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Arithmetic Properties For (r, s)-Regular Partition Functions With Distinct Parts

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Abstract. *For any relatively prime integers r and s, let $a_{r,s}(n)$ denote the number of (r, s)-regular partitions of a positive integer of n into distinct parts. Prasad and Prasad (2018) proved many infinite families of congruences modulo 2 for $a_{3,5}(n)$. In this paper, we establish families of congruences modulo 2 and 4 for $a_{r,s}(n)$ with $(r, s) \in \{(2, 5), (2, 7), (4, 5), (4, 9)\}$. For example, we show that for all $\beta \geq 0$ and $n \geq 0$, we have*

$$a_{2,5}\left(4 \cdot 5^{2\beta+1}n + \frac{37 \cdot 5^{2\beta} - 1}{6}\right) \equiv 0 \pmod 4.$$

1. Introduction

The partition of a positive integer n is a non-increasing sequence of positive integers whose sum is equal to n. If $p(n)$ denote the number of partitions of a positive integer n and we adopt the convention $p(0) = 1$, then the generating function for $p(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q; q)_\infty}, \tag{1.1}$$

where

$$(a; q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n). \tag{1.2}$$

Throughout this paper, we write

$$f_k := (q^k; q^k)_\infty, \quad \text{for any integer} \quad k \geq 1.$$

Ramanujan [11] established the following beautiful congruences for all $n \geq 0$:

$$p(5n+4) \equiv 0 \pmod 5, \quad p(7n+5) \equiv 0 \pmod 7 \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11}.$$

Ramanujan's congruences on $p(n)$ have motivated many mathematicians to seek similar results for restricted partition functions. One example is the ℓ-regular

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partition function $b_\ell(n)$, which counts the number of partitions of n in which no part is divisible by ℓ and whose generating function satisfies the identity

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{f_\ell}{f_1}. \tag{1.3}$$

Many results on the arithmetic of $b_\ell(n)$ have been established (see, for example, [2, 9, 12]).

For relatively prime integers r and s, an (r, s)-regular partition is one in which none of the parts is divisible by r or s. Denote by $a_{r,s}(n)$, the number of (r, s)-regular partitions of n into distinct parts. For example, $a_{2,5}(13) = 2$ since the $(2, 5)$-regular partitions of 13 into distinct parts are 13 and 9+3+1. The generating function for $a_{r,s}(n)$ satisfies the identity

$$\sum_{n=0}^{\infty} a_{r,s}(n)q^n = \frac{(-q;q)_\infty(-q^{rs};q^{rs})_\infty}{(-q^r;q^r)_\infty(-q^s;q^s)_\infty}. \tag{1.4}$$

Prasad and Prasad [10] proved many infinite families of congruences modulo 2 for $a_{3,5}(n)$.

In this paper, we establish families of congruences modulo 2 for $a_{2,5}(n)$, $a_{2,7}(n)$, $a_{4,5}(n)$ and $a_{4,9}(n)$. We also prove congruences modulo 4 for $a_{2,5}(n)$. The congruences are listed in the following theorems:

Theorem 1.1. *For every $n \geq 0$, we have*

$$a_{2,5}(4n+2) \equiv 0 \pmod 2 \tag{1.5}$$

and

$$a_{2,5}(4n) \equiv \begin{cases} 1 \pmod 2, & \text{if } n \text{ is a pentagonal number} \\ 0 \pmod 2, & \text{otherwise.} \end{cases} \tag{1.6}$$

Theorem 1.2. *Let $p > 5$ be a prime with $\left(\frac{-10}{p}\right) = -1$ and $1 \leq j \leq p-1$. Then for all $\gamma \geq 0$, we have*

$$\sum_{n=0}^{\infty} a_{2,5}\Big(4\cdot p^{2\gamma}n + \frac{7\cdot p^{2\gamma}-1}{6}\Big)q^n \equiv f_2f_5 \pmod 2, \tag{1.7}$$

$$a_{2,5}\Big(4\cdot p^{2\gamma+1}(pn+j) + \frac{7\cdot p^{2\gamma+2}-1}{6}\Big) \equiv 0 \pmod 2, \tag{1.8}$$

$$\sum_{n=0}^{\infty} a_{2,5}\Big(20\cdot p^{2\gamma}n + \frac{55\cdot p^{2\gamma}-1}{6}\Big)q^n \equiv f_1f_{10} \pmod 2 \tag{1.9}$$

and

$$a_{2,5}\Big(20\cdot p^{2\gamma+1}(pn+j) + \frac{55\cdot p^{2\gamma+2}-1}{6}\Big) \equiv 0 \pmod 2. \tag{1.10}$$

