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Stability properties of dissipative evolution equations with nonautonomous and nonlinear damping

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Abstract. *In this paper, we obtain some stability results of (abstract) dissipative evolution equations with a nonautonomous and nonlinear damping using the exponential stability of the retrograde problem with a linear and autonomous feedback and a comparison principle. We then illustrate our abstract statements for different concrete examples, where new results are achieved. In a preliminary step, we prove some well-posedness results for some nonlinear and nonautonomous evolution equations.*

1. Introduction

Stability of evolution equations of hyperbolic type with linear or nonlinear autonomous feedbacks has been the object of many works. Let us quote the stability of the wave equation [30, 31, 33, 34, 36, 41, 66], of the elastodynamic system [1, 8, 19, 20, 21, 23, 37, 63], of the Petrovsky system [18, 34, 35], of Mawxell's system [5, 16, 32, 55, 61] or combination of them [14, 26, 54], see also the references cited in the aforementioned works. On the contrary the case of nonautonomous damping is less considered in the literature, let us quote [15, 24, 46, 47, 48, 50, 51, 52, 62] for the wave equation and [6, 7] for the Lamé system.

In the nonautonomous case, even if some similarities appear in the long time behavior of the solution, the proof is always made for each particular examples. Hence, our main idea is to treat the stability of (abstract) evolution equations of hyperbolic type with nonautonomous and nonlinear damping by adapting an approach that was successfully used in the autonomous case in [53, 55], namely use Liu's principle and a comparison principle that goes back to [40] and was improved in [15]. Liu's principle consists in estimating the energy of the direct system by some terms related to the feedbacks using a retrograde system with final data equal to the final data of the direct system. These terms are then estimated using the exponential stability of the inverse (retrograde) problem with a linear feedback (based on Russell's principle) and a comparison principle. This principle consists in estimating the energy of the system by the solution of a nonlinear and nonautonomous ODE. Furthermore, our goal is to present an abstract setting leading to the stability of the abstract (non linear and non autonous) system as soon as the retrograde linear and autonomous system is exponentially stable. Our setting is

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chosen as large as possible to include all examples of the aforementioned papers concerning nonautonomous damping and allowing new applications. The strength of our approach lies in the fact that the stability results (with general feedbacks) are only based on the exponential stability of the retrograde system with a linear and autonomous feedback, property that may be checked for an explicit problem by different techniques, like the multiplier method, microlocal analysis or any method entering in a linear framework (like nonharmonic analysis for instance). We further illustrate our approach by considering different examples for which new stability results are obtained. Many other examples, like the Petrovsky system or the thermoelastic system, may be treated using the exponential stability of the retrograde system with a linear and autonomous feedback, we do not present them for the sake of shortness.

Let us notice that existence results for evolution equations of hyperbolic type with nonlinear and nonautonomous feedbacks are no fully direct, because the domain of the operator may depend on the time variable. Hence, in a preliminary step, we prove a well-posedness result for a class of nonlinear and nonautonomous evolution equations, extending a result from [27] and then specializes it to evolution equations of hyperbolic type.

The paper is organized as follows: in Section 2 we give a well posedness result for nonlinear and nonautonomous evolution equations. In Section 3, we use this result to obtain some well posedness results for nonlinear and nonautonomous evolution equations of hyperbolic type. Section 4 is devoted to the stability results for a class of nonautonomous and nonlinear feedbacks adapting Liu's principle. Finally in Section 5 different illustrative examples are treated.

2. Well-posedness of nonlinear nonautonomous evolution equations

Let us first recall some useful definitions, see *e.g.* [9, 59].

Definition 2.1. *Let H be a Hilbert space with inner product $(\cdot,\cdot)_H$ and let A be a (single-valued) nonlinear operator on H with domain $D(A)$. Then*
1. A is called monotone if and only if

$$\Re(Au - Av, u - v)_H \geq 0, \quad \forall u, v \in D(A).$$

2. A is called maximal monotone if and only if A is monotone and there exists $\lambda > 0$ such that the range of $A + \lambda\mathbb{I}$ is equal to H.
3. A is called maximal quasi-monotone if and only if there exists a non negative real number ω such that $A + \omega\mathbb{I}$ is maximal monotone.

All examples that we will present below can be reduced to a nonlinear evolution equation in a Hilbert space X of the form

$$\begin{cases} \frac{dU}{dt}(t) + A(t)U(t) = 0, \text{ in } X, \\ U(0) = U_0, \end{cases}$$

where U is the unknown, $U_0 \in X$ and $A(t)$ is a (single-valued) nonlinear operators on X. A general theory of such equations with linear operators $A(t)$ has been developed using semigroup theory in [29, 28, 60] for instance. For nonlinear operators $A(t)$ similar results exist but for maximal quasi-monotone operators $A(t)$ (for one inner product independent of t), see [27, 13, 17, 43] or for maximal monotone operators $A(t)$ for a time-dependent inner product depending "smoothly" on t, see [56]. For our systems we need a variant of such results for maximal quasi-monotone operators $A(t)$ for a time-dependent inner product depending "smoothly" on t (see Remarks 4 and 5 in [27]). More precisely the next result holds.

Theorem 2.2. *Let X be a Hilbert space. For a fixed $T > 0$ and any $t \in [0, T]$ we assume that there exists an inner product $(\cdot, \cdot)_t$ on X depending "smoothly" on t in the following sense: there exists $c > 0$ such that*

$$\frac{d}{dt}(u, u)_t \leq 2c(u, u)_t, \quad \forall u \in X, \ t \in [0, T]. \tag{2.1}$$

For each $t \in [0, T]$, denote by $\| \cdot \|_t$ the norm associated with this inner product, namely

$$\|u\|_t^2 = (u, u)_t, \quad \forall u \in X.$$

Furthermore, assume that
(i) For all $t \in [0, T]$, $A(t)$ is single-valued and is a maximal quasi-monotone operator for the inner product $(\cdot, \cdot)_t$, in other words, there exists a non negative real number ω (independent of $t \in [0, T]$) such that $A(t) + \omega \mathbb{I}$ is a maximal monotone operator for the inner product $(\cdot, \cdot)_t$,
(ii) the domain $D(A(t)) = D$ of $A(t)$ is independent of t, for all $t \in [0, T]$,
(iii) there exists a positive constant L such that

$$\|A(t)u - A(s)u\| \leq L|t - s|(1 + \|u\| + \|A(s)u\|), \forall u \in D, s, t \in [0, T], \tag{2.2}$$

where for shortness $\| \cdot \|_0$ is denoted by $\| \cdot \|$.

Then for all $a \in D$ the Cauchy problem

$$\begin{cases} \dfrac{du}{dt}(t) + A(t)u(t) = 0, \ \text{for } 0 \leq t \leq T, \\ u(0) = a, \end{cases} \tag{2.3}$$

has a unique solution $u \in C([0, T]; X)$ such that $u(t)$ belongs to D for all $t \in [0, T]$, its strong derivative $\frac{du}{dt}(t) = -A(t)u(t)$ exists and is continuous except at a countable number of points t.

Note that the condition (2.1) and Gronwall's inequality imply that

$$\|u\|_t \leq e^{c|t-s|}\|u\|_s\,, \quad \forall u \in X, \ s, t \in [0, T]. \tag{2.4}$$

This estimate implies in particular that the norms $\| \cdot \|_t$ are equivalent and gives the variation of the norm $\| \cdot \|_t$ with respect to t.

Remark 2.3. In the linear case the conditions (2.1) and (i) to (iii) imply that the triplet $\{A, X, D\}$ forms a CD-system in the sense of [29, 28].

Proof. The proof is fully similar to the one in [27]; so, we only give its main steps. First we recall that $A(t) + \omega\mathbb{I}$ is a monotone operator for the inner product $(\cdot,\cdot)_t$ if and only if

$$\Re(A(t)u - A(t)v + \omega(u - v), u - v)_t \geq 0, \forall u, v \in D, \tag{2.5}$$

or equivalently (see [27, Lemma 1.1])

$$\|(1 + \alpha\omega)(u - v) + \alpha(A(t)u - A(t)v)_t \geq \|u - v\|_t, \forall u, v \in D, \alpha > 0.$$

By dividing this estimate by $1 + \alpha\omega$ and setting $\lambda = \frac{\alpha}{1+\alpha\omega}$ (that is clearly $< \omega^{-1}$ if $\omega > 0$), this is equivalent to

$$\|u - v + \lambda(A(t)u - A(t)v)_t \geq (1 - \lambda\omega)\|u - v\|_t, \forall u, v \in D, \lambda > 0 \text{ such that } \lambda\omega < 1. \tag{2.6}$$

Hence, we can apply Lemmas 1.1 and 1.2 of [13] to $A(t)$ for the norm t. In particular for all $n \in \mathbb{N}$ such that $n > \omega$, $\mathbb{I} + n^{-1}A(t)$ is invertible and if we set

$$J_n(t) = (\mathbb{I} + n^{-1}A(t))^{-1}, \quad A_n(t) = A(t)J_n(t), \forall n \in \mathbb{N} \text{ such that } n > \omega,$$

then the following estimates hold

$$\|J_n(t)x - J_n(t)y\|_t \leq (1 - n^{-1}\omega)^{-1}\|x - y\|_t, \quad \forall x, y \in X,$$
$$\|A_n(t)x - A_n(t)y\|_t \leq n(1 + (1 - n^{-1}\omega)^{-1})\|x - y\|_t, \quad \forall x, y \in X,$$
$$\|A_n(t)x\|_t \leq (1 - n^{-1}\omega)^{-1}\|A(t)x\|_t, \quad \forall x \in D.$$

Using (2.4), they are equivalent to

$$\|J_n(t)x - J_n(t)y\| \leq (1 - n^{-1}\omega)^{-1}e^{2cT}\|x - y\|, \quad \forall x, y \in X, \tag{2.7}$$
$$\|A_n(t)x - A_n(t)y\| \leq n(1 + (1 - n^{-1}\omega)^{-1})e^{2cT}\|x - y\|, \quad \forall x, y \in X, \tag{2.8}$$
$$\|A_n(t)x\| \leq (1 - n^{-1}\omega)^{-1}e^{2cT}\|A(t)x\|, \quad \forall x \in D, \tag{2.9}$$

that respectively correspond to the estimates (2.4) and (2.5) of [27] and are valid for all $n \in \mathbb{N}$ such that $n > \omega$. As the factor $(1 - n^{-1}\omega)^{-1}e^{2cT}$ is uniformly bounded in n as n goes to infinity, Lemmas 2.4 and 2.5 from [27] remain valid. Furthermore, by the estimate (2.9) and our assumption (2.2), we have (see the proof of Lemma 4.1 from [27])

$$\|A_n(t)x - A_n(s)x\| \leq (1 - n^{-1}\omega)^{-1}e^{2cT}L|t - s|(1 + \|u\| + (1 + n^{-1})\|A_n(s)u\|),$$
$$\forall u \in D,\ s, t \in [0, T],\ n > \omega, \tag{2.10}$$

that corresponds to the estimate (4.2) of [27]. Since D is dense in X, this estimate shows that $A_n(t)$ is Lipschitz continuous in t for all $x \in X$, while (2.8) means that the map $x \to A(t)x$ is Lipschitz continuous for a fixed $t \in [0, T]$, uniformly in x and t. Thus the approximated problem

$$\begin{cases} \dfrac{du_n}{dt}(t) + A_n(t)u_n(t) = 0, \;\text{ for } 0 \le t \le T, \\ u_n(0) = a, \end{cases} \tag{2.11}$$

has a unique solution $u_n \in C^1([0, T]; X)$ for all $a \in X$. We now show that the statements of Lemma 4.2 of [27] hold if $a \in D$, namely there exists a positive constant K (that depends on c, ω, T, and $\|a\| + \|A(0)a\|$ but not on n) such that

$$\|u_n(t)\| \le K, \forall t \in [0, T], n > \omega, \tag{2.12}$$

$$\|u_n'(t)\| = \|A_n(t)u_n(t)\| \le K, \forall t \in [0, T], n > \omega, \tag{2.13}$$

where for shortness we write $\frac{du_n}{dt} = u_n'$. Indeed for $t \in [0, T)$, let us fix h in $[0, T-t]$ and set $x_n(t) := u_n(t+h) - u_n(t)$. As x_n is differentiable in t and using (2.1), we have

$$2\|x_n(t)\|_t \frac{d}{dt}\|x_n(t)\|_t = \frac{d}{dt}\|x_n(t)\|_t^2 \le 2c\|x_n(t)\|_t^2 + 2\Re(x_n'(t), x_n(t))_t.$$

Using (2.11), we get

$$\begin{aligned} \|x_n(t)\|_t \frac{d}{dt}\|x_n(t)\|_t &\le c\|x_n(t)\|_t^2 - \Re(A_n(t+h)u_n(t+h) - A_n(t)u_n(t), x_n(t))_t \\ &\le c\|x_n(t)\|_t^2 - \Re(A_n(t+h)u_n(t+h) - A_n(t+h)u_n(t), x_n(t))_t \\ &- \Re(A_n(t+h)u_n(t) - A_n(t)u_n(t), x_n(t))_t. \end{aligned}$$

Using (2.5) and (2.10), we obtain

$$\begin{aligned} \|x_n(t)\|_t \frac{d}{dt}\|x_n(t)\|_t &\le (c+\omega)\|x_n(t)\|_t^2 \\ &+ (1-n^{-1}\omega)^{-1}e^{2cT}Lh(1+\|u_n(t)\|+(1+n^{-1})\|u_n'(t)\|)\|x_n(t)\|_t, \end{aligned}$$

Simplifying by $\|x_n(t)\|_t$ (see [27, p. 515]), we find

$$\begin{aligned} \frac{d}{dt}\|x_n(t)\|_t &\le (c+\omega)\|x_n(t)\|_t \\ &+ (1-n^{-1}\omega)^{-1}e^{2cT}Lh(1+\|u_n(t)\|+(1+n^{-1})\|u_n'(t)\|). \end{aligned}$$

This estimate directly implies that

$$\frac{d}{dt}\left(e^{-(c+\omega)t}\|x_n(t)\|_t\right) \le L_1 h(1+\|u_n(t)\|+\|u_n'(t)\|),$$

for a positive constant L_1 that depends on c, ω and T but is independent of n. Integrating this estimate in $(0,t)$, we find

$$e^{-(c+\omega)t}\|x_n(t)\|_t - \|x_n(0)\|_0 \le L_1 h \int_0^t (1 + \|u_n(s)\| + \|u_n'(s)\|)\, ds.$$

By (2.4), we find

$$\|x_n(t)\| \le L_2(\|x_n(0)\| + h \int_0^t (1 + \|u_n(s)\| + \|u_n'(s)\|)\, ds),$$

for a positive constant L_2 that depends on c, ω and T but is independent of n. Dividing by h and letting h goes to zero, we obtain

$$\|u_n'(t)\| \le L_2(\|u_n'(0)\| + \int_0^t (1 + \|u_n(s)\| + \|u_n'(s)\|)\, ds).$$

As $u_n'(0) = A(0)a$ and

$$u_n(t) = a + \int_0^t u_n'(s)\, ds,$$

we find as in [27, p. 516] that

$$\|u_n(t)\| + \|u_n'(t)\| \le L_3 \left(1 + \int_0^t (\|u_n(s)\| + \|u_n'(s)\|)\, ds\right),$$

for a positive constant L_3 that depends on c, ω, T and $\|a\| + \|A(0)a\|$ but is independent of n. By Gronwall's Lemma, we deduce that (2.12) and (2.13) hold.

