, \end{aligned}$$

and using (5.24) we get

$$\begin{aligned} &\lim_{n\to\infty} \int_{\{|u_n - w_j| \leq k\}} a\left(x, T_\rho(u_n), \nabla T_\rho(u_n)\right) \nabla w_j dx \\ &\quad = \int_{\{|u - w_j| \leq k\}} a\left(x, T_\rho(u), \nabla T_\rho(u)\right) \nabla w_j dx. \end{aligned}$$

We conclude that

$$\begin{aligned} &\int_\Omega a\left(x, T_\rho(u), \nabla T_\rho(u)\right) \nabla T_k(u - w_j)\, dx \\ &\quad \leq \liminf_{n\to\infty} \int_\Omega a\left(x, T_\rho(u_n), \nabla T_\rho(u_n)\right) \nabla T_k(u_n - w_j)\, dx, \end{aligned}$$

as above we obtain $\Phi_n(x, T_\rho(u_n)) \to \Phi(x, T_\rho(u))$ in $E_{\bar{\varphi}}(\Omega)$ as $n \to \infty$, and using the fact that $\nabla T_k(u_n - w_j) \rightharpoonup \nabla T_k(u - w_j)$ as $n \to \infty$, we can easily see that

$$\int_\Omega \Phi(x, T_\rho(u_n)) \nabla T_k(u_n - w_j)\, dx \to \int_\Omega \Phi(x, T_\rho(u)) \nabla T_k(u - w_j)\, dx$$

and

$$\int_\Omega F \nabla T_k(u_n - w_j)\, dx \to \int_\Omega F \nabla T_k(u - w_j)\, dx,$$

since $T_k(u_n - w_j) \to T_k(u - w_j)$ weakly in $L^\infty(\Omega)$ as $n \to \infty$, we have

$$\int_\Omega f_n T_k(u_n - w_j)\, dx \to \int_\Omega f T_k(u - w_j)\, dx \text{ as } n \to \infty.$$

This allows us to let $n \to \infty$ in both sides of (5.25), to obtain

$$\int_\Omega a(x, u, \nabla u) \nabla T_k(u - w_j)\, dx + \int_\Omega \Phi(x, u) \nabla T_k(u - w_j)\, dx \\ \leq \int_\Omega f T_k(u - w_j)\, dx,$$

in which we can easily pass to the limit as $j \to \infty$ to get

$$\int_\Omega a(x, u, \nabla u) \nabla T_k(u - v) dx + \int_\Omega \Phi(x, u) \nabla T_k(u - v) dx \\ \leq \int_\Omega f T_k(u - v) dx + \int_\Omega F \nabla T_k(u - v) dx,$$

for all $k > 0$; we can deduce that u is an entropy solution of the problem (1.1).This completes the proof of Theorem 4.1 .

Acknowledgements. The authors are grateful to Prof. Abdelmoujib Benkirane for his comments and suggestions.

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Received: 25 June 2020/Accepted: 10 June 2021/Published online: 28 June 2021

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