Theorem 1.3. *If* $w_1 \in \{13, 37\}$ *and* $w_3 \in \{41, 89\}$*. Then for all* $\beta \geq 0$*, we have*

$$\sum_{n=0}^{\infty} a_{2,5}\Big(4 \cdot 5^{2\beta} n + \frac{13 \cdot 5^{2\beta} - 1}{6}\Big) q^n \equiv 2q^2 f_1 f_{20}^3 \pmod{4}, \tag{1.11}$$

$$\sum_{n=0}^{\infty} a_{2,5}\Big(4 \cdot 5^{2\beta+1} n + \frac{17 \cdot 5^{2\beta+1} - 1}{6}\Big) q^n \equiv 2f_4^3 f_5 \pmod{4}, \tag{1.12}$$

$$a_{2,5}\Big(4 \cdot 5^{2\beta+1} n + \frac{w_1 \cdot 5^{2\beta} - 1}{6}\Big) \equiv 0 \pmod{4} \tag{1.13}$$

and

$$a_{2,5}\Big(4 \cdot 5^{2(\beta+1)} n + \frac{w_3 \cdot 5^{2\beta+1} - 1}{6}\Big) \equiv 0 \pmod{4}. \tag{1.14}$$

Theorem 1.4. *Let* $p > 7$ *be a prime with* $\Big(\frac{-14}{p}\Big) = -1$ *and* $1 \leq j \leq p-1$*. Then for all* $\alpha \geq 0$*, we have*

$$\sum_{n=0}^{\infty} a_{2,7}\Big(2 \cdot p^{2\alpha} n + \frac{5 \cdot p^{2\alpha} - 1}{4}\Big) q^n \equiv f_1 f_{14} \pmod{2}, \tag{1.15}$$

$$a_{2,7}\Big(2 \cdot p^{2\alpha+1}(pn + j) + \frac{5 \cdot p^{2\alpha+2} - 1}{4}\Big) \equiv 0 \pmod{2}, \tag{1.16}$$

$$\sum_{n=0}^{\infty} a_{2,7}\Big(14 \cdot p^{2\alpha} n + \frac{21 \cdot p^{2\alpha} - 1}{4}\Big) q^n \equiv f_2 f_7 \pmod{2} \tag{1.17}$$

and

$$a_{2,7}\Big(14 \cdot p^{2\alpha+1}(pn + j) + \frac{21 \cdot p^{2\alpha+2} - 1}{4}\Big) \equiv 0 \pmod{2}. \tag{1.18}$$

Theorem 1.5. *If* $w \in \{13, 17\}$*, then for all* $\alpha \geq 0$*, we have*

$$\sum_{n=0}^{\infty} a_{4,5}\Big(2 \cdot 5^{\alpha} n + \frac{5^{\alpha} - 1}{2}\Big) q^n \equiv f_1 f_5 \pmod{2} \tag{1.19}$$

and

$$a_{4,5}\Big(2 \cdot 5^{\alpha+1} n + \frac{w \cdot 5^{\alpha} - 1}{2}\Big) \equiv 0 \pmod{2}. \tag{1.20}$$

Theorem 1.6. *Let* $p > 5$ *be a prime with* $\Big(\frac{-5}{p}\Big) = -1$ *and* $1 \leq j \leq p-1$*. Then for all* $\alpha \geq 0$*, we have*

$$\sum_{n=0}^{\infty} a_{4,5}\Big(2 \cdot p^{2\alpha} n + \frac{p^{2\alpha} - 1}{2}\Big) q^n \equiv f_1 f_5 \pmod{2} \tag{1.21}$$

and

$$a_{4,5}\Big(2 \cdot p^{2\alpha+1}(pn + j) + \frac{p^{2\alpha+2} - 1}{2}\Big) \equiv 0 \pmod{2}. \tag{1.22}$$