We now show that the statements of Lemma 4.3 of [27] hold, namely for $a \in D$, the strong limit $u(t) = \lim_{n\to\infty} u_n(t)$ exists uniformly for $t \in [0,T]$ and u is Lipschitz continuous. Indeed for all $m,n \in \mathbb{N}$ such that $m,n > \omega$, we set $x_{mn}(t) = u_m(t) - u_n(t)$ and as before we have

$$\frac{d}{dt}\|x_{mn}(t)\|_t^2 \le 2c\|x_{mn}(t)\|_t^2 + 2\Re(x_{mn}'(t), x_{mn}(t))_t.$$

Using (2.11) and (2.5), we find

$$\begin{aligned}\frac{d}{dt}\|x_{mn}(t)\|_t^2 \le{}& 2c\|x_{mn}(t)\|_t^2 + 2\omega\|y_{mn}(t)\|_t^2\\ &+ 2\Re(A_m(t)x_m(t) - A_n(t)x_n(t), y_{mn}(t) - x_{mn}(t))_t,\end{aligned}$$

where $y_{mn}(t) = J_m(t)u_m(t) - J_n(t)u_n(t)$. By the triangle inequality, we then have

$$\begin{aligned}\frac{d}{dt}\|x_{mn}(t)\|_t^2 \le{}& 2(c+2\omega)\|x_{mn}(t)\|_t^2 + 4\omega\|y_{mn}(t) - x_{mn}(t)\|_t^2\\ &+ 2\Re(A_m x_m(t) - A_n(t)x_n(t), y_{mn}(t) - x_{mn}(t))_t,\end{aligned}$$

Using the estimate (2.13) and (2.4), we arrive at

$$\frac{d}{dt}\|x_{mn}(t)\|_t^2 \leq 2(c+2\omega)\|x_{mn}(t)\|_t^2+K_1\|y_{mn}(t)-x_{mn}(t)\|^2+K_1\|y_{mn}(t)-x_{mn}(t)\|,$$

for a positive constant K_1 that depends on c, ω, T, and $\|a\|+\|A(0)a\|$ but not on m, n. Obviously, this is equivalent to

$$\frac{d}{dt}\left(e^{-2(c+2\omega)t}\|x_{mn}(t)\|_t^2\right) \leq K_1(\|y_{mn}(t)-x_{mn}(t)\|^2+\|y_{mn}(t)-x_{mn}(t)\|),$$

and integrating it between 0 and t, we find (as $x_{mn}(0)=0$)

$$e^{-2(c+2\omega)t}\|x_{mn}(t)\|_t^2 \leq K_1\int_0^t (\|y_{mn}(s)-x_{mn}(s)\|^2+\|y_{mn}(s)-x_{mn}(s)\|)\,ds.$$

This finally leads to

$$\|x_{mn}(t)\|^2 \leq e^{2(3c+2\omega)T}K_1\int_0^t (\|y_{mn}(s)-x_{mn}(s)\|^2+\|y_{mn}(s)-x_{mn}(s)\|\,ds.$$

As

$$\begin{aligned} y_{mn}(s)-x_{mn}(s) &= J_m(s)u_m(s)-u_m(s)+J_n(s)u_n(s)-u_n(s) \\ &= n^{-1}A_n(s)u_n(s)-m^{-1}A_m(s)u_m(s), \end{aligned}$$

by (2.13), we obtain

$$\|y_{mn}(s)-x_{mn}(s)\| \leq K(m^{-1}+n^{-1}).$$

Inserting this estimate in the previous one, we arrive at

$$\|x_{mn}(t)\|^2 \leq K_2(m^{-1}+n^{-1}), \quad \forall t\in[0,T],$$

for a positive constant K_2 that depends on c, ω, T, and $\|a\|+\|A(0)a\|$ but not on m, n. Thus the strong limit $u(t)=\lim_{n\to\infty}u_n(t)$ exists uniformly in $t\in[0,T]$. The Lipschitz continuity of u follows from the uniform Lipschitz property of the u_n, that is consequence of (2.13).

The remainder of the proof is the same as in [27] since it is based on the properties proved before. □

Remark 2.4. Obviously, the previous Theorem remains valid if X is a real Hilbert space.

3. Abstract hyperbolic setting

In this section we describe a general abstract setting of hyperbolic type inspired from [53] that will be used later on. It is motivated by the examples (and other ones) given in Section 5 which all enter in this setting.

3.1. General assumptions

Let us fix two real Hilbert spaces $\mathcal{H}$, $\mathcal{V}$ with respective inner products $(.,.)_{\mathcal{H}}$, $(.,.)_{\mathcal{V}}$ and such that $\mathcal{V}$ is densely and continuously embedded into $\mathcal{H}$. Identifying $\mathcal{H}$ with its dual $\mathcal{H}'$ we have the standard diagram

$$\mathcal{V} \hookrightarrow \mathcal{H} = \mathcal{H}' \hookrightarrow \mathcal{V}'.$$

The duality pairing between $\mathcal{V}'$ and $\mathcal{V}$ will be denoted by $\langle \cdot, \cdot \rangle$, so that

$$\langle u, v \rangle = (u, v)_{\mathcal{H}}, \quad \forall u, v \in \mathcal{H}.$$

We suppose that $\mathcal{V}$ is continuously embedded into a control space U, that is supposed to be in the form

$$U = \prod_{j=1}^{J} U_j, \tag{3.1}$$

where for all $j = 1, \dots, J \in \mathbb{N}^\star := \mathbb{N} \setminus \{0\}$, U_j is a closed subspace of $L^2(X_j, \mu_j)^{N_j}$, with $N_j \in \mathbb{N}^\star$, when X_j is a metric space, and $(X_j, \mathcal{A}_j, \mu_j)$ is a measure space such that $\mu_j(X_j) < \infty$.

For all $j = 1, \dots, J$, we suppose given a mapping $\alpha_j \in C([0, \infty) \times X_j; (0, \infty))$ and locally Lipschitz with respect to the time variable, in the sense that for all T, there exist a positive constant $\kappa(T)$ (that may depend on T) such that

$$|\alpha_j(t, x) - \alpha_j(t, x)| \leq \kappa(T)|t - s|, \forall t \in [0, T], x \in X_j, \tag{3.2}$$

and a continuous mapping $g_j : \mathbb{R}^{N_j} \to \mathbb{R}^{N_j}$ such that:

$$(g_j(x) - g_j(y)) \cdot (x - y) \geq 0, \forall x, y \in \mathbb{R}^{N_j} \text{ (monotonicity)}, \tag{3.3}$$

$$g_j(0) = 0, \tag{3.4}$$

$$|g_j(x)| \leq M(1 + |x|), \forall x \in \mathbb{R}^{N_j}, \tag{3.5}$$

for some positive constant M.

We further define the (nonlinear) time-dependent operator $B(t)$ from $\mathcal{V}$ into $\mathcal{V}'$ by

$$\langle B(t)u, v \rangle = \sum_{j=1}^{J} \int_{X_j} \alpha_j(t, x_j) g_j((I_U u)_j(x_j)) \cdot (I_U v)_j(x_j) \, d\mu_j(x_j), \forall u, v \in \mathcal{V}, \tag{3.6}$$

where I_U denotes the embedding from $\mathcal{V}$ to U and therefore, $(I_U u)_j$ is the j^{th} component of $I_U u$.

We finally suppose given a bounded linear operator A_1 from $\mathcal{V}$ into $\mathcal{V}'$ and consider the evolution equation

$$\begin{cases} \frac{dx}{dt}(t) + A_1 x(t) + B(t)x(t) = 0 \text{ in } \mathcal{H}, t \geq 0, \\ x(0) = x_0. \end{cases} \tag{3.7}$$

This system clearly involves the (nonlinear) and time-dependent operator $\mathcal{A}_B(t)$ defined by

$$D(\mathcal{A}_B(t)) = \{v \in \mathcal{V} | (A_1 + B(t))v \in \mathcal{H}\}, \tag{3.8}$$

$$\mathcal{A}_B(t) = (A_1 + B(t))v, \forall v \in D(\mathcal{A}_B(t)). \tag{3.9}$$

In its full generality, the domain of $\mathcal{A}_B(t)$ depends on the time variable. Consequently we cannot apply Theorem 2.2. Nevertheless there are two cases treated below for which this Theorem applies. In both cases, if $x_0 \in D(\mathcal{A}_B(0))$, we will show that a unique solution x exists with the following properties:

$$\begin{cases} x \in C([0,\infty), \mathcal{H}) \text{ is such that } x(t) \in D(\mathcal{A}_B(t)), \text{ for all } t \in [0,\infty) \\ \text{and } x'(t) = -\mathcal{A}_B(t)x(t) \text{ exists in } \mathcal{H} \text{ and is continuous except at} \\ \text{a countable number of points } t. \end{cases} \tag{3.10}$$

Before going on let us show that under the additional assumption that

$$\langle A_1 u, u \rangle = 0, \forall u \in \mathcal{V}, \tag{3.11}$$

system (3.7) is dissipative.

Lemma 3.1. *Under the above assumptions, for all $t \geq 0$, the operator $A_B(t)$ is monotone for the natural inner product of $\mathcal{H}$, namely*

$$(A_B(t)u - A_B(t)v, u - v)_{\mathcal{H}} = \langle B(t)u - B(t)v, u - v \rangle \geq 0, \forall u \in D(\mathcal{A}_B(t)). \tag{3.12}$$

Consequently if x is a solution of (3.7) with the regularity (3.10), its associated energy

$$\mathcal{E}(t) = \frac{1}{2} ||x(t)||^2_{\mathcal{H}} \tag{3.13}$$

is non-increasing; moreover, we have

$$\mathcal{E}(S) - \mathcal{E}(T) = \int_S^T \langle B(t)u(t), u(t) \rangle \, dt, \forall 0 \leq S < T < \infty, \tag{3.14}$$

$$\frac{d}{dt}\mathcal{E}(t) = -\langle B(t)u(t), u(t) \rangle \leq 0, \text{ for a. a. } t \geq 0. \tag{3.15}$$

Proof. Let us first show that $A_B(t)$ is monotone. Indeed for any $u, v \in D(\mathcal{A}_B(t))$, by the definition of $A_B(t)$ and the property (3.11), we have

$$(A_B(t)u - A_B(t)v, u - v)_{\mathcal{H}} = \langle A_1(u - v), u - v\rangle \\ + \langle B(t)u - B(t)v, u - v\rangle = \langle B(t)u - B(t)v, u - v\rangle.$$

Finally by the definition of $B(t)$ and then (3.3) and recalling that $\alpha_j(t, x) > 0$, we have

$$\langle B(t)u - B(t)v, u - v\rangle = \sum_{j=1}^{J} \int_{X_j} \alpha_j(t, x_j)\,(g_j((I_U u)_j(x_j)) - g_j((I_U v)_j(x_j))) \\ \cdot ((I_U v)_j(x_j) - (I_U v)_j(x_j))\, d\mu_j(x_j) \\ \geq 0,$$

which proves (3.12).

For the second assertion it suffices to show (3.15) since (3.14) follows by integration between S and T. By the regularity assumptions on x, we have

$$\frac{d}{dt}\mathcal{E}(t) = (x'(t), x(t))_{\mathcal{H}} = -(\mathcal{A}_B(t)u(t), u(t))_{\mathcal{H}}, \text{ for a. a. } t \geq 0,$$

by (3.7). By our assumption (3.4), we have $A_B(t)0 = 0$ and consequently by (3.12), we get (3.15). □

3.2. The "bounded" case

We here assume that $\mathcal{H}$ is continuously embedded into U. As we shall see below this assumption implies that $B(t)$ becomes a (nonlinear) operator from $\mathcal{H}$ into itself and therefore, the domain of $\mathcal{A}_B(t)$ does not depend on t anymore.

Theorem 3.2. *In addition to the previous assumptions, assume that $\mathcal{H}$ is continuously embedded into U, and that there exists a positive real number λ such that the range $R(\lambda\mathbb{I} + \mathcal{A}_B(t))$ is equal to $\mathcal{H}$. Then*

$$D(\mathcal{A}_B(t)) = D = \{u \in \mathcal{V} | A_1 u \in \mathcal{H}\}, \forall t \geq 0, \tag{3.16}$$

and for any $x_0 \in D$, problem (3.7) has a unique solution x satisfying (3.10).

Proof. We first show that (3.16) holds. Indeed as $\mathcal{H}$ is continuously embedded into U, the mapping I_U extends to a linear and continuous operator from $\mathcal{H}$ into U; therefore, there exists a positive constant C such that

$$\|I_U u\|_{\mathcal{H}} \leq C\|u\|_{\mathcal{H}}, \forall u \in \mathcal{H}. \tag{3.17}$$

By our assumption (3.5) and the definition of $B(t)$, we then have

$$|\langle B(t)u, v\rangle| \leq M \sum_{j=1}^{J} \int_{X_j} \alpha_j(t, x_j)(1 + |(I_U u)_j(x_j)|)|I_U v)_j(x_j|\, d\mu_j(x_j), \forall u, v \in \mathcal{V}.$$

By the continuity property of $\alpha_j(t,\cdot)$, Cauchy-Schwarz's inequality and the estimate (3.17), we obtain

$$|\langle B(t)u, v\rangle| \leq C(t)(1 + \|u\|_{\mathcal{H}})\|v\|_{\mathcal{H}}, \forall u, v \in \mathcal{V},$$

where $C(t)$ is a positive constant that depends on M, C, and t. As $\mathcal{V}$ is dense in $\mathcal{H}$, for a fixed $u \in \mathcal{V}$, the linear mapping

$$\mathcal{V} \to \mathbb{R} : v \to \langle B(t)u, v\rangle,$$

can be extended to a linear and continuous form to the whole $\mathcal{H}$. By the Riesz's representation theorem, there exists $h(t) \in \mathcal{H}$ such that

$$\langle B(t)u, v\rangle = (h(t), v)_{\mathcal{H}}, \forall v \in \mathcal{H}.$$

In other words, for $u \in \mathcal{V}$, $B(t)u$ can be identified with $h(t)$ and therefore, $(A_1 + B(t))u \in \mathcal{H}$ if and only if $A_1 u \in \mathcal{H}$, which proves (3.16).

By Lemma 3.1 and our additional assumption $R(\lambda\mathbb{I} + \mathcal{A}_B(t)) = \mathcal{H}$, for some $\lambda > 0$, we deduce that the assumption (i) of Theorem 2.2 holds.

Let us end up with the third assumption. Fix $T > 0$ and let $u \in D$, and $t, s \in [0, T]$, then we clearly have

$$\mathcal{A}_B(t)u - \mathcal{A}_B(s)u = B(t)u - B(s)u.$$

Therefore, for any $v \in \mathcal{H}$, by the definition of $B(t)$ and our previous considerations, we may write

$$\begin{aligned}(\mathcal{A}_B(t)u &- \mathcal{A}_B(s)u, v)_{\mathcal{H}} \\ &= \sum_{j=1}^{J} \int_{X_j} (\alpha_j(t, x_j) - \alpha_j(s, x_j)) g_j((I_U u)_j(x_j)) \cdot (I_U v)_j(x_j)\, d\mu_j(x_j).\end{aligned}$$

By our assumptions (3.2) to (3.5), we obtain

$$\begin{aligned}|(\mathcal{A}_B(t)u - \mathcal{A}_B(s)u, v)_{\mathcal{H}}| &\leq \kappa(T)|t-s| \sum_{j=1}^{J} \int_{X_j} g_j((I_U u)_j(x_j)) \cdot (I_U v)_j(x_j)\, d\mu_j(x_j) \\ &\leq \kappa(T)|t-s| \sum_{j=1}^{J} \int_{X_j} (1 + |(I_U u)_j(x_j)|) \cdot (I_U v)_j(x_j)\, d\mu_j(x_j).\end{aligned}$$

Cauchy-Schwarz's inequality and the estimate (3.17) allow to conclude that

$$|(\mathcal{A}_B(t)u - \mathcal{A}_B(s)u, v)_{\mathcal{H}}| \leq \sqrt{2} C\kappa(T)|t-s|(\sum_{j=1}^{J} \mu_j(X_j) + C\|u\|_{\mathcal{H}})\|v\|_{\mathcal{H}}).$$

Since this estimate is valid for all $v \in \mathcal{H}$, this means that

$$\|\mathcal{A}_B(t)u - \mathcal{A}_B(s)u\|_{\mathcal{H}} \leq \sqrt{2} C\kappa(T)|t-s|(\sum_{j=1}^{J} \mu_j(X_j) + C\|u\|_{\mathcal{H}}),$$

and proves that the assumption (iii) of Theorem 2.2 holds.