Theorem 1.7. *Let $w_1 \in \{3,5\}$, $w_2 \in \{13,25,37\}$ and $\alpha \geq 0$. Then*

$$a_{4,9}\Big(6n+w_1\Big) \equiv 0 \pmod 2, \tag{1.23}$$

$$a_{4,9}\Big(24n+19\Big) \equiv 0 \pmod 2, \tag{1.24}$$

$$a_{4,9}\Big(6\cdot 4^{\alpha+2}n + 20\cdot 4^{\alpha+1} - 1\Big) \equiv 0 \pmod 2, \tag{1.25}$$

$$a_{4,9}\Big(48n+w_2\Big) \equiv 0 \pmod 2 \tag{1.26}$$

and

$$a_{4,9}(48n+1) \equiv \begin{cases} 1 \pmod 2 & \textit{if } n \textit{ is a pentagonal number,} \\ 0 \pmod 2 & \textit{otherwise.} \end{cases} \tag{1.27}$$

2. Preliminaries

In this section, we collect the q-series identities that are used in our proofs. Recall that Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{2.1}$$

Important special cases of $f(a,b)$ [1, p. 36, Entry 22 (i), (ii), (iii)] are the theta-functions $\phi(q)$, $\psi(q)$ and $f(-q)$, which satisfy the identities

$$\phi(q) := f(q,q) = \sum_{n=0}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{{f_2}^5}{{f_1}^2{f_4}^2}, \tag{2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{{f_2}^2}{f_1}, \tag{2.3}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=0}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{2.4}$$

In terms of $f(a,b)$, Jacobi's triple product identity [1, Entry 19, p.35] is given by

$$f(a,b) = (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty. \tag{2.5}$$

Lemma 2.1 ([3, Theorem 2.2]). *For any prime $p \geq 5$, we have*

$$\begin{aligned} f_1 = &\sum_{\substack{k=-(p-1)/2 \\ k\neq (p^*-1)/6}}^{k=(p-1)/2} (-1)^k q^{(3k^2+k)/2} f\left(-q^{(3p^2+(6k+1)p)/2}, -q^{(3p^2-(6k+1)p)/2}\right) \\ &+ (-1)^{(p^*-1)/6} q^{(p^2-1)/24} f_{p^2}, \end{aligned} \tag{2.6}$$

where

$$p^* = \begin{cases} p, & if\ p \equiv 1 \pmod 6 \\ -p, & if\ p \equiv 5 \pmod 6). \end{cases}$$

Furthermore, if

$$\frac{-(p-1)}{2} \le k \le \frac{(p-1)}{2} \ and\ k \ne \frac{(p^*-1)}{6},$$

then

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

Lemma 2.2 ([1, p. 303, Entry 17(v)]). *We have that*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.7}$$

where $A(q) = f(-q^3, -q^4), B(q) = f(-q^2, -q^5)$ *and* $C(q) = f(-q, -q^6)$.

Lemma 2.3 ([5]). *We have that*

$$f_1 = f_{25}(R(q^5) - q - q^2R(q^5)^{-1}), \tag{2.8}$$

where

$$R(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}.$$

Lemma 2.4. *We have*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.9}$$

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}, \tag{2.10}$$

$$f_1 f_5^3 = f_2^3 f_{10} - q\frac{f_2^2 f_{10}^2 f_{20}}{f_4} + 2q^2 f_4 f_{20}^3 - 2q^3\frac{f_4^4 f_{10} f_{40}^2}{f_2 f_8^2}, \tag{2.11}$$

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \tag{2.12}$$

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.13}$$

$$f_1 f_7 = \frac{f_2 f_{14} f_{16}^2 f_{56}^5}{f_4 f_8 f_{28}^3 f_{112}^2} - qf_4 f_{28} + q^6\frac{f_2 f_8^5 f_{14} f_{112}^2}{f_4^3 f_{16}^2 f_{28} f_{56}}. \tag{2.14}$$

For the proof of (2.9), see Hirschhorn [7, p.40]. Equation (2.10) was proved by Xia and Yao [13]. For the proof of (2.11), see Naika et.al [8]. Equation (2.12) was proved by Hirschhorn and Sellers [6]. Equation (2.13) was proved by Hirschhorn et.al [4]. Equation (2.14) was proved by Xia [14, Lemma 3.14].

To end this section, we record the following congruence which can be easily proved using the binomial theorem: For all positive integers t and m we have

$${f_t}^{2m} \equiv {f_{2t}}^m \pmod 2. \tag{2.15}$$

3. Proof of Theorems 1.1-1.3

Proof of Theorem 1.1:

Proof. Setting $(r,s)=(2,5)$ in (1.4) and using elementary q-operations, we obtain

$$\sum_{n=0}^{\infty} a_{2,5}(n)q^n = \frac{f_2^2 f_5 f_{20}}{f_1 f_4 f_{10}^2}. \tag{3.1}$$

Combining (2.12) and (3.1), we find that

$$\sum_{n=0}^{\infty} a_{2,5}(n)q^n = \frac{f_8 f_{20}^3}{f_4 f_{10}^2 f_{40}} + q\frac{f_4^2 f_{40}}{f_2 f_8 f_{10}}. \tag{3.2}$$

Extracting the terms involving even powers of q of (3.2), we obtain

$$\sum_{n=0}^{\infty} a_{2,5}(2n)q^n = \frac{f_4 f_{10}^3}{f_2 f_5^2 f_{20}}. \tag{3.3}$$

In view of (2.15), (3.3) can be written as

$$\sum_{n=0}^{\infty} a_{2,5}(2n)q^n \equiv f_2 \pmod 2. \tag{3.4}$$

Extracting the terms involving odd powers of q from (3.4) yields (1.5). Finally, extracting the terms involving even powers of q from both sides of (3.4) and using (2.4) yields (1.6). □

Proof of Theorem 1.2:

Proof. Extracting the terms involving odd powers of q from both sides of (3.2), we obtain

$$\sum_{n=0}^{\infty} a_{2,5}(2n+1)q^n = \frac{f_2^2 f_{20}}{f_1 f_4 f_5}. \tag{3.5}$$

In view of (2.15), we can rewrite (3.5) as

$$\sum_{n=0}^{\infty} a_{2,5}(2n+1)q^n \equiv \frac{f_1 f_5^3}{f_2} \pmod 2. \tag{3.6}$$

Combining (2.11) and (3.6), we find that

$$\sum_{n=0}^{\infty} a_{2,5}(2n+1)q^n \equiv f_2^2 f_{10} - q\frac{f_2 f_{10}^2 f_{20}}{f_4} \pmod 2. \tag{3.7}$$

Extracting the terms involving even powers of q from both sides of (3.7), we obtain

$$\sum_{n=0}^{\infty} a_{2,5}(4n+1)q^n \equiv f_2 f_5 \pmod 2. \tag{3.8}$$

Equation (3.8) is the $\gamma = 0$ case of (1.7). Now suppose that (1.7) holds for some $\gamma \geq 0$. Using (2.6) in (1.7), we deduce that

$$\begin{aligned} &\sum_{n\geq 0}^{\infty} a_{2,5}\Big(4\cdot p^{2\gamma}n + \frac{7\cdot p^{2\gamma}-1}{6}\Big)q^n \\ &\equiv \Bigg[\sum_{\substack{k=-(p-1)/2 \\ k\neq (p^*-1)/6}}^{k=(p-1)/2} q^{3k^2+k} f\big(-q^{3p^2+(6k+1)p}, -q^{3p^2-(6k+1)p}\big) + q^{(p^2-1)/12} f_{2p^2}\Bigg] \\ &\times \Bigg[\sum_{\substack{m=-(p-1)/2 \\ m\neq (p^*-1)/6}}^{m=(p-1)/2} q^{5(3m^2+m)/2} f\Big(-q^{5(3p^2+(6m+1)p)/2}, -q^{5(3p^2-(6m+1)p)/2}\Big) + q^{5(p^2-1)/24} f_{5p^2}\Bigg] \pmod 2. \end{aligned} \tag{3.9}$$