In conclusion by Theorem 2.2 for $x_0 \in D$ and any $T > 0$, there exists a unique solution $u_T \in C([0,T];\mathcal{H})$ of problem

$$\begin{cases} \dfrac{dx_T}{dt}(t) + A_1 x_T(t) + B(t)x_T(t) = 0 \text{ in } \mathcal{H}, t \in [0,T], \\ x_T(0) = x_0, \end{cases} \tag{3.18}$$

such that $x_T(t)$ belongs to D for all $t \in [0,T]$, its strong derivative $\dfrac{dx_T}{dt}(t) = -A(t)x_T(t)$ exists and is continuous except at a countable number of points t.

By uniqueness, for $T' > T$, the restriction of $x_{T'}$ to $[0,T]$ coincides with x_T. Therefore, a unique global solution $x \in C([0,\infty);\mathcal{H})$ of (3.7) exists with the properties (3.10). □

3.3. The "unbounded" case

Here we assume that the mappings α_j do not depend on the variable x_j and coincide, namely there exists a mapping $\alpha \in C^1([0,\infty;(0,\infty))$ such that α' is locally Lipschitz (in the sense that for all $T > 0$, there exists a positive constant $\nu(T)$ such that

$$|\alpha'(t) - \alpha'(s)| \le \nu(T)|t-s|,$$

for all $s, t \in [0,T]$) such that

$$\alpha_j(t, x_j) = \alpha(t), \forall x_j \in X_j, t \ge 0. \tag{3.19}$$

Due to (3.6), this means that $B(t) = \alpha(t)B_1$, where

$$\langle B_1 u, v\rangle = \sum_{j=1}^{J} \int_{X_j} g_j((I_U u)_j(x_j)) \cdot (I_U v)_j(x_j)\, d\mu_j(x_j), \quad \forall u, v \in \mathcal{V}. \tag{3.20}$$

Theorem 3.3. *In addition to the assumptions made in subsection 3.1, assume that (3.19) holds, that $A_1 + B_1$ is maximal quasi-monotone with a dense domain in $\mathcal{H}$, and that there exist two mappings $D \in C^1([0,\infty);\mathcal{L}(\mathcal{H}))$ and $\tilde{D} \in C([0,\infty);\mathcal{L}(\mathcal{H}))$ such that D' and $\tilde{D}$ are locally Lipschitz and for all $t \ge 0$, $D(t)$ and $\tilde{D}(t)$ are invertible, $D(t)\tilde{D}(t)$ is symmetric positive definite and for all $T > 0$, there exists a positve constant c_T such that*

$$(\tilde{D}(t)^{-1}D(t)^{-1}x, x)_{\mathcal{H}} \ge c_T\|x\|^2_{\mathcal{H}}, \forall x \in \mathcal{H}, \forall t \in [0,T], \tag{3.21}$$

and finally

$$(A_1 + \alpha(t)B_1)D(t)^{-1} = \tilde{D}(t)(A_1 + B_1), \forall t \ge 0. \tag{3.22}$$

Then for all $x_0 \in D(\mathcal{A}_B(0))$, problem (3.7) has a unique solution x satisfying (3.10).

Proof. Assuming that the solution x of problem (3.7) exists and is smooth enough, we perform the change of unknown

$$\tilde{x}(t) = D(t)x(t).$$

Hence, as $\tilde{x}'(t) = D'(t)x(t) + D(t)x'(t)$ and by (3.7) , we get

$$\tilde{x}'(t) = D'(t)D(t)^{-1}\tilde{x}(t) - D(t)(A_1 + \alpha(t)B_1)D(t)^{-1}\tilde{x}(t).$$

With our assumption (3.22), we arrive at

$$\tilde{x}'(t) = D'(t)D(t)^{-1}\tilde{x}(t) - D(t)\tilde{D}(t)(A_1 + B_1)\tilde{x}(t). \tag{3.23}$$

This corresponds to (2.3) with the operator

$$A(t) = D(t)\tilde{D}(t)(A_1 + B_1) - D'(t)D(t)^{-1},$$

whose domain is clearly

$$D(A(t)) = D(A_1 + B_1),$$

and is independent of t, due to our assumptions on $D(t)$ and $\tilde{D}(t)$.

In order to apply Theorem 2.2 we introduce the time dependent inner product

$$(x, \tilde{x})_t = (\tilde{D}(t)^{-1}D(t)^{-1}x, \tilde{x})_{\mathcal{H}}, \forall x, \tilde{x} \in \mathcal{H}.$$

Our assumptions on D and $\tilde{D}$ guarantee that it is indeed an inner product on $\mathcal{H}$ whose associated norm is equivalent to the standard one, namely for a fixed T, we have

$$\sqrt{c_T}\|x\|_{\mathcal{H}} \leq \|x\|_t \leq C_T\|x\|_{\mathcal{H}}, \quad \forall x \in \mathcal{H}, \tag{3.24}$$

for some positive constant C_T, and that the property (2.1) also holds. From its definition, we see that $A(t)$ is quasi-monotone for this inner product. Indeed from it definition, for any $x, y \in D(A_1 + B_1)$, and $t \in [0, T]$, we have

$$\begin{aligned}(A(t)x - A(t)y, x - y)_t = {} & ((A_1 + B_1)x - (A_1 + B_1)y, x - y)_{\mathcal{H}} \\ & - (\tilde{D}(t)^{-1}D(t)^{-1}D'(t)D(t)^{-1}(x - y), x - y)_{\mathcal{H}}.\end{aligned}$$

Hence, as $A_1 + B_1$ is quasi-monotone in $\mathcal{H}$ (i.e. $A_1 + B_1 + \omega_1\mathbb{I}$ is monotone for some $\omega_1 \geq 0$), and due to our assumptions on D and $\tilde{D}$, we then have

$$(A(t)x - A(t)y, x - y)_t \geq -\omega_T\|x - y\|^2_{\mathcal{H}},$$

for some $\omega_T > 0$ (depending on T). Due to the equivalence (3.24), we arrive at

$$(A(t)x - A(t)y, x - y)_t \geq -\omega_T C_T^2\|x - y\|^2_t,$$

which yields the quasi-monotonicity of $A(t)$. Let us now show the maximality property. Indeed for $\lambda > 0$ large enough, we want to show that

$$E(t) := \lambda\mathbb{I} + D(t)\tilde{D}(t)(A_1 + B_1) - D'(t)D(t)^{-1}$$

is surjective. But as $D(t)\tilde{D}(t)$ is an isomorphism, this is equivalent to the surjectivity of

$$E(t) := \lambda(D(t)\tilde{D}(t))^{-1} + A_1 + B_1 - (D(t)\tilde{D}(t))^{-1}D'(t)D(t)^{-1}.$$

Now we take advantage of Theorem 1 of [10] by considering the previous operator as a perturbation of $T_1 := A_1 + B_1 + \omega_1\mathbb{I}$ (that satisfies the assumption of this Theorem). Due to the linearity of

$$T_2(t) := \lambda(D(t)\tilde{D}(t))^{-1} - (D(t)\tilde{D}(t))^{-1}D'(t)D(t)^{-1} - \omega_1\mathbb{I},$$

it is clearly hemicontinous and due to the assumption (3.21), for $\lambda > 0$ large enough, $T_2(t)$ will be monotone, bounded and coercive. Using the above Theorem, we deduce that $E(t) = T_1 + T_2(t)$ is surjective.

In summary assumption (i) of Theorem 2.2 holds and it remains to check the assumption (iii) of this Theorem. For that purpose, let us fix $x \in D(A_1 + B_1)$ and $s, t \in [0, T]$, then by definition we have

$$A(t)x - A(s)x = (D(t)\tilde{D}(t) - D(s)\tilde{D}(s))(A_1 + B_1)x + (D'(s)D(s)^{-1} - D'(t)D(t)^{-1})x.$$

By the local Lipschitz property of D, $\tilde{D}$ and of the derivative of D, we get

$$\|A(t)x - A(s)x\|_{\mathcal{H}} \le K(T)|t - s|(\|(A_1 + B_1)x\|_{\mathcal{H}} + \|x\|_{\mathcal{H}}).$$

We now transform

$$(A_1 + B_1)x = (D(s)\tilde{D}(s))^{-1}(D(s)\tilde{D}(s)(A_1 + B_1)x - D'(s)D(s)^{-1}) + (D(s)\tilde{D}(s))^{-1}D'(s)D(s)^{-1}x,$$

use the triangle inequality and use the continuity of D, $\tilde{D}$ and D' to find

$$\|A(t)x - A(s)x\|_{\mathcal{H}} \le K_1(T)|t - s|(\|A(s)x\|_{\mathcal{H}} + \|x\|_{\mathcal{H}}).$$

for a positive constant $K_1(T)$, which implies that (2.2) is valid.

In conclusion by Theorem 2.2, there exists a unique solution $\tilde{x}$ of (3.23) with initial condition $\tilde{x}(0) = D(0)x_0$ (that belongs to $D(A_1 + B_1)$ by the assumption on x_0) satisfying $\tilde{x} \in C([0, \infty), \mathcal{H})$, $x(t) \in D(A_1 + B_1)$, for all $t \in [0, \infty)$ and $x'(t) = -A(t)x(t)$ exists in $\mathcal{H}$ and is continuous except at a countable number of points t. Setting $x(t) = D(t)^{-1}\tilde{x}(t)$, we readily check that it is the unique solution of problem (3.7) and that it satisfies (3.10). □

4. Stability results in the nonlinear and nonautonomous case

Here we use Liu's principle [44] and a comparison principle with a nonlinear and nonautonomous ODE from [15] (see also [40]) to deduce decay rates of the energy using appropriate nonlinear and nonautonomous feedbacks.

We first recall the comparison principle obtained in [15] (compare with [40, Theorem 2 and Corollary 2])).

Theorem 4.1. *Let β be a continuous mapping from $[0,\infty)$ to $(0,\infty)$ and p a strictly increasing convex mapping from $[0,+\infty)$ to $[0,+\infty)$ such that $p(0)=0$. Let $\mathcal{E}\colon [0,+\infty)\to[0,+\infty)$ be a non-increasing mapping satisfying*

$$\beta((n+1)T)p(\mathcal{E}(nT))+\mathcal{E}((n+1)T)\leq \mathcal{E}(nT), \forall n\in\mathbb{N}, \tag{4.1}$$

for some $T>0$. Then

$$\mathcal{E}(t)\leq \psi^{-1}\left(\int_T^t \beta(s)\,ds\right), \forall t\geq T, \tag{4.2}$$

where ψ is defined by

$$\psi(x)=\int_x^{\mathcal{E}(0)}\frac{1}{p(s)}\,ds, \forall x>0. \tag{4.3}$$

Proof. Let us shortly recall the proof from [15]. Since (4.2) trivially holds if $\mathcal{E}(0)=0$ (because in such a case $\mathcal{E}(t)=0$, for all $t\geq 0$), we can assume that $\mathcal{E}(0)>0$. First by [15, Lemma 4.2], the next comparison principle holds

$$\mathcal{E}(t)\leq S(t-T), \forall t\geq T, \tag{4.4}$$

where S is the unique solution of the nonlinear and nonautonomous ODE

$$S'(t)+\beta(t+T)p(S(t))=0, \forall t\geq 0, \quad S(0)=\mathcal{E}(0). \tag{4.5}$$

Such a solution exists and remains positive for all $t>0$ due the Cauchy-Lipschitz Theorem (because the assumptions on p garantee that it is locally Lipschitz in $[0,\infty)$).

With the help of [15, Lemma 4.2/2.] (the properties on p guarantee that the assumption (24) from [15] holds with $m=p(1)^{-1}$), we deduce that

$$S(t)\leq \psi^{-1}\left(\int_0^t \beta(s+T)\,ds\right), \quad \forall t\geq 0,$$

with ψ defined by (4.3) (and is meaningful because $\lim_{x\to 0+}\psi(x)=+\infty$ reminding that $p(x)\leq p(1)x$, for all $x\in[0,1]$). This estimate combined with (4.4) yields the result. □

Let us now recall Russell's principle that yields an exact controllability result for the evolution equation associated with the operator $-A_1$ with controls in $L^2(]0,T[;U)$ provided $A_1 - I_U$ generates a semigroup of contractions and $-A_1 - I_U$ generates an exponentially stable semigroup of contractions in $\mathcal{H}$, see [53, Theorem 4.1].

Theorem 4.2. *Assume that $A_1 - I_U$ generates a semigroup of contractions in $\mathcal{H}$ and that $-A_1 - I_U$ generates a semigroup of contractions $S(t)$ in $\mathcal{H}$ that is exponentially stable in the sense that there exist two positive constants C and ω such that*

$$\|S(t)x_0\|_{\mathcal{H}} \leq Ce^{-\omega t}\|x_0\|_{\mathcal{H}}, \forall x_0 \in \mathcal{H}. \tag{4.6}$$

Then there exists $T > 0$ large enough, such that for any $p_0 \in \mathcal{H}$, there exists a control $K \in L^2((0,T);U)$ such that the solution $p \in C([0,T];\mathcal{H})$ of

$$\begin{cases} \frac{\partial p}{\partial t} + A_1 p = K \textit{ in } \mathcal{V}', t \in [0,T], \\ p(T) = p_0, \end{cases} \tag{4.7}$$

satisfies

$$p(0) = 0. \tag{4.8}$$

Furthermore, there exists a positive constant $D > 1$ depending only on T, and the constants C and ω such that

$$\int_0^T \|K(t)\|_U^2 \, dt + \int_0^T \|I_U p(t)\|_U^2 \, dt \leq 2D\|p_0\|_{\mathcal{H}}^2. \tag{4.9}$$

We now give the consequence of this result to our system (3.7) in three different cases of functions α_j: non-increasing with respect to t, non-decreasing with respect to t and oscillating with respect to t. But first we give an energy estimate valid in all cases.

Lemma 4.3. *Under the assumptions of Theorem 4.2, any solution x of (3.7) with the regularity* (3.10)*, satisfies*

$$\mathcal{E}(T) \leq D\left(\sum_{j=1}^J \int_0^T \int_{X_j} \{|(I_U x(t))_j(x_j)|^2 + \alpha_j(t,x_j)^2 |g_j((I_U x(t))_j(x_j))|^2\} \, d\mu_j(x_j) \, dt\right). \tag{4.10}$$

Proof. Let x be the unique solution of (3.7) satisfying (3.10) and consider p the solution of problem (4.7) and (4.8) with $p_0 = x(T) \in \mathcal{H}$ with $T > 0$ from Theorem 4.2. Owing to [53, Remark 4.2], consider a sequence

$$p_\epsilon \in W^{1,\infty}([0,\infty),\mathcal{H}) \cap L^\infty([0,\infty),\mathcal{V})$$

of strong solution of (4.7) with final data $p_{0\epsilon}$ tending to p in $C([0,T],\mathcal{H})$ as ϵ goes to zero and satisfying

$$K_\epsilon \to K \text{ in } L^2(]0,T[;U) \text{ as } \epsilon \to 0, \tag{4.11}$$

$$I_U p_\epsilon \to I_U p \text{ in } L^2(]0,T[;U) \text{ as } \epsilon \to 0. \tag{4.12}$$

By (3.7) and (4.7) we may write

$$\langle \partial_t x + A_1 x + B(t)x, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} + \langle \partial_t p_\epsilon + A_1 p_\epsilon - K_\epsilon, x\rangle_{\mathcal{V}',\mathcal{V}} = 0, \text{ for a.a. } t \in [0,T].$$

As the assumption (3.11) yields

$$\langle A_1 x, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} + \langle A_1 p_\epsilon, x\rangle_{\mathcal{V}',\mathcal{V}} = 0,$$

the above identity reduces to

$$(\partial_t x, p_\epsilon)_{\mathcal{H}} + (\partial_t p_\epsilon, x)_{\mathcal{H}} + \langle B(t)x, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} - \langle K_\epsilon, x\rangle_{\mathcal{V}',\mathcal{V}} = 0, \text{ for a.a. } t \in [0,T].$$

Integrating this identity for $t \in (0,T)$, we get

$$(x(T), p_\epsilon(T))_{\mathcal{H}} - (x(0), p_\epsilon(0))_{\mathcal{H}} + \int_0^T (\langle B(t)x, p_\epsilon\rangle_{\mathcal{V}',\mathcal{V}} - \langle K_\epsilon, x\rangle_{\mathcal{V}',\mathcal{V}})\, dt = 0.$$

By the definitions of K_ϵ and $B(t)$ we arrive at

$$\begin{aligned}&(x(T), p_\epsilon(T))_{\mathcal{H}} - (x(0), p_\epsilon(0))_{\mathcal{H}}\\ &= \int_0^T \Big((K_\epsilon, I_U x)_U - \sum_{j=1}^J \int_{X_j} \alpha_j(t,x_j) g_j((I_U x)_j(x_j)) \cdot (I_U p_\epsilon)_j(x_j)\, d\mu_j(x_j)\Big)\, dt.\end{aligned}$$

Passing to the limit in ϵ and using the initial and final conditions on p, we obtain

$$2\mathcal{E}(T) = \int_0^T \left((K, I_U x)_U - \sum_{j=1}^J \int_{X_j} \alpha_j(t,x_j) g_j((I_U x)_j(x_j)) \cdot (I_U p)_j(x_j)\, d\mu_j(x_j)\right) dt.$$

Cauchy-Schwarz's inequality leads finally to

$$\begin{aligned}2\mathcal{E}(T) &\le \|K\|_{L^2(0,T;U)} \|I_U x\|_{L^2(0,T;U)} \\ &\quad + \|I_U p\|_{L^2(0,T;U)} \left(\sum_{j=1}^J \int_0^T \int_{X_j} \alpha_j(t,x_j)^2 |g_j((I_U x)_j(x_j))|^2\, d\mu_j(x_j)\, dt \right)^{1/2}.\end{aligned} \tag{4.13}$$

Using the estimate (4.9) (recalling that $p_0 = x(T)$), we have

$$\int_0^T \|K(t)\|_U^2\, dt + \int_0^T \|I_U p(t)\|_U^2\, dt \le 4D\mathcal{E}(T).$$

Using this estimate in the previous one, we arrive at (4.10). □

Corollary 4.4. *Under the assumptions of Theorem 4.2, any solution x of (3.7) with the regularity* (3.10)*, satisfies*

$$\begin{aligned}&\mathcal{E}((n+1)T)\\ &\le D\Big(\sum_{j=1}^J \int_{nT}^{(n+1)T} \int_{X_j} \{ |(I_U x(t))_j(x_j)|^2 + \alpha_j(t,x_j)^2 |g_j((I_U x(t))_j(x_j))|^2 \}\, d\mu_j(x_j)\, dt\Big),\end{aligned} \tag{4.14}$$

for all $n \in \mathbb{N}$.

Proof. We apply the previous Lemma to x_n (instead of x) defined by

$$x_n(t) = x(t + nT), \forall t \geq 0,$$

that is still solution of (3.7) with the regularity (3.10), where the (nonlinear) and time-dependent operator B is replaced by $B_n(t) = B(t+nT)$. The estimate (4.10) applied to x_n yields

$$\mathcal{E}((n+1)T) \leq D\Big(\sum_{j=1}^{J} \int_0^T \int_{X_j} \{|(I_U x(t+nT))_j(x_j)|^2 + \alpha_j(t+nT, x_j)^2 |g_j((I_U x(t+nT))_j(x_j))|^2\} \, d\mu_j(x_j)\, dt\Big),$$

that is nothing else than (4.14) by a simple change of variable. □

4.1. The non-increasing case

Theorem 4.5. *In addition to the previous assumptions on g_j and α_j, $j = 1, \ldots, J$, suppose that g_j satisfies*

$$g_j(x) \cdot x \geq m|x|^2, \forall x \in \mathbb{R}^{N_j} : |x| \geq 1, \tag{4.15}$$

$$|x|^2 + |g_j(x)|^2 \leq G(g_j(x) \cdot x), \forall x \in \mathbb{R}^{N_j} : |x| \leq 1, \tag{4.16}$$

for some positive constant m and a concave strictly increasing function $G\colon [0,\infty) \to [0,\infty)$ such that $G(0) = 0$. Furthermore, we assume that for all $j = 1, \ldots, J$ and all $x_j \in X_j$, the mapping

$$\alpha_j(\cdot, x_j) : [0,\infty) \to (0,\infty) : t \to \alpha_j(t, x_j) \text{ is non-increasing,} \tag{4.17}$$

$$\tilde{\alpha} := \max_{1\leq j\leq J} \sup_{x_j \in X_j} \alpha_j(0, x_j) < \infty, \tag{4.18}$$

and

$$\alpha(t) = \min_{1\leq j\leq J} \inf_{x_j \in X_j} \alpha_j(t, x_j) > 0, \forall t \in [0,\infty). \tag{4.19}$$

Under the assumptions of Theorem 4.2, there exists $c > 0$ (depending on T, C, ω (from Theorem 4.2), $\max_j \mu_j(X_j)$, $\tilde{\alpha}$, M, and m) such that

$$\mathcal{E}(t) \leq \psi^{-1}\left(T\mu \int_T^t \alpha(s)\, ds\right), \quad \forall t \geq T, \tag{4.20}$$

for all solution x of (3.7) satisfying (3.10), where $\mu = \min_j \mu_j(X_j)$, ψ is given by (4.3) with $p = h^{-1}$, and h is defined by

$$h(x) = c(x + G(x)), \forall x \geq 0. \tag{4.21}$$

Proof. Let x be the unique solution of (3.7) satisfying (3.10) and let n be an arbitrary nonnegative integer. Using (4.14) and the definition of $\tilde{\alpha}$, we get

$$\mathcal{E}((n+1)T) \\ \leq C_1\Big(\sum_{j=1}^{J}\int_{nT}^{(n+1)T}\int_{X_j}\{|(I_Ux)_j(x_j)|^2+\alpha_j(t,x_j)|g_j((I_Ux)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt\Big), \tag{4.22}$$

with $C_1 = D\max\{1,\tilde{\alpha}\}$.

We now estimate the right-hand side of (4.22) as follows: For all $j = 1,\dots,J$, introduce

$$\Sigma_{j,n}^{+} = \{(x,t)\in X_j\times(nT,(n+1)T)\ :\ |(I_Ux)_j(x,t)|>1\}, \tag{4.23}$$

$$\Sigma_{j,n}^{-} = \{(x,t)\in X_j\times(nT,(n+1)T)\ :\ |(I_Ux)_j(x,t)|\leq 1\}, \tag{4.24}$$

and split up

$$\int_{nT}^{(n+1)T}\int_{X_j}\{|(I_Ux)_j(x_j)|^2+\alpha_j(t,x_j)|g_j((I_Ux)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt = I_{j,n}^{+}+I_{j,n}^{-},$$

where

$$I_{j,n}^{+} := \int_{\Sigma_{j,n}^{+}}\{|(I_Ux)_j(x_j)|^2+\alpha_j(t,x_j)|g_j((I_Ux)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt,$$

$$I_{j,n}^{-} := \int_{\Sigma_{j,n}^{-}}\{|(I_Ux)_j(x_j)|^2+\alpha_j(t,x_j)|g_j((I_Ux)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt.$$

For the estimation of $I_{j,n}^{+}$, we first notice that the assumption (3.5) leads to

$$I_{j,n}^{+} \leq \int_{\Sigma_{j,n}^{+}}(1+2M\alpha_j(t,x_j))|(I_Ux)_j(x_j)|^2\,d\mu_j(x_j)\,dt,$$

and by (4.15) and (4.18) we get

$$I_{j,n}^{+} \leq m^{-1}(1+2M\tilde{\alpha})\int_{\Sigma_{j,n}^{+}}(I_Ux)_j(x_j)\cdot g_j((I_Ux)_j(x_j))\,d\mu_j(x_j)dt.$$

As $\alpha_j(\cdot,x_j)$ is non-increasing, and using (4.18)-(4.19), we have

$$1\leq\frac{\alpha_j(t,x_j)}{\alpha_j((n+1)T,x_j)}\leq\frac{\alpha_j(t,x_j)}{\alpha((n+1)T)},\forall x_j\in X_j, t\in[nT,(n+1)T], \tag{4.25}$$

which allows to obtain

$$I_{j,n}^{+}\leq m^{-1}(1+2M\tilde{\alpha})\alpha((n+1)T)^{-1}\int_{\Sigma_{j,n}^{+}}\alpha_j(t,x_j)(I_Ux)_j(x_j)\cdot g_j((I_Ux)_j(x_j))d\mu_j(x_j)dt. \tag{4.26}$$

Since (3.3) and (3.4) yield

$$g_j(x) \cdot x \geq 0, \forall x \in \mathbb{R}^{N_j}, \tag{4.27}$$

and since $\alpha_j(t, x_j) > 0$ for all t and $x_j \in X_j$, we have

$$\begin{aligned} &\int_{\Sigma^+_{j,n}} \alpha_j(t, x_j)(I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j))\, d\mu_j(x_j) dt \\ &\quad \leq \int_{nT}^{(n+1)T} \int_{X_j} \alpha_j(t, x_j)(I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j))\, d\mu_j(x_j) dt \\ &\quad \leq (\mathcal{E}(nT) - \mathcal{E}((n+1)T)), \end{aligned} \tag{4.28}$$

this last estimate following from (3.14). Using this estimate in (4.26), we arrive at

$$I^+_{j,n} \leq c_1 \alpha((n+1)T)^{-1} (\mathcal{E}(nT) - \mathcal{E}((n+1)T)), \tag{4.29}$$

for some positive constant c_1 depending only on $\tilde{\alpha}$, M and m.

Similarly by the assumption (4.16) and the monotonicity of G and α we have

$$\begin{aligned} I^-_{j,n} &\leq \max\{1, \tilde{\alpha}\} \int_{\Sigma^-_{j,n}} G((I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j)))\, d\mu_j(x_j) dt \\ &\leq \max\{1, \tilde{\alpha}\} \int_{nT}^{(n+1)T} \int_{X_j} G((I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j)))\, d\mu_j(x_j) dt. \end{aligned}$$

Jensen's inequality then yields

$$I^-_{j,n} \leq \max\{1, \tilde{\alpha}\} T \mu_j(X_j) G \left(\frac{1}{T \mu_j(X_j)} \int_{nT}^{(n+1)T} \int_{X_j} (I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j))\, d\mu_j(x_j) dt \right).$$

As G is strictly increasing and again using (4.25), we obtain

$$I^-_{j,n} \leq KG \left(\frac{1}{T \mu_j(X_j) \alpha((n+1)T)} \int_{nT}^{(n+1)T} \int_{X_j} \alpha_j(t, x_j)(I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j)) d\mu_j(x_j) dt \right),$$

where $K = \max\{1, \tilde{\alpha}\} T \max_j \mu_j(X_j)$. By (3.14), we arrive at

$$I^-_{j,n} \leq \; KG \left(\frac{\mathcal{E}(nT) - \mathcal{E}((n+1)T)}{T \mu_j(X_j) \alpha((n+1)T)} \right). \tag{4.30}$$

The estimates (4.29) and (4.30) into the estimate (4.22) and the monotonicity of G give

$$\mathcal{E}((n+1)T) \leq c_2 \left\{ \frac{\mathcal{E}(nT) - \mathcal{E}((n+1)T)}{T \mu \alpha((n+1)T)} + G\left(\frac{\mathcal{E}(nT) - \mathcal{E}((n+1)T)}{T \mu \alpha((n+1)T)} \right) \right\},$$

for some positive constant c_2 (depending on T, C, ω (from Thm. 4.2), $\max_j \mu_j(X_j)$, $\tilde{\alpha}$, M and m), recalling that $\mu = \min_j \mu_j(X_j)$. As (4.18)-(4.19) imply that $\alpha((n+1)T) \leq \tilde{\alpha}$, this finally leads to

$$\begin{aligned}\mathcal{E}(nT) &= \mathcal{E}(nT) - \mathcal{E}((n+1)T) + \mathcal{E}((n+1)T)\\ &\leq \max\{\mu T\tilde{\alpha}, c_2\}\left\{\frac{\mathcal{E}(nT) - \mathcal{E}((n+1)T)}{T\mu\alpha((n+1)T)} + G\left(\frac{\mathcal{E}(nT) - \mathcal{E}((n+1)T)}{T\mu\alpha((n+1)T)}\right)\right\}.\end{aligned}$$

With $c = \max\{\mu T\tilde{\alpha}, c_2\}$, and the definition (4.21) of h, we have found that

$$\mathcal{E}(nT) \leq h\left(\frac{\mathcal{E}(nT) - \mathcal{E}((n+1)T)}{T\mu\alpha((n+1)T)}\right),$$

which can be equivalently written as

$$T\mu\alpha((n+1)T)h^{-1}(\mathcal{E}(nT)) + \mathcal{E}((n+1)T) \leq \mathcal{E}(nT). \tag{4.31}$$

Since this estimate is valid for all $n \in \mathbb{N}$, we conclude by Theorem 4.1 with the choice $\beta(t) = T\mu\alpha(t)$. □

Note that the conditions (3.5) and (4.15) mean that g_j is linearly bounded at infinity; therefore, the decay rate in (4.20) is guided by the behaviour of g_j near zero and by the behavior of $\int_0^t \alpha(s)\,dx$ as t goes to ∞. Since we are mainly interested in the influence of the time dependency on the decay rate, we restrict ourselves to examples of functions g_j that are linear, sublinear or superlinear near 0 (compare with subsection 3.2.1 and Example 1 of [15]).

Example 4.6. Suppose that g_j satisfies (3.3) to (3.5) and (4.15) as well as

$$x \cdot g_j(x) \geq c_0|x|^{p+1},\ |g_j(x)| \leq C_0|x|^{\gamma}, \forall |x| \leq 1, \tag{4.32}$$

for some positive constants c_0, C_0, $\gamma \in (0,1]$ and $p \geq \gamma$. Then g_j satisfies (4.16) with $G(x) = x^{\frac{2}{q+1}}$ and $q = \frac{p+1}{\gamma} - 1$ (which is ≥ 1).

If $p = \gamma = 1$ (then $q = 1$), that corresponds to a linear behavior of g_j near 0, we have $G(x) = x$ and, hence, $h(x) = 2cx$. Therefore, under the other assumptions of Theorem 4.5 we get the decay

$$\mathcal{E}(t) \leq K\mathcal{E}(0)e^{-L\int_0^t \alpha(s)\,ds}, \quad \forall t \geq 0,$$

for some positive constants K and L, since $\psi^{-1}(t) = \mathcal{E}(0)e^{-\frac{t}{2c}}$.

On the contrary if $p + 1 > 2\gamma$ (corresponding to the sublinear case if $p = 2$ and to the superlinear case if $\gamma = 1$ and $p > 1$), then we get the decay $K(\mathcal{E}(0))\left(\int_0^t \alpha(s)\,ds\right)^{-\frac{2\gamma}{p+1-2\gamma}}$ (since $\psi^{-1}(t)$ is equivalent to $t^{\frac{2}{1-q}}$ for t large), with $K(\mathcal{E}(0)) = K(1 + \mathcal{E}(0)^{-\frac{p+1-2\gamma}{2\gamma}})^{-\frac{2\gamma}{p+1-2\gamma}}$, with a positive constant K.

Note that in both cases, the energy tends to zero as soon as

$$\int_0^t \alpha(s)\,ds \to \infty, \text{ as } t \to \infty.$$

In particular, if $\alpha(t) = \frac{1}{(1+t)^\sigma}$, with $\sigma > 0$, in both cases, we get

$$\mathcal{E}(t) \le K(\mathcal{E}(0))t^{-r},$$

for some $r > 0$ (with $K(\mathcal{E}(0)) = K\mathcal{E}(0)$ in the linear case) that, in the linear case, translates an underdamped phenomenon.