Consider the congruence

$$3k^2 + k + 5\frac{(3m^2+m)}{2} \equiv \frac{7(p^2-1)}{24} \pmod p,$$

which is equivalent to

$$(12k+2)^2 + 10(6m+1)^2 \equiv 0 \pmod p.$$

Since $\left(\frac{-10}{p}\right) = -1$, the only solution of this congruence is $k = m = \frac{(p^*-1)}{6}$. Therefore, extracting the terms involving $q^{pn+7(p^2-1)/24}$ from both sides of (3.9), dividing by $q^{7(p^2-1)/24}$ and then replacing q^p by q, we find that

$$\sum_{n=0}^{\infty} a_{2,5}\Big(4\cdot p^{2\gamma+1}n + \frac{7\cdot p^{2\gamma+2}-1}{6}\Big)q^n \equiv f_{2p} f_{5p} \pmod 2, \tag{3.10}$$

which yields

$$\sum_{n=0}^{\infty} a_{2,5}\Big(4\cdot p^{2(\gamma+1)}n + \frac{7\cdot p^{2(\gamma+1)}-1}{6}\Big)q^n \equiv f_2 f_5 \pmod 2, \tag{3.11}$$

which is the $\gamma+1$ case of (1.7). On the other hand, extracting the terms involving q^{pn+j} $(1 \leq j \leq p-1)$ from (3.10), we arrive at (1.8). Employing (2.8) in (1.7) and then extracting the terms involving q^{5n+2} yields (1.9). Next, using (2.6) in (1.9) and proceeding as in the proof of (1.7), we arrive at

$$\sum_{n=0}^{\infty} a_{2,5}\Big(20 \cdot p^{2\gamma+1} n + \frac{55 \cdot p^{2\gamma+2} - 1}{6}\Big) q^n \equiv f_p f_{10p} \pmod{2}. \tag{3.12}$$

Finally, (1.10) follows from extracting the terms involving q^{pn+j} $(1 \leq j \leq p-1)$ from (3.12). □

Proof of Theorem 1.3:

Proof. Combining (2.9) and (3.3), we find that

$$\sum_{n=0}^{\infty} a_{2,5}(2n) q^n = \frac{f_4 f_{40}^5}{f_2 f_{10}^2 f_{20} f_{80}^2} + 2q^5 \frac{f_4 f_{20} f_{80}^2}{f_2 f_{10}^2 f_{40}}. \tag{3.13}$$

Extracting the terms involving odd powers of q from both sides of (3.13), we obtain

$$\sum_{n=0}^{\infty} a_{2,5}(4n+2) q^n = 2q^2 \frac{f_2 f_{10} f_{40}^2}{f_1 f_5^2 f_{20}}. \tag{3.14}$$

In view of (2.15), (3.14) can be written as

$$\sum_{n=0}^{\infty} a_{2,5}(4n+2) q^n \equiv 2q^2 f_1 f_{20}^3 \pmod{4}, \tag{3.15}$$

which is the $\beta = 0$ case of (1.11). Now assume that (1.11) holds for some $\beta \geq 0$. Employing (2.8) in (1.11), we arrive at

$$\sum_{n=0}^{\infty} a_{2,5}\Big(4 \cdot 5^{2\beta} n + \frac{13 \cdot 5^{2\beta} - 1}{6}\Big) q^n \equiv 2q^2 f_{20}^3 f_{25} \Big(R(q^5) - q - q^2 R(q^5)^{-1}\Big) \pmod{4}. \tag{3.16}$$

Extracting the terms involving q^{5n+3} from (3.16) yields (1.12). Next, using (2.8) in (1.12) and then extracting the terms involving q^{5n+2}, we obtain

$$\sum_{n=0}^{\infty} a_{2,5}\Big(4 \cdot 5^{2(\beta+1)} n + \frac{13 \cdot 5^{2(\beta+1)} - 1}{6}\Big) q^n \equiv 2q^2 f_1 f_{20}^3 \pmod{4}, \tag{3.17}$$

which is the $\beta+1$ case of (1.11). Employing (2.8) in (1.11) and then extracting the terms involving q^{5n+j} for $j \in \{0, 1\}$ yields (1.13). Finally, using (2.8) in (1.12) and then extracting terms involving q^{5n+j} for $j \in \{1, 3\}$ yields (1.14). □

4. Proof of Theorem 1.4

Proof. Setting $(r,s)=(2,7)$ in (1.4) and using elementary q-operations, we have

$$\sum_{n=0}^{\infty} a_{2,7}(n)q^n = \frac{f_2^2 f_7 f_{28}}{f_1 f_4 f_{14}^2}. \tag{4.1}$$

In view of (2.15), we can rewrite (4.1) as

$$\sum_{n=0}^{\infty} a_{2,7}(n)q^n \equiv \frac{f_1 f_7}{f_2} \pmod 2. \tag{4.2}$$