A function g satisfying all these assumptions is given by

$$g(x) = \begin{cases} |x|^{\gamma-1}x & \text{if } |x| \le 1, \\ x & \text{if } |x| \ge 1, \end{cases}$$

for some $\gamma \in (0,1]$. In that case (4.32) holds for $p = \gamma$.

4.2. The non-decreasing case

Theorem 4.7. *In addition to the assumptions on g_j and $\alpha_j, j = 1,\dots,J$, from subsection 3.1, suppose that g_j satisfies (4.15) and (4.16) for some positive constant m and a concave strictly increasing function $G\colon [0,\infty) \to [0,\infty)$ such that $G(0) = 0$ and satisfying the additional assumption*

$$\exists \delta \ge 2, C_G > 0 :\ \beta^2 G(x) \le C_G G(\beta^\delta x),\ \forall x, \beta \in (0,\infty). \tag{4.33}$$

Furthermore, we assume that for all $j = 1,\dots,J$ and all $x_j \in X_j$, the mapping

$$\alpha_j(\cdot, x_j)\colon [0,\infty) \to (0,\infty) : t \to \alpha_j(t, x_j) \text{ is non-decreasing,} \tag{4.34}$$

and that for all $t \in [0,\infty)$

$$\alpha(t) = \max_{1\le j\le J} \sup_{x_j\in X_j} \alpha_j(t, x_j) < \infty, \tag{4.35}$$

and

$$\alpha(0) > 0, \tag{4.36}$$

so that the mapping

$$\alpha\colon [0,\infty) \to (0,\infty) : \quad t \mapsto \alpha(t)$$

is non-decreasing. We finally suppose that there exists $c_0 \in (0,1]$ such that

$$c_0\alpha(t) \le \alpha_j(t, x_j) \le \alpha(t), \quad \forall t \in [0,\infty),\ x_j \in X_j,\ j = 1,\dots,J. \tag{4.37}$$

Under the assumptions of Theorem 4.2, there exists $c > 0$ (depending on T, C, ω (from Theorem 4.2), $\max_j \mu_j(X_j)$, $\alpha(0)$, c_0, M, and m) such that

$$\mathcal{E}(t) \le \psi^{-1}\left(T\mu c_0 \int_T^t \alpha(s-T)\alpha(s)^{-\delta}\, ds\right), \quad \forall t \ge T, \tag{4.38}$$

for all solution x of (3.7) satisfying (3.10), where $\mu = \min_j \mu_j(X_j)$, ψ is given by (4.3) with $p = h^{-1}$, and h is defined by (4.21).

Proof. Let x be the unique solution of (3.7) satisfying (3.10) and let n be an arbitrary nonnegative integer. We estimate the right-hand side of (4.14) as follows: Using the sets $\Sigma^+_{j,n}$ and $\Sigma^-_{j,n}$ defined by (4.23) and (4.24) respectively, we split up

$$\int_{nT}^{(n+1)T}\int_{X_j}\{|(I_Ux)_j(x_j)|^2+\alpha_j(t,x_j)^2|g_j((I_Ux)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt = I^+_{j,n}+I^-_{j,n},$$

where

$$I^+_{j,n} := \int_{\Sigma^+_{j,n}}\{|(I_Ux)_j(x_j)|^2+\alpha_j(t,x_j)^2|g_j((I_Ux)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt,$$
$$I^-_{j,n} := \int_{\Sigma^-_{j,n}}\{|(I_Ux)_j(x_j)|^2+\alpha_j(t,x_j)^2|g_j((I_Ux)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt.$$

For the estimation of $I^+_{j,n}$, we first notice that the assumptions (4.36) and (4.37) lead to

$$\alpha_j(t,x_j)\le\alpha((n+1)T), \forall t\in[nT,(n+1)T], \tag{4.39}$$
$$c_0\alpha(0)\le\alpha_j(t,x_j), \forall t\ge 0. \tag{4.40}$$

Therefore, using the assumption (3.5) on g_j, we have

$$I^+_{j,n}\le(\frac{1}{c_0^2\alpha(0)^2}+2M)\alpha((n+1)T)\int_{\Sigma^+_{j,n}}\alpha_j(t,x_j)|(I_Ux)_j(x_j)|^2\,d\mu_j(x_j)\,dt,$$

and by (4.15) we get

$$I^+_{j,n}\le$$
$$m^{-1}(\frac{1}{c_0\alpha(0)}+2M)\alpha((n+1)T)\int_{\Sigma^+_{j,n}}\alpha_j(t,x_j)|(I_Ux)_j(x_j)\cdot g_j((I_Ux)_j(x_j))d\mu_j(x_j)dt.$$

Since the estimate (4.28) remains valid, we obtain

$$I^+_{j,n}\le c_1\alpha((n+1)T)(\mathcal{E}(nT)-\mathcal{E}((n+1)T)), \tag{4.41}$$

for some positive constant c_1 depending only on c_0, $\alpha(0)$, M and m.

Let us go on with the estimation of $I^-_{j,n}$. First using (4.39)-(4.40), we may write

$$I^-_{j,n}\le C_1\alpha((n+1)T)^2\int_{\Sigma^-_{j,n}}\{|(I_Ux)_j(x_j)|^2+|g_j((I_Ux)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt,$$

where $C_1=\max\{\frac{1}{c_0^2\alpha(0)^2},1\}$. Hence, by the assumption (4.16) and the monotonicity of G and the positivity of α_j, as before we have

$$I^-_{j,n}\le C_1\alpha((n+1)T)^2\int_{nT}^{(n+1)T}\int_{X_j}\alpha_j(t,x_j)G((I_Ux)_j(x_j)\cdot g_j((I_Ux)_j(x_j)))\,d\mu_j(x_j)dt.$$

Jensen's inequality then yields

$$I_{j,n}^{-} \leq C_1 \alpha((n+1)T)^2 T \mu_j(X_j)$$
$$G\left(\frac{1}{T\mu_j(X_j)} \int_{nT}^{(n+1)T} \int_{X_j} (I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j))\, d\mu_j(x_j) dt\right).$$

Now (4.36) and (4.37) yield

$$c_0 \alpha(nT) \leq \alpha_j(t, x_j), \forall t \in [nT, (n+1)T],$$

and since G is strictly increasing, we then obtain

$$I_{j,n}^{-} \leq C_2 \alpha((n+1)T)^2$$
$$G\left(\frac{1}{T\mu c_0 \alpha(nT)} \int_{nT}^{(n+1)T} \int_{X_j} \alpha_j(t, x_j)(I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j))\, d\mu_j(x_j) dt\right),$$

where $C_2 = C_1 T \max_j \mu_j(X_j)$. By (3.14), we arrive at

$$I_{j,n}^{-} \leq \; C_2 \alpha((n+1)T)^2 G\left(\frac{\mathcal{E}(nT) - \mathcal{E}((n+1)T)}{T\mu c_0 \alpha(nT)}\right). \tag{4.42}$$

At this stage, we take advantage of the property (4.33) to conclude that

$$I_{j,n}^{-} \leq \; C_2 G\left(\frac{\alpha((n+1)T)^\delta (\mathcal{E}(nT) - \mathcal{E}((n+1)T))}{T\mu c_0 \alpha(nT)}\right). \tag{4.43}$$

The estimates (4.41) (as $\alpha((n+1)T) \leq \frac{\alpha((n+1)T)^\delta}{\alpha(0)^{\delta-2}\alpha(nT)}$ because α is non-decreasing and $\delta \geq 2$) and (4.43) into the estimate (4.14) give

$$\mathcal{E}((n+1)T) \leq c_2 \left\{ \frac{\alpha((n+1)T)^\delta (\mathcal{E}(nT) - \mathcal{E}((n+1)T))}{T\mu c_0 \alpha(nT)} \right.$$
$$\left. + G\left(\frac{\alpha((n+1)T)^\delta (\mathcal{E}(nT) - \mathcal{E}((n+1)T))}{T\mu c_0 \alpha(nT)}\right)\right\},$$

for some positive constant c_2 (depending on T, $\max_j \mu_j(X_j)$, c_0, $\alpha(0)$, δ, C, ω, M and m). As the non-decreasing property of α implies that $\frac{\alpha((n+1)T)^\delta}{\alpha(nT)} \geq \alpha(0)^{\delta-1}$, this finally leads to

$$\mathcal{E}(nT) = \mathcal{E}(nT) - \mathcal{E}((n+1)T) + \mathcal{E}((n+1)T)$$
$$\leq c \left\{ \frac{\alpha((n+1)T)^\delta (\mathcal{E}(nT) - \mathcal{E}((n+1)T))}{T\mu c_0 \alpha(0)} \right.$$
$$\left. + G\left(\frac{\alpha((n+1)T)^\delta (\mathcal{E}(nT) - \mathcal{E}((n+1)T))}{T\mu c_0 \alpha(0)}\right)\right\},$$

where $c = \max\{\frac{T\mu c_0}{\alpha(0)^{\delta-2}}, c_2\}$. By the definition (4.21) of h, we have found that

$$\mathcal{E}(nT) \leq h\left(\frac{\alpha((n+1)T)^{\delta}(\mathcal{E}(nT)-\mathcal{E}((n+1)T))}{T\mu c_0 \alpha(nT)}\right),$$

which can be equivalently written as

$$T\mu c_0 \alpha(nT)\alpha((n+1)T)^{-\delta} h^{-1}(\mathcal{E}(nT)) + \mathcal{E}((n+1)T) \leq \mathcal{E}(nT). \tag{4.44}$$

Since this estimate is valid for all $n \in \mathbb{N}$, we conclude by Theorem 4.1 with the choice $\beta(t) = T\mu c_0 \alpha(t-T)\alpha(t)^{-\delta}$. □

Example 4.8. If g_j satisfies the assumptions from Example 4.6, G is given by $G(x) = x^{\frac{2}{q+1}}$ and $q = \frac{p+1}{\gamma} - 1 \geq 1$; hence, it satisfies the assumption (4.33) with $C_G = 1$ and $\delta = q+1 = \frac{p+1}{\gamma}$.

If $p = \gamma = 1$ (then $q = 1$), that corresponds to a linear behavior of g_j near 0, we have $G(x) = x$ and, hence, $h(x) = 2cx$. Under the other assumptions of Theorem 4.7 we then get the decay

$$\mathcal{E}(t) \leq K\mathcal{E}(0)e^{-L\int_T^t \alpha(s-T)\alpha(s)^{-2}\,ds}, \forall t \geq T,$$

for some positive constants K and L, since $\psi^{-1}(t) = \mathcal{E}(0)e^{-\frac{t}{2c}}$.

On the contrary if $p+1 > 2\gamma$ (corresponding to the sublinear case if $p = 2$ and to the superlinear case if $\gamma = 1$ and $p > 1$), then we get the decay $K(\mathcal{E}(0))\left(\int_T^t \alpha(s-T)\alpha(s)^{-\frac{p+1}{\gamma}}\,ds\right)^{-\frac{2\gamma}{p+1-2\gamma}}$ (since $\psi^{-1}(t)$ is equivalent to $t^{\frac{2}{1-q}}$ for t large).

Note that in both cases, the energy tends to zero as soon as

$$\int_T^t \alpha(s-T)\alpha(s)^{-\delta}\,ds \to \infty, \text{ as } t \to \infty.$$

In particular, if $\alpha(t) = (1+t)^{\sigma}$, with $0 < \sigma \leq \frac{1}{\delta-1} = \frac{\gamma}{p+1-\gamma}$, in both cases, we get

$$\mathcal{E}(t) \leq K(\mathcal{E}(0))t^{-r},$$

for some $r > 0$, that, in the linear case, translates an overdamping phenomenon.

4.3. The oscillating case

Theorem 4.9. *In addition to the assumptions on g_j and $\alpha_j, j = 1, \ldots, J$, from subsection 3.1, suppose that g_j satisfies (4.15) and (4.16) for some positive constant m and a concave strictly increasing function $G\colon [0,\infty) \to [0,\infty)$ such that $G(0) = 0$ Furthermore, we assume that there exists two positive constants α_0 and $\tilde{\alpha}$ such that*

$$\alpha_0 \leq \alpha_j(t, x_j) \leq \tilde{\alpha}, \quad \forall t \in [0,\infty),\ x_j \in X_j,\ j = 1, \ldots, J. \tag{4.45}$$

Under the assumptions of Theorem 4.2, there exists $c > 0$ *(depending on* T, C, ω *(from Theorem 4.2),* $\max_j \mu_j(X_j)$, $\tilde{\alpha}$, α_0, M, *and* m*) such that*

$$\mathcal{E}(t) \leq \psi^{-1}\left(T\mu\alpha_0(t-T)\right), \forall t \geq T, \tag{4.46}$$

for all solution x *of (3.7) satisfying (3.10), where* $\mu = \min_j \mu_j(X_j)$, ψ *is given by (4.3) with* $p = h^{-1}$, *and* h *is defined by (4.21).*

Proof. Let x be the unique solution of (3.7) satisfying (3.10) and let n be an arbitrary nonnegative integer. Using (4.14) and the assumption (4.45), we get

$$\mathcal{E}((n+1)T) \leq C_1\Big(\sum_{j=1}^{J}\int_{nT}^{(n+1)T}\int_{X_j}\{|(I_U x)_j(x_j)|^2+|g_j((I_U x)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt\Big), \tag{4.47}$$

with $C_1 = D\max\{1, \tilde{\alpha}^2\}$.

We now estimate the right-hand side of (4.47) as follows: Using the sets $\Sigma^+_{j,n}$ and $\Sigma^-_{j,n}$ from (4.23) and (4.24), we split up

$$\int_{nT}^{(n+1)T}\int_{X_j}\{|(I_U x)_j(x_j)|^2 + |g_j((I_U x)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt = I^+_{j,n} + I^-_{j,n},$$

where

$$I^+_{j,n} := \int_{\Sigma^+_{j,n}}\{|(I_U x)_j(x_j)|^2 + |g_j((I_U x)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt,$$
$$I^-_{j,n} := \int_{\Sigma^-_{j,n}}\{|(I_U x)_j(x_j)|^2 + \alpha_j(t,x_j)^2|g_j((I_U x)_j(x_j))|^2\}\,d\mu_j(x_j)\,dt.$$

For the estimation of $I^+_{j,n}$, we first notice that the assumption (3.5) leads to

$$I^+_{j,n} \leq (1+2M)\int_{\Sigma^+_{j,n}}(|(I_U x)_j(x_j)|^2\,d\mu_j(x_j)\,dt,$$

and by (4.15) and (4.18) we get

$$I^+_{j,n} \leq m^{-1}(1+2M)\int_{\Sigma^+_{j,n}}(I_U x)_j(x_j)\cdot g_j((I_U x)_j(x_j))\,d\mu_j(x_j)dt.$$

By the assumption (4.45), we directly obtain

$$I^+_{j,n} \leq m^{-1}(1+2M)\alpha_0^{-1}\int_{\Sigma^+_{j,n}}\alpha_j(t,x_j)(I_U x)_j(x_j)\cdot g_j((I_U x)_j(x_j))\,d\mu_j(x_j)dt.$$

By (4.28), we arrive at

$$I^+_{j,n} \leq m^{-1}(1+2M)\alpha_0^{-1}(\mathcal{E}(nT) - \mathcal{E}((n+1)T)). \tag{4.48}$$

Similarly by the assumption (4.16) and the monotonicity of G and of α we have

$$I_{j,n}^{-} \leq \int_{nT}^{(n+1)T} \int_{X_j} G((I_U x)_j(x_j) \cdot g_j((I_U x)_j(x_j)))\, d\mu_j(x_j)dt.$$

Jensen's inequality then yields

$$I_{j,n}^{-} \leq T\mu_j(X_j)G\left(\frac{1}{T\mu_j(X_j)}\int_{nT}^{(n+1)T}\int_{X_j}(I_U x)_j(x_j)\cdot g_j((I_U x)_j(x_j))\, d\mu_j(x_j)dt\right).$$

As G is strictly increasing and again using (4.45), we obtain

$$I_{j,n}^{-} \leq T\mu_j(X_j)G\left(\frac{1}{T\mu_j(X_j)\alpha_0}\int_{nT}^{(n+1)T}\int_{X_j}\alpha_j(t,x_j)(I_U x)_j(x_j)\cdot g_j((I_U x)_j(x_j))d\mu_j(x_j)dt\right).$$

By (3.14), we arrive at

$$I_{j,n}^{-} \leq \ T\mu_j(X_j)G\left(\frac{\mathcal{E}(nT)-\mathcal{E}((n+1)T)}{T\mu_j(X_j)\alpha_0}\right). \tag{4.49}$$

The estimates (4.48) and (4.49) into the estimate (4.22) and the monotonicity of G give

$$\mathcal{E}((n+1)T) \leq c_2\left\{\frac{\mathcal{E}(nT)-\mathcal{E}((n+1)T)}{T\mu\alpha_0} + G\left(\frac{\mathcal{E}(nT)-\mathcal{E}((n+1)T)}{T\mu\alpha_0}\right)\right\},$$

for some positive constant c_2 (depending on T, C, ω (from Thm. 4.2), $\max_j \mu_j(X_j)$, $\tilde{\alpha}$, α_0, M and m). Hence,

$$\begin{aligned}\mathcal{E}(nT) &= \mathcal{E}(nT)-\mathcal{E}((n+1)T)+\mathcal{E}((n+1)T)\\ &\leq \max\{T\mu\alpha_0, c_2\}\left\{\frac{\mathcal{E}(nT)-\mathcal{E}((n+1)T)}{T\mu\alpha_0} + G\left(\frac{\mathcal{E}(nT)-\mathcal{E}((n+1)T)}{T\mu\alpha_0}\right)\right\}.\end{aligned}$$

With $c = \max\{T\mu\alpha_0, c_2\}$, and the definition (4.21) of h, we have found that

$$T\mu\alpha_0 h^{-1}(\mathcal{E}(nT)) + \mathcal{E}((n+1)T) \leq \mathcal{E}(nT).$$

Since this estimate is valid for all $n \in \mathbb{N}$, we conclude by Theorem 4.1 with the choice $\beta(t) = T\mu\alpha_0$. □

Example 4.10. If g_j satisfies the assumptions from Example 4.6, then in the linear case (i.e., if $p = \gamma = 1$) we get the exponential decay

$$\mathcal{E}(t) \leq K\mathcal{E}(0)e^{-Lt}, \quad \forall t \geq 0,$$

for some positive constants K and L. On the contrary if $p+1 > 2\gamma$, then we get the decay $K(\mathcal{E}(0))t^{-\frac{2\gamma}{p+1-2\gamma}}$. In both cases, the decay rate is the same as the one of the autonomous case.

5. Illustrative examples

5.1. Second order evolution equations

Some examples given below enter in the following framework: Let H and V be two real separable Hilbert spaces such that V is densely and continuously embedded into H. Define the linear operator A_2 from V into V' by

$$\langle A_2 u, v\rangle_{V'-V} = (u, v)_V, \forall u, v \in V, \tag{5.1}$$

and suppose given a (nonlinear) and time-dependent mapping $B_2(t)$ from V into V' as follows: We assume that V is continuously embedded into a control space U in the form (3.1) with the same assumptions on U_j, $j = 1, \dots, J$. Similarly, we suppose given mappings g_j and α_j satisfying the same assumptions than in subsection 3.1. We then define the (nonlinear) operator $B_2(t)$ from V into V' by

$$\langle B_2(t)u, v\rangle = \sum_{j=1}^{J} \int_{X_j} \alpha_j(t,x) g_j((J_U u)_j(x_j)) \cdot (J_U v)_j(x_j)\, d\mu_j(x_j), \forall u, v \in V, \tag{5.2}$$

where J_U denotes the embedding from V to U (hence, $(J_U u)_j$ is the j^{th} component of $J_U u$).

With these data, we consider the second order evolution equation

$$\begin{cases} \dfrac{d^2u}{dt^2}(t) + A_2 u(t) + B_2(t)\dfrac{du}{dt}(t) = 0 \text{ in } H, t \geq 0, \\ u(0) = u_0, \dfrac{du}{dt}(0) = u_1. \end{cases} \tag{5.3}$$

This system is reduced to the first order system (3.7) using the standard argument of reduction of order: setting $\mathcal{H} = V \times H$, $\mathcal{V} = V \times V$ with their natural inner products,

$$x = (u, v)^\top,$$

with $v = \frac{du}{dt}$ and introducing the operators

$$A_1 x = (-v, A_2 u)^\top, B(t)x = (0, B_2(t)v)^\top. \tag{5.4}$$

Note that $B(t)$ is indeed in the form (3.6) with $I_U(u, v)^\top = J_U v$, for all $(u, v)^\top \in V \times V$.

With this definition, we see that x is solution of (3.7), assuming that u exists and is sufficiently regular. But in its full generality, the domain of $A_B(t)$ is time-dependent; so, again we distinguish between two cases.

Before going on, let us notice that the above operator A_1 trivially satisfies (3.11) due to (5.1). Consequently the (nonlinear) operator $A_B(t) = A_1 x + B(t)$ corresponding to (5.4) satisfies all assumptions of subsection 3.1.

Let us finally remark that Theorem 6.1 of [53] shows that $A_1 - I_U$ and $-A_1 - I_U$ generates a C_0-semigroup of contractions in $\mathcal{H}$.

5.1.1. The bounded case

Theorem 5.1. *In addition to the above assumptions, if we assume that H is continuously embedded into the control space U, then for all $(u_0, u_1) \in D(A_2) \times V$ problem (5.3) has a unique solution $u \in C([0, \infty), V) \cap C^1([0, \infty), H)$ such that its second derivative $u''(t) = -A_2 u(t) - B_2(t)u'(t)$ exists and is continuous in H, except at a countable number of points t.*

Proof. We show that $A_B(t) = A_1 x + B(t)$ satisfies the assumptions of Theorem 3.2. First as $\mathcal{H} = V \times H$, it is clearly embedded into U as $H \hookrightarrow U$ and that $D(A_B(t)) = D(A_2) \times V$. Hence, it suffices to show that there exists a positive real number λ such that $R(\lambda\mathbb{I} + \mathcal{A}_B(t)) = \mathcal{H}$. But this properties is proved in [53, Theorem 6.1] for $\lambda = 1$. We then conclude by Theorem 3.2 that for any $(u_0, u_1) \in D(A_2) \times V$, there exists a unique solution x of (3.7) with the properties (3.10). We now come back to the original system by noticing that $x(t) = (u(t), v(t))^\top$ satisfies

$$(u'(t), v'(t))^\top = (v(t), -A_2 u(t) - B_2(t)v(t))^\top, \forall t \geq 0.$$

Hence, $u \in C^1([0, \infty), H)$, $v = u'$ and the second components of the above identity yields $u''(t) = -A_2 u(t) - B_2(t)u'(t)$.

The proof is complete. □

5.1.2. The unbounded case

Here in order to avoid the time-dependency of the domain of $A_B(t)$, we suppose that the mappings α_j satisfy (3.19) for some $\alpha \in C([0, \infty), (0, \infty))$. In such a case, the operator $B_2(t)$ defined in (5.4) will be in the form $B(t) = \alpha(t)B_1$, where $B_1(u, v)^\top = (0, B_2 v)^\top$, with (compare with (3.20))

$$\langle B_2 u, v \rangle = \sum_{j=1}^{J} \int_{X_j} g_j((J_U u)_j(x_j)) \cdot (J_U v)_j(x_j)\, d\mu_j(x_j), \forall u, v \in V. \tag{5.5}$$

Under the previous assumptions on A_2 and B_2, with the help of Theorem 3.3 we can prove the next existence result for problem (5.3).

Theorem 5.2. *In addition to the above assumptions, we assume that the mappings α_j satisfy (3.19) for some $\alpha \in C^1([0, \infty), (0, \infty)$ such that α' is locally Lipschitz. Then for all $(u_0, u_1) \in V \times V$ such that $A_2 u_0 + \alpha(0)B_2 u_1 \in H$, problem (5.3) has a unique solution $u \in C([0, \infty), V) \cap C^1([0, \infty), H)$ such that its second derivative $u''(t) = -A_2 u(t) - \alpha(t)B_2 u'(t)$ exists and is continuous in H, except at a countable number of points t.*

Proof. We first recall that $x = (u, v)^\top$ is solution of (3.7) with A_1 and $B(t)$ from (5.4) (and $B_2(t) = \alpha(t)B_2$) if and only if

$$u'(t) = v(t),$$
$$v'(t) = -A_2 u(t) - \alpha(t)v(t).$$

We now perform the following change of unknowns (assuming that u, v exist and are sufficiently regular)

$$\tilde{u}(t) = \alpha(t)^{-1}u(t), \quad \tilde{v}(t) = v(t). \tag{5.6}$$

Then setting $\tilde{x} = (\tilde{u}(t), \tilde{v}(t))^\top$, we see that it satisfies

$$\tilde{x}' = \begin{pmatrix} -\alpha(t)^{-2}\alpha'(t)u(t) + \alpha(t)^{-1}v(t) \\ -(A_2u(t) + \alpha(t)B_2v(t)) \end{pmatrix} = \begin{pmatrix} \alpha(t)^{-1}(-\alpha'(t)\tilde{u}(t) + \tilde{v}(t) \\ -\alpha(t)(A_2\tilde{u}(t) + B_2\tilde{v}(t)) \end{pmatrix}. \tag{5.7}$$

This means that as operator $D(t) \in \mathcal{L}(\mathcal{H})$, we here choose

$$D(t)(u, v)^\top = (\alpha(t)^{-1}u, v)^\top, \forall (u, v)^\top \in V \times H. \tag{5.8}$$

From the previous identity (5.7), we see that the assumption (3.22) holds with

$$\tilde{D}(t)(u, v)^\top = (u, \alpha(t)v)^\top, \forall (u, v)^\top \in V \times H. \tag{5.9}$$

From the assumptions on α and their definitions, we readily check that all other assumptions from Theorem 3.3 on D and $\tilde{D}$ are satisfied. Finally Theorem 6.1 of [53] (since B_2 defined above satisfies the assumption of this Theorem) guarantees that $A_1 + B_1$ is maximal monotone and has a dense domain in $\mathcal{H}$. In conclusion, by Theorem 3.3, if $(u_0, u_1) \in D(A_B(0))$ (or equivalently if $(u_0, u_1) \in V \times V$ is such that $A_2u_0 + \alpha(0)B_2u_1 \in H$), there exists a unique solution x of (3.7) with the properties (3.10). □

In the remainder of this section Ω is a bounded domain of $\mathbb{R}^n$, $n \geq 1$ with a Lipschitz boundary Γ. Some restrictions will be specified later on when they will be necessary. We further denote by ν the unit outward normal vector along Γ.

5.2. Nonlinear and nonautonomous stabilization of the wave equation

5.2.1. Interior damping

Consider the wave equation with interior damping and Dirichlet boundary condition

$$\begin{cases} \partial_t^2 u - \Delta u + \sigma \sum_{j=1}^{J} \alpha_j(t, \cdot)g_j(\partial_t u) = 0 \text{ in } Q := \Omega\times]0, +\infty[, \\ u = 0 \text{ on } \Sigma := \Gamma\times]0, +\infty[, \\ u(0) = u_0,\ \partial_t u(0) = u_1 \text{ in } \Omega, \end{cases} \tag{5.10}$$

where σ is a non-negative function that belongs to $L^\infty(\Omega)$ such that that there exists a positive constant σ_0 such that

$$\sigma \geq \sigma_0 \text{ on } \mathcal{O}, \tag{5.11}$$

for some open and non empty subset $\mathcal{O}$ of the support X_σ of σ. For all $j = 1, \dots, J$, the functions α_j and g_j satisfy the assumptions of subsection 3.1 with $U_j = L^2(X_j)$, X_j being an open and non empty subset of X_σ such that

$$X_j \cap X_k = \emptyset, \text{ for } j \neq k, \text{ and } \cup_{j=1}^J \bar{X}_j = X_\sigma. \tag{5.12}$$

The stability of this problem in the autonomous case, namely for $\alpha_j = 1$, was extensively studied in the litterature, let us cite the papers [22, 33, 42, 45, 49, 65] and the references cited there. Both papers are restricted to some particular choices of σ and g_j leading to some exponential or polynomial decay rates of the energy of the solution of (5.10). On the contrary the nonautonomous case is less considered in the literature and with the exception of [51] all papers concern interior damping acting on the whole domain (i.e. $\sigma = 1$), see [15, 24, 46, 47, 48, 50, 52, 62]. Using the results of the previous section, and under the assumption that the autonomous linear system is exponentially stable, we obtain new decay results for a large class of functions g_j and α_j.

The first point is that problem (5.10) enters in the framework of problem (5.3) from subsection 5.1 once we take

$$\begin{aligned}
H &= L^2(\Omega),\\
V &= H_0^1(\Omega),\\
(u, v)_V &= \int_\Omega \nabla u \cdot \nabla v \, dx, \forall u, v \in V,\\
\langle B_2(t)u, v\rangle_{V'-V} &= \sum_{j=1}^J \int_{X_j} \alpha_j(t, x)\sigma(x) f_j(u(x))v(x)\, dx, \forall u, v \in V.
\end{aligned}$$

Let us notice that the inner product $(\cdot, \cdot)_V$ induces a norm on V equivalent to the usual one due to Poincaré inequality. Furthermore, the condition (3.5) allows to show that $B_2(t)$ is well-defined from V to V'.

As $L^2(\Omega)$ is clearly embedded into $U = \prod_{j=1}^J L^2(X_j)$ (that is clearly identical with $L^2(X_\sigma)$), the assumptions of Theorem 5.1 are satisfied and therefore, there exists a unique solution u of (5.10) such that $(u, u')^\top$ satisfies (3.10).

In order to deduce some stability results for our system (5.10) with the help of Theorem 4.5 we need that $-A_1 - I_U$ generates an exponentially stable semigroup in $\mathcal{H}$, with the control space $U = L^2(X_\sigma)$. This property is equivalent to the exponential decay of the solution of the autonous and linear problem

$$\left\{\begin{array}{l}
\partial_t^2 u - \Delta u + \sigma \partial_t u = 0 \text{ in } Q := \Omega \times]0, +\infty[,\\
u = 0 \text{ on } \Sigma,\\
u(0) = u_0,\, \partial_t u(0) = u_1 \text{ in } \Omega.
\end{array}\right. \tag{5.13}$$

Note that the exponential stability of (5.13) holds in many different situations, see [22, 65] in the case of a C^2 boundary and $\mathcal{O}$ being a neighborhood of

$$\Gamma_+ := \{x \in \Gamma : (x - x_0) \cdot \nu(x) > 0\}, \tag{5.14}$$

for some $x_0 \in \mathbb{R}^n$, or [42] in the case of a domain Ω with an analytical boundary, σ smooth and $\mathcal{O}$ satisfying a geometrical control condition. Note that in the case $d = 1$, this assumption is valid as soon as $\mathcal{O}$ contains an open interval of Ω, see [22, Exemple 1]. Moreover, if the linear damping acts on the whole domain, namely if $\sigma = 1$ in (5.13) a simple spectral analysis shows that (5.13) is exponentially stable without any assumption on the regularity of the boundary of Ω. In all these situations, if g_j and α_j satisfy the additional assumptions of Theorem 4.5, 4.7 or 4.9, then the energy of our system will satisfy (4.20), (4.38) or (4.46). This allows to recover and extend some results from [62, 52, 50, 24, 15, 46, 47, 48]. Particular cases not covered by the previous references are the case when we have only a local damping, namely $X_\sigma \neq \bar{\Omega}$, and/or a factor $\alpha(t)$ piecewise variables, for instance

$$\alpha_j(t,x) = \alpha_j(t), \forall x \in X_j.$$

5.2.2. Boundary damping

Consider the wave equation with a boundary damping

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \text{ in } Q := \Omega\times]0,+\infty[, \\ u = 0 \text{ on } \Sigma_0 := \Gamma_0\times]0,+\infty[, \\ \partial_\nu u + au + \alpha(t)k(x)g(\partial_t u) = 0 \text{ on } \Sigma_1 := \Gamma_1\times]0,+\infty[, \\ u(0) = u_0,\ \partial_t u(0) = u_1 \text{ in } \Omega, \end{cases} \tag{5.15}$$

where Γ_0 is an open subset of Γ, $\Gamma_1 = \Gamma \setminus \bar{\Gamma}_0$, $a, k \in L^\infty(\Gamma_1)$ are two non negative real-valued functions. The function g is a non-decreasing continuous function from $\mathbb{R}$ into itself such that $g(0) = 0$ and satisfying (3.5), while the function $\alpha \in C^1([0,\infty;(0,\infty))$ and α' is locally Lipschitz.