Combining (2.14) and (4.2), we find that

$$\sum_{n=0}^{\infty} a_{2,7}(n)q^n \equiv \frac{f_{14} f_{16}^2 f_{56}^5}{f_4 f_8 f_{28}^3 f_{112}^2} - q\frac{f_4 f_{28}}{f_2} + q^6 \frac{f_8^5 f_{14} f_{112}^2}{f_4^3 f_{16}^2 f_{28} f_{56}} \pmod 2. \tag{4.3}$$

Extracting the terms involving odd powers of q from both sides of (4.3), we obtain

$$\sum_{n=0}^{\infty} a_{2,7}(2n+1)q^n \equiv f_1 f_{14} \pmod 2, \tag{4.4}$$

which is the $\alpha = 0$ case of (1.15). Using (2.6) in (1.15) and proceeding as in the proof of (1.7), we arrive at

$$\sum_{n=0}^{\infty} a_{2,7}\Big(2\cdot p^{2\alpha+1}n + \frac{5\cdot p^{2\alpha+2}-1}{4}\Big)q^n \equiv f_p f_{14p} \pmod 2, \tag{4.5}$$

which yields

$$\sum_{n=0}^{\infty} a_{2,7}\Big(2\cdot p^{2(\alpha+1)}n + \frac{5\cdot p^{2(\alpha+1)}-1}{4}\Big)q^n \equiv f_1 f_{14} \pmod 2, \tag{4.6}$$

which is the $\alpha+1$ case of (1.15). On the other hand, extracting the terms involving q^{pn+j} $(1 \le j \le p-1)$ from (4.5), we arrive at (1.16). Next, using (2.7) in (1.15) and then extracting the terms involving q^{7n+2} yields (1.17). Now employing (2.6) in (1.17) and proceeding as in the proof of (1.15), we arrive at

$$\sum_{n=0}^{\infty} a_{2,7}\Big(14\cdot p^{2\gamma+1}n + \frac{21\cdot p^{2\gamma+2}-1}{4}\Big)q^n \equiv f_{2p} f_{7p} \pmod 2. \tag{4.7}$$

Finally, (1.18) follows from extracting the terms involving $q^{pn+j}(1 \le j \le p-1)$ from (4.7). ∎

5. Proof of Theorems 1.5-1.6

Proof of Theorem 1.5:

Proof. Setting $(r,s)=(4,5)$ in (1.4) and using elementary q-operations, we obtain

$$\sum_{n=0}^{\infty} a_{4,5}(n)q^n = \frac{f_2f_4f_5f_{40}}{f_1f_8f_{10}f_{20}}. \tag{5.1}$$

Combining (2.12) and (5.1), we find that

$$\sum_{n=0}^{\infty} a_{4,5}(n)q^n = \frac{f_4f_{20}}{f_2f_{10}} + q\frac{f_4^4f_{40}^2}{f_2^2f_8^2f_{20}^2}. \tag{5.2}$$

Extracting the terms involving even powers of q from both sides of (5.2), we obtain

$$\sum_{n=0}^{\infty} a_{4,5}(2n)q^n = \frac{f_2f_{10}}{f_1f_5}. \tag{5.3}$$

In view of (2.15), we can rewrite (5.3) as

$$\sum_{n=0}^{\infty} a_{4,5}(2n)q^n \equiv f_1f_5 \pmod{2}, \tag{5.4}$$

which is the $\alpha=0$ case of (1.19). Now assume that (1.19) holds for some $\alpha \geq 0$. Using (2.8) in (1.19), we find that

$$\sum_{n=0}^{\infty} a_{4,5}\Big(2\cdot 5^{\alpha}n+\frac{5^{\alpha}-1}{2}\Big)q^n \equiv f_5f_{25}\Big(R(q^5)-q-q^2R(q^5)^{-1}\Big) \pmod{2}. \tag{5.5}$$

Extracting the terms involving q^{5n+1} from both sides of (5.5), we arrive at

$$\sum_{n=0}^{\infty} a_{4,5}\Big(2\cdot 5^{\alpha+1}n+\frac{5^{\alpha+1}-1}{2}\Big)q^n \equiv f_1f_5 \pmod{2}, \tag{5.6}$$

which is the $\alpha+1$ case of (1.19). Finally, using (2.8) in (1.19) and then extracting the terms involving q^{5n+j} for $j\in\{3,4\}$ yields (1.20). □