For the sake of simplicity we suppose that

$$\text{either } \Gamma_0 \text{ is not empty or } a \not\equiv 0. \tag{5.16}$$

As previously, the stability of this problem in the autonomous case, namely for $\alpha = 1$, was extensively studied in the litterature, let us cite the papers [4, 11, 12, 30, 31, 34, 36, 38, 41, 64, 66] and the references cited there. Both papers are restricted to some particular choices of Γ_0, a, and g leading to some exponential or polynomial decay rates of the energy of the solution of (5.10). On the other hand to the best of our knowledge the nonautonomous case is only considered in [51]. Using the results of the previous section, and under the assumption that the autonomous linear system is exponentially stable, we obtain new decay results for a large class of functions g and α.

As before problem (5.15) enters in the framework of problem (5.3) from subsection 5.1 once we take:

$$\begin{aligned} H &= L^2(\Omega), \\ V &= \{v \in H^1(\Omega) | v = 0 \text{ on } \Gamma_0\}, \\ (u,v)_V &= \int_\Omega \nabla u \cdot \nabla v \, dx + \int_{\Gamma_1} au \cdot v \, d\sigma, \forall u, v \in V, \\ U &= L^2(\Gamma_1), \\ \langle B_2(t)u, v\rangle_{V'-V} &= \alpha(t) \int_{\Gamma_1} k(x) g(u(x)) v(x) \, d\sigma(x), \forall u, v \in V. \end{aligned}$$

Let us remark that the assumption (5.16) implies that the inner product $(\cdot,\cdot)_V$ induces a norm on V equivalent to the usual one, while our condition (3.5) implies that $B_2(t)$ is well-defined.

We readily check that these assumptions guarantee that $B_2(t)$ fulfils all the assumptions of Theorem 5.2; hence, (5.15) has a unique solution u such that $(u, u')^\top$ satisfies (3.10).

In order to take advantage of Theorem 4.5 we need that $-A_1 - I_U$ generates an exponentially stable semigroup in $\mathcal{H}$. For this particular example this property is equivalent to the exponential decay of the solution of the autonous and linear problem

$$\left\{ \begin{array}{l} \partial_t^2 u - \Delta u = 0 \text{ in } Q := \Omega \times]0, +\infty[, \\ u = 0 \text{ on } \Sigma_0 := \Gamma_0 \times]0, +\infty[, \\ \partial_\nu u + au + k\partial_t u = 0 \text{ on } \Sigma_1 := \Gamma_1 \times]0, +\infty[, \\ u(0) = u_0, \, \partial_t u(0) = u_1 \text{ in } \Omega. \end{array} \right. \tag{5.17}$$

The exponential stability of (5.17) was obtained in many different situations, let us quote [12, 11], where $a = 0$, $k \in L^\infty(\Gamma_1)$ such that

$$k \geq k_0 \text{ on } \Gamma_1, \tag{5.18}$$

for some positive constant k_0 and under the assumptions that

$$m \cdot \nu \leq 0 \text{ on } \Gamma_0, \tag{5.19}$$

$$m \cdot \nu \geq \gamma > 0 \text{ on } \Gamma_1, \tag{5.20}$$

where γ is a positive constant and m is the standard multiplier defined by

$$m(x) = x - x_0, \forall x \in \mathbb{R}^n, \tag{5.21}$$

for some point $x_0 \in \mathbb{R}^n$.

This result was generalized in [38, 64] to a more general class of multipliers $m \in C^2(\bar{\Omega})$ for which the matrix $(\partial_j m_i + \partial_i m_j)_{1 \leq i,j \leq n}$ is uniformly positive definite in $\bar{\Omega}$ but still under the assumptions $a = 0$, $k \in L^\infty(\Gamma_1)$ satisfying (5.18) and the geometrical constraints (5.19)-(5.20).

Let us observe that conditions (5.19)-(5.20) force to have

$$\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset. \tag{5.22}$$

This constraint has been removed in [41] since condition (5.20) has been removed, while the other conditions from [38, 64] remain. Alternatively, in [36, 39], the choice $k = m\nu$ (with m in the form (5.21) and then as in [38, 64]) allows to replace the condition (5.20) by

$$m \cdot \nu > 0 \text{ on } \Gamma_1,$$

under the conditions $a = 0$ and Γ_0 non empty, see also [30, 49] for the case $a \not\equiv 0$.

Let us finally notice that microlocal analysis arguments from [4] allow to suppress the condition (5.19) if Γ is analytic, the condition (5.22) holds, a and k are smooth, and Γ_1 satisfies the geometrical control condition that it must meet each ray in a nondiffractive point.

Since in [51], it is assumed that $a = 0$, $k = 1$, that (5.19)-(5.20) hold with m in the form (5.21) and that (5.22) holds, Theorems 4.5, 4.7 and 4.9 allow to improve significantly the result from [51] by obtaining different decay rates of the solution of system (5.15) with appropriated choices of α and g using the above mentioned results about the exponential decay of system (5.17).

5.2.3. Pointwise interior damping

In this subsection, we consider the large time behavior of the solution of a homogenous string equation with a homogenous Dirichlet boundary condition at the left end, a Neuman boundary condition at the right end, and subject to a time-dependent and nonlinear pointwise interior actuator. More precisely, we condider the problem

$$\left\{ \begin{array}{l} \partial_t^2 u - \partial_x^2 u + \alpha(t) g(\partial_t u)\, \delta_\xi = 0 \text{ in } (0,\pi) \times \mathbb{R}, \\ u(0,t) = \partial_x u(\pi,t) = 0,\ t > 0, \\ u(\cdot,0) = u_0,\ \partial_t u(\cdot,0) = u_1 \text{ in } (0,\pi), \end{array} \right. \tag{5.23}$$

where ξ is a fixed point of $(0,\pi)$, the functions g is a non-decreasing continuous function from $\mathbb{R}$ into itself such that $g(0) = 0$ and satisfying (3.5), and the function $\alpha \in C^1([0,\infty;(0,\infty))$ is such that α' is locally Lipschitz.

The stability of this problem in the autonomous and linear case, namely for $\alpha = g = 1$ was considered in [2] (see also [3]), and to the best of our knowledge, the case of a nonautonomous and nonlinear pointwise damping has not been analyzed.

Let us notice that problem (5.23) enters in the framework of problem (5.3) from subsection 5.1 once we take:

$$\begin{aligned} H &= L^2(0,\pi), \\ V &= \{v \in H^1(0,\pi) | v(0) = 0\}, \\ (u,v)_V &= \int_0^\pi u_x v_x \, dx, \forall u, v \in V, \\ U &= \mathbb{R}, \\ \langle B_2(t)u, v\rangle_{V'-V} &= \alpha(t) g(u(\xi)) v(\xi), \forall u, v \in V. \end{aligned}$$

These assumptions guarantee that $B_2(t)$ fulfils all the assumptions of Theorem 5.2; hence, (5.23) has a unique solution u such that $(u, u')^\top$ satisfies (3.10).

As Theorem 1.2 of [2] guarantees the exponential decay of the solution of (5.23) with $\alpha = g = 1$ if $\frac{\xi}{\pi} = \frac{p}{q}$ with $p \in \mathbb{N}^*$ odd and $q \in \mathbb{N}^*$, we can apply Theorem 4.5, 4.7 or 4.9 to obtain different decay rates of the solution of system (5.23) under this assumption on ξ and if α and g satisfy the additional assumptions from Theorem 4.5, 4.7 or 4.9.

5.3. Nonlinear and nonautonomous stabilization of the elastodynamic system

With the notation of the above subsubsection 5.2.2, we consider the following elastodynamic system:

$$\begin{cases} \partial_t^2 u - \nabla\sigma(u) + \sigma \sum_{j=1}^J \alpha_j(t,\cdot) g_j(\partial_t u) = 0 \text{ in } Q, \\ u = 0 \text{ on } \Sigma_0, \\ \sigma(u)\cdot\nu + au + kg(\partial_t u) = 0 \text{ on } \Sigma_1, \\ u(0) = u_0,\ \partial_t u(0) = u_1 \text{ in } \Omega. \end{cases} \tag{5.24}$$

As usual $u(x,t)$ is the displacement field at the point $x \in \Omega$ at time t and $\sigma(u) = (\sigma_{ij}(u))_{i,j=1}^n$ is the stress tensor given by (here and in the sequel we shall use the summation convention for repeated indices)

$$\sigma_{ij}(u) = a_{ijkl}\epsilon_{kl}(u),$$

where $\epsilon(u) = (\epsilon_{ij}(u))_{i,j=1}^n$ is the strain tensor given by

$$\epsilon_{ij}(u) = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}),$$

and the tensor $(a_{ijkl})_{i,j,k,l=1,\dots,n}$ is made of $W^{1,\infty}(\Omega)$ entries such that

$$a_{ijkl} = a_{jikl} = a_{klij},$$

and satisfying the ellipticity condition

$$a_{ijkl}\epsilon_{ij}\epsilon_{kl} \geq \alpha\epsilon_{ij}\epsilon_{ij},$$

for every symmetric tensor (ϵ_{ij}) and some $\alpha > 0$. Hereabove and below $\nabla\sigma(u)$ is the vector field defined by

$$\nabla\sigma(u) = (\partial_j \sigma_{ij}(u))_{i=1}^n.$$

Finally a and k are two nonnegative real numbers. As before we assume that

$$g_j = 0, \forall j = 1, \dots, J \text{ or } \Gamma_1 = \emptyset. \tag{5.25}$$

This last assumption means that we stabilizate our system either by a boundary feedback or by an internal feedback with only Dirichlet boundary conditions. In case of a boundary damping, we also suppose that (5.16) holds.

In case of an internal feedback, as in subsubsection 5.2.1, σ is a non-negative function that belongs to $L^\infty(\Omega)$ satisfying (5.11) for some open and non empty subset $\mathcal{O}$ of the the support X_σ of σ. For all $j = 1, \dots, J$, the functions α_j and g_j satisfy the assumptions of subsection 3.1 with $U_j = L^2(X_j)^n$, X_j being an open and non empty subset of X_σ such that (5.12) holds.

In case of a boundary feedback, the functions α and g satisfy the assumptions of subsection 3.1 with $U = L^2(\Gamma_1)^n$ and we suppose, moreover, that $\alpha \in C^1([0, \infty; (0, \infty))$ is such that α' is locally Lipschitz.

The stability of the system (5.24) was considered in [1, 8, 19, 20, 21, 23, 37, 63] in the autonomous case under some particular hypotheses on Γ_0, Γ_1, a, g_j and g leading to exponential or polynomial decay of the energy of the solution of (5.24). The nonautonomous case with internal feedback and for the Lamé system (corresponding to $n = 3$ and to the choice $a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, where λ and μ are positive constants, called Lamé parameters) was treated in [6, 7].

As in the above subsection, problem (5.24) may be expressed in the form (5.3) from subsection 5.1 with the choices:

$$\begin{aligned}
H &= L^2(\Omega)^n,\\
V &= \{v \in H^1(\Omega)^n | v = 0 \text{ on } \Gamma_0\},\\
(u, v)_V &= \int_\Omega \sigma_{ij}(u)\epsilon_{ij}(v)\, dx + a \int_{\Gamma_1} u \cdot v\, d\sigma, \forall u, v \in V,
\end{aligned}$$

and

$$\langle B_2(t)u, v\rangle_{V'-V} = \sum_{j=1}^J \int_{X_j} \alpha_j(t, x)\sigma(x) f_j(u(x)) \cdot v(x)\, dx, \forall u, v \in V,$$

in case of an interior damping and

$$\langle B_2(t)u, v\rangle_{V'-V} = \alpha(t) \int_{\Gamma_1} g(u) \cdot v\, d\sigma, \quad \forall u, v \in V,$$

otherwise.

In the case of an interior damping (resp. boundary damping), all the assumptions of Theorem 5.1 (resp. Theorem 5.2) are satisfied and therefore, we have a unique solution u of (5.24) such that $(u, u')^\top$ satisfies (3.10).

For stability results concerning (5.24), we need to check that $-A_1 - I_U$ generates an exponentially stable semigroup in $V \times H$, where the control space U is defined by

$$U = L^2(X_\sigma)^n \text{ if } \Gamma_1 = \emptyset,$$
$$U = L^2(\Gamma_1)^n \text{ if } g_j = 0, \forall j = 1, \ldots, J.$$

As before, this is equivalent to the exponential decay of the autonomous and linear system (5.24), i.e. corresponding to $\Gamma_1 = \emptyset$, $\alpha_j = 1$ and $g_j(s) = s$ in the first case and to $g_j = 0$, $\alpha = 1$ and $g(s) = s$ in the second case.

In the first case (i. e., $\Gamma_1 = \emptyset$), this exponential decay was proved in [21, Theorem 1.1] (see also [19] for the case $X_\sigma = \Omega$) under the assumption that $\mathcal{O}$ is a neighborhood of Γ_+ defined by (5.14). Hence, in the setting of one of these papers, under the additional assumptions on α_j and g_j from Theorem 4.5, 4.7 or 4.9, different decay rates of the solution of system (5.24) (with $\Gamma_1 = \emptyset$) are available.

In the second case (i.e., $g_j = 0$, for all $j = 1, \ldots, J$), the exponential decay of the autonomous and linear system (5.24) was proved in [1, 8, 23, 37] under some geometric assumptions. In the setting of one of these papers, we then obtain different decay rates of the solution of system (5.24) (with $g_j = 0$, for all $j = 1, \ldots, J$) if g and α satisfy the assumptions from Theorem 4.5, 4.7 or 4.9.

5.4. Nonlinear and nonautonomous stabilization of Maxwell's equations

We consider Maxwell's equations in $\Omega \subset \mathbb{R}^3$ with a smooth boundary with either a nonlinear and nonautonomous internal feedback or a nonlinear and nonautonomous boundary feedback. To the best of our knowledge, the analysis of Maxwell's system with nonautonomous and nonlinear damping has not been analyzed.

To clarify the presentation, we treat these two cases separately.

5.4.1. Interior damping

Here we consider the problem

$$\left\{\begin{array}{l} \varepsilon \frac{\partial E}{\partial t} - \operatorname{curl} H + \sigma \sum_{j=1}^J \alpha_j(t,\cdot) g_j(E) = 0 \text{ in } Q, \\ \mu \frac{\partial H}{\partial t} + \operatorname{curl} E = 0 \text{ in } Q, \\ \operatorname{div}\, (\mu H) = 0 \text{ in } Q, \\ E \times \nu = 0, H \cdot \nu = 0 \text{ on } \Sigma := \Gamma \times]0, +\infty[, \\ E(0) = E_0,\, H(0) = H_0 \text{ in } \Omega. \end{array}\right. \tag{5.26}$$

As usual ϵ and μ are real, positive functions of class $C^1(\bar{\Omega})$, while σ is a nonnegative function that belongs to $L^\infty(\Omega)$ satisfying (5.11) for some open and non empty subset $\mathcal{O}$ of the the support X_σ of σ. For all $j = 1, \ldots, J$, the functions α_j and g_j satisfy the assumptions of subsection 3.1 with $U_j = L^2(X_j)^3$, X_j being an open subset of Ω such that (5.12) holds.

The stability of this system was studied in [57, 58, 61] with a linear and autonomous feedback $g_j(E) = E$ and $\alpha_j = 1$, where some exponential decay results were obtained under some constraints on ϵ, μ and σ. The nonlinear and autonomous case was treated in [53].

Contrary to the above examples this system is not a second order system but it enters in the setting of (3.7) once we set

$$\begin{aligned}
\mathcal{H} &= L^2(\Omega)^3 \times \hat{J}(\Omega,\mu),\\
\hat{J}(\Omega,\mu) &= \{H \in L^2(\Omega)^3 : \operatorname{div}(\mu H) = 0 \text{ in } \Omega, H\cdot\nu = 0 \text{ on } \Gamma\},\\
((E,H),(E',H'))_{\mathcal{H}} &= \int_\Omega (\epsilon E\cdot E' + \mu H\cdot H')\,dx, \forall (E,H),(E',H') \in \mathcal{H},\\
\mathcal{V} &= V \times \hat{J}(\Omega,\mu),\\
V &= \{E \in L^2(\Omega)^3 : \operatorname{curl} E \in L^2(\Omega)^3, E\times\nu = 0 \text{ on } \Gamma\},\\
\langle A_1(E,H),(E',H')\rangle &= \int_\Omega (\operatorname{curl} E\cdot H' - H\cdot \operatorname{curl} E')\,dx, \forall (E,H),(E',H') \in \mathcal{V},\\
\langle B(t)(E,H),(E',H')\rangle &= \sum_{j=1}^J \int_\Omega \alpha_j(t,x) g_j(E)\cdot E'\,dx, \forall (E,H),(E',H') \in \mathcal{V}.
\end{aligned}$$

As $\mathcal{H}$ is continuously embedded into $U = L^2(\Omega)^3 \times \{0\}$ (with $I_U(E,H)^\top = (E,0)^\top$), $\mathcal{A}_B(t) = A_1 + B(t)$ satisfies (3.16). Furthermore, one readily checks (as in [16, §3]) that $A_1 + B(t)$ is maximal monotone for the inner product $(\cdot,\cdot)_{\mathcal{H}}$, since the bilinear form

$$\int_\Omega (\mu^{-1} \operatorname{curl} E\cdot \operatorname{curl} E' + \epsilon E\cdot E')\,dx$$

is clearly coercive on V. Hence, by Theorem 5.1 system (5.26) has a unique solution $(E,H)^\top$ of (5.26) satisfying (3.10).

As before $\pm A_1 - I_U$ generates an exponentially stable semigroup in $\mathcal{H}$ if and only if system (5.26) with a linear and autonomous feedback is exponentially stable. As Theorems 5.1 and 5.5 of [61] (resp. Theorem 4.1 of [57] and Remark 5.2 of [58]) imply that such an exponential stability holds if ε and μ are constant (resp. sufficiently smooth) and under some conditions on $\mathcal{O}$, we may conclude some decays of the solution of (5.26) in the setting of one of these papers, as soon as g_j and α_j satisfy the additional assumptions from Theorem 4.5, 4.7 or 4.9.

5.4.2. Boundary damping

Let us go on with Maxwell's equations with a nonlinear and nonautonous boundary feedback

$$\left\{\begin{array}{l}
\varepsilon \frac{\partial E}{\partial t} - \operatorname{curl} H = 0 \text{ in } Q := \Gamma\times]0,+\infty[,\\
\mu \frac{\partial H}{\partial t} + \operatorname{curl} E = 0 \text{ in } Q,\\
\operatorname{div}\,(\varepsilon E) = \operatorname{div}\,(\mu H) = 0 \text{ in } Q,\\
H\times\nu + \alpha(t) g(E\times\nu)\times\nu = 0 \text{ on } \Sigma := \Gamma\times]0,+\infty[,\\
E(0) = E_0,\ H(0) = H_0 \text{ in } \Omega,
\end{array}\right. \tag{5.27}$$

where the functions α and g satisfy the assumptions of subsection 3.1 with $U = L^2(\Gamma)^3$ and $\alpha \in C^1([0,\infty;(0,\infty))$ is such that α' is locally Lipschitz.

The autonomous case was studied in [5, 16, 25, 32, 34, 55, 56, 61], where different decay rates are avalaible under different conditions on ϵ, μ and Γ and appropriated assumptions on g.

Let us now show that (5.27) enters in the framework of subsection 3.3 if we take (see [55, §2])

$$
\begin{aligned}
\mathcal{H} &= J(\Omega,\varepsilon) \times J(\Omega,\mu),\\
J(\Omega,\mu) &= \{H \in L^2(\Omega)^3 : \operatorname{div}(\mu H) = 0 \text{ in } \Omega\},\\
((E,H),(E',H'))_{\mathcal{H}} &= \int_\Omega (\epsilon E\cdot E' + \mu H\cdot H')\,dx,\\
\mathcal{V} &= V \times J(\Omega,\mu), \forall (E,H),(E',H') \in \mathcal{H},\\
V &= \{E \in J(\Omega,\varepsilon) : \operatorname{curl} E \in L^2(\Omega)^3, E\times\nu \in L^2(\Gamma)^3\},\\
U &= L^2(\Gamma)^3,\\
\langle A_1(E,H),(E',H')\rangle &= \int_\Omega (\operatorname{curl} E\cdot H' - H\cdot \operatorname{curl} E')\,dx, \forall (E,H),(E',H') \in \mathcal{V},\\
B(t) &= \alpha(t)B_1,\\
\langle B_1(E,H),(E',H')\rangle &= \int_\Gamma g(E\times\nu)\cdot(E'\times\nu)\,d\sigma(x), \forall (E,H),(E',H') \in \mathcal{V}.
\end{aligned}
$$

Note first that $B(t)$ is well-defined with the embedding $I_U(E,H)^\top = E\times\nu$, while by its definition A_1 directly satisfies (3.11). Hence, all assumptions of subsection 3.1 are satisfied. Now in order to apply Theorem 3.3, for all $t \geq 0$, we introduce the bounded linear operators $D(t)$ and $\tilde{D}$ from $\mathcal{H}$ into itself by

$$
D(t)(E,H)^\top = (E,\alpha(t)^{-1}H)^\top, \quad \tilde{D}(t)(E,H)^\top = (\alpha(t)E,H)^\top
$$

that, due to the assumptions on α, satisfy the requested regularity assumptions and the condition (3.21) from Theorem 3.3. Furthermore, simple calculations shows that (3.22) holds. As Lemma 2.3 of [55] guarantees that the domain of $A_1 + B_1$ is dense in $\mathcal{H}$ and Lemma 2.3 of [55] shows that $A_1 + B_1$ is maximal monotone in $\mathcal{H}$, we can apply Theorem 3.3 to obtain the well posedness of problem (5.27).

Here again $\pm A_1 - I_U$ generates an exponentially stable semigroup in $\mathcal{H}$ if and only if system (5.27) with a linear and autonomous feedback is exponentially stable. Such a stability property was obtained in many papers, let us quote [25, 32, 34, 61]. Hence, if system (5.27) with a linear and autonomous feedback is exponentially stable and if additionally α and g satisfy the additional assumptions from Theorem 4.5, 4.7 or 4.9, we may conclude some decays of the solution of (5.27).

References

[1] Alabau, F., Komornik V.: Boundary observability, controllability, and stabilization of linear elastodynamic systems. SIAM J. Control Optim. **37(2)**, 521–542 (1999)

[2] Ammari, K., Henrot, A., Tucsnak, M.: Asymptotic behaviour of the solutions and optimal location of the actuator for the pointwise stabilization of a string. Asymptot. Anal. **28(3-4)**, 215–240 (2001)

[3] Ammari, K., Tucsnak, M.: Stabilization of second order evolution equations by a class of unbounded feedbacks. ESAIM Control Optim. Calc. Var. **6**, 361–386 (2001)

[4] Bardos, C., Lebeau G., Rauch, J.: Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary. SIAM J. Control Optim. **30(5)**, 1024–1065 (1992)

[5] Barucq, H., Hanouzet, B.: Étude asymptotique du système de Maxwell avec la condition aux limites absorbante de Silver-Müller II. C. R. Acad. Sci. Paris Sér. I **316**, 1019–1024 (1993)

[6] Bchatnia, A., Daoulatli, M.: Behavior of the energy for Lamé systems in bounded domains with nonlinear damping and external force. Electron. J. Differential Equations **2013**, No. 01 (2013)

[7] Bellassoued, M.: Energy decay for the elastic wave equation with a local time-dependent nonlinear damping. Acta Math. Sin. (Engl. Ser.) **24(7)**, 1175–1192 (2008)

[8] Bey, R., Heminna, A., Lohéac, J.-P. : Boundary stabilization of the linear elastodynamic system by a Lyapunov-type method. Rev. Mat. Complut. **16(2)**, 417–441 (2003)

[9] Brézis, H.: Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Mathematics Studies, **5**. North-Holland Publishing Co., Amsterdam-London (1973).

[10] Browder, F. E.: Nonlinear maximal monotone operators in Banach space. Math. Ann. **175**, 89–113 (1968)

[11] Chen, G.: Control and stabilization for the wave equation in a bounded domain. SIAM J. Control Optim., **17(1)**, 66–81 (1979)

[12] Chen, G.: Energy decay estimates and exact boundary value controllability for the wave equation in a bounded domain. J. Math. Pures Appl. **58(3)**, 249–273 (1979)

[13] Crandall, M. G., Pazy, A.: Nonlinear evolution equations in Banach spaces. Israel J. Math. **11**, 57–94 (1972)

[14] da Luz, C. R., Perla Menzala, G. A.: Uniform decay rates of coupled anisotropic elastodynamic/Maxwell equations with nonlinear damping. Port. Math. **68(2)**, 205–238 (2011)

[15] Daoulatli, M.: Rates of decay for the wave systems with time dependent damping. Discrete Contin. Dyn. Syst. **31(2)**, 407–443 (2011)

[16] Eller, M., Lagnese, J. E., Nicaise, S.: Decay rates for solutions of a Maxwell system with nonlinear boundary damping. Comp. and Appl. Math. **21**, 135–165 (2002)

[17] Evans, L. C.: Nonlinear evolution equations in an arbitrary banach space. Israel J. Math. **26**, 1–42 (1977)

[18] Guesmia, A.: Existence globale et stabilisation interne non linéaire d'un système de Petrovsky. Bull. Belg. Math. Soc. Simon Stevin **5(4)**, 583–594 (1998)

[19] Guesmia, A.: Observability, controllability and boundary stabilization of some linear elasticity systems. Acta Sci. Math. (Szeged) **64(1-2)**, 109–119 (1998)

[20] Guesmia, A.: Existence globale et stabilisation frontière non linéaire d'un système d'élasticité. Portugal. Math. **56(3)**, 361–379 (1999)

[21] Guesmia, A.: On the decay estimates for elasticity systems with some localized dissipations. Asymptot. Anal. **22(1)**, 1–13 (2000)

[22] Haraux, A.: Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps. Portugal. Math. **46(3)**, 245–258 (1989)

[23] Horn, M. A.: Implications of sharp trace regularity results on boundary stabilization of the system of linear elasticity. J. Math. Anal. Appl. **223(1)**, 126–150 (1998)

[24] Jiao, Z., Xiao, T.-J.: Convergence and speed estimates for semilinear wave systems with nonautonomous damping. Math. Methods Appl. Sci. **39(18)**, 5465–5474 (2016)

[25] Kapitanov, B. V.: Stabilization and exact boundary controllability for Maxwell's equations. SIAM J. Control Optim. **32**, 408–420 (1994)

[26] Kapitanov, B. V., Raupp, M. A.: Exact boundary controllability in problems of transmission for the system of electromagneto-elasticity. Math. Meth. Appl. Sci. **24**, 193–207 (2001)

[27] Kato, T.: Nonlinear semigroups and evolution equations. J. Math. Soc. Japan **19**, 508–520 (1967)

[28] Kato, T.: Abstract differential equations and nonlinear mixed problems. Lezioni Fermiane. [Fermi Lectures]. Scuola Normale Superiore, Pisa, Accademia Nazionale dei Lincei, Rome (1985)

[29] Kato, T.: Linear and quasi-linear equations of evolution of hyperbolic type. In: Hyperbolicity, volume 72 of C.I.M.E. Summer Sch., pages 125–191. Springer, Heidelberg (2011)

[30] Komornik, V.: Rapid boundary stabilization of the wave equation. SIAM J. Control Optim. **29**,197–208 (1991)

[31] Komornik, V.: On the nonlinear boundary stabilization of the wave equation. Chinese Ann. Math. Ser. B **14**, 153–164 (1993)

[32] Komornik, V.: Boundary stabilization, observation and control of Maxwell's equations. PanAm. Math. J. **4**,47–61 (1994)

[33] Komornik, V.: Decay estimates for the wave equation with internal damping. In: Control and estimation of distributed parameter systems: nonlinear phenomena (Vorau, 1993), volume 118 of Internat. Ser. Numer. Math., pages 253–266. Birkhäuser, Basel (1994)

[34] Komornik, V.: Exact controllability and stabilization. The multiplier method. Research in Applied Mathematics **36**. Masson, Paris (1994) .

[35] Komornik, V.: On the nonlinear boundary stabilization of Kirchhoff plates. NoDEA Nonlinear Differential Equations Appl. **1(4)**, 323–337 (1994)

[36] Komornik, V., Zuazua, E.: A direct method for the boundary stabilization of the wave equation. J. Math. Pures Appl. **69(1)**, 33–54 (1990)

[37] Lagnese, J. E.: Boundary stabilization of linear elastodynamic systems. SIAM J. Control Optim. **21(6)**, 968–984 (1983)

[38] Lagnese, J. E.: Decay of solutions of wave equations in a bounded region with boundary dissipation. J. Differential Equations **50(2)**, 163–182 (1983)

[39] Lagnese, J. E.: Note on boundary stabilization of wave equations. SIAM J. Control Optim. **26(5)**, 1250–1256 (1988)

[40] Lasiecka, I., Tataru, D.: Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping. Differential Integral Equations **6(3)**, 507–533 (1993)

[41] Lasiecka, I. , Triggiani R.: Uniform stabilization of the wave equation with Dirichlet or Neumann feedback control without geometrical conditions. Appl. Math. Optim. **25(2)**, 189–224 (1992)

[42] Lebeau, G.: Équation des ondes amorties. In: Algebraic and geometric methods in mathematical physics (Kaciveli, 1993), volume 19 of Math. Phys. Stud., pages 73–109. Kluwer Acad. Publ., Dordrecht (1996)

[43] Lin, C.-Y.: Time-dependent nonlinear evolution equations. Differential Integral Equations **15**, 257–270 (2002)

[44] Liu, K., Liu, Z.: Boundary stabilization of a nonhomogeneous beam with rotatory inertia at the tip. J. Comput. Appl. Math. **114(1)**, 1–10 (2000)

[45] Liu, W.-J., Zuazua, E.: Decay rates for dissipative wave equations. Ricerche Mat. **48**, 61–75 (1999)

[46] Luo, J.-R., Xiao,T.-J.: Decay rates for second order evolution equations in Hilbert spaces with nonlinear time-dependent damping. Evol. Equ. Control Theory **9(2)**, 359–373 (2020)

[47] Luo, J.-R., Xiao,T.-J.: Decay rates for semilinear wave equations with vanishing damping and Neumann boundary conditions. Math. Methods Appl. Sci. **44(1)**, 303–314 (2021)

[48] Luo, J.-R., Xiao,T.-J.: Optimal energy decay rates for abstract second order evolution equations with non-autonomous damping. ESAIM Control Optim. Calc. Var. **27**, Paper No. 59 (2021)

[49] Martinez, P.: A new method to obtain decay rate estimates for dissipative systems with localized damping. Rev. Mat. Comp. Madrid **12**, 251–283 (1999)

[50] Martinez, P.: Precise decay rate estimates for time-dependent dissipative systems. Israel J. Math. **119**, 291–324 (2000)

[51] Mustafa, M. I.: Uniform decay for wave equations with weakly dissipative