Proof of Theorem 1.6:

Proof. Congruence (5.4) is the $\alpha=0$ case of (1.21). Now suppose that (1.21) holds for some $\alpha \geq 0$. Using (2.6) in (1.21) and proceeding as in the proof of (1.7), we arrive at

$$\sum_{n=0}^{\infty} a_{4,5}\Big(2\cdot p^{2\alpha+1}n+\frac{p^{2\alpha+2}-1}{2}\Big)q^n \equiv f_pf_{5p} \pmod{2}, \tag{5.7}$$

which yields

$$\sum_{n=0}^{\infty} a_{4,5}\Big(2 \cdot p^{2(\alpha+1)}n + \frac{p^{2(\alpha+1)}-1}{2}\Big)q^n \equiv f_1 f_5 \pmod{2}, \tag{5.8}$$

which is the case $\alpha+1$ of (1.21). On the other hand, extracting the terms involving $q^{pn+j}(1 \le j \le p-1)$ from (5.7), we arrive at (1.22). □

6. Proof of Theorem 1.7

Proof. Setting $(r, s) = (4, 9)$ in (1.4) and using elementary q-operations, we obtain

$$\sum_{n=0}^{\infty} a_{4,9}(n)q^n = \frac{f_2 f_4 f_9 f_{72}}{f_1 f_8 f_{18} f_{36}}. \tag{6.1}$$

Combining (2.10) and (6.1), we find that

$$\sum_{n=0}^{\infty} a_{4,9}(n)q^n = \frac{f_4 f_{12}^3 f_{72}}{f_2 f_6 f_8 f_{36}^2} + q\frac{f_4^3 f_6 f_{72}}{f_2^2 f_8 f_{12} f_{18}}. \tag{6.2}$$

Extracting the terms involving odd powers of q from (6.2) and then employing (2.15), we obtain

$$\sum_{n=0}^{\infty} a_{4,9}(2n+1)q^n \equiv \frac{f_9^3}{f_3} \pmod{2}. \tag{6.3}$$

Comparing the terms involving q^{3n+j}, for $j \in \{1, 2\}$ from both sides of (6.3) yields (1.23). Next, extracting the terms involving q^{3n} from both sides of (6.3), we obtain

$$\sum_{n=0}^{\infty} a_{4,9}(6n+1)q^n \equiv \frac{f_3^3}{f_1} \pmod{2}. \tag{6.4}$$

Combining (2.13) and (6.4), we find that

$$\sum_{n=0}^{\infty} a_{4,9}(6n+1)q^n \equiv \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4} \pmod{2}. \tag{6.5}$$

Extracting the terms involving odd powers of q from (6.5), we obtain

$$\sum_{n=0}^{\infty} a_{4,9}(12n+7)q^n \equiv \frac{f_6^3}{f_2} \pmod{2}. \tag{6.6}$$

Extracting the terms involving odd powers of q from (6.6) yields (1.24). Next, extracting the terms involving even powers of q from both sides of (6.6), we obtain

$$\sum_{n=0}^{\infty} a_{4,9}(24n+7)q^n \equiv \frac{f_3^3}{f_1} \pmod{2}. \tag{6.7}$$

Combining (6.4) and (6.7), we find that

$$a_{4,9}(24n+7) \equiv a_{4,9}(6n+1) \pmod 2. \tag{6.8}$$

From (6.8) and by mathematical induction, we have

$$a_{4,9}\Big(6\cdot 4^{\alpha+1}n + 2\cdot 4^{\alpha+1} - 1\Big) \equiv a_{4,9}(6n+1) \pmod 2. \tag{6.9}$$

Using (6.9) and congruence (1.24), we arrive at (1.25). On the other hand, extracting the terms involving even powers of q from both sides of (6.5), we obtain

$$\sum_{n=0}^{\infty} a_{4,9}(12n+1)q^n \equiv f_4 \pmod 2. \tag{6.10}$$

Extracting the terms involving q^{4n+j} for $j \in \{1,2,3\}$ from (6.10) yields (1.26). On the other hand, extracting the terms involving q^{4n} from both sides of (6.10) and using (2.4) yields (1.27). □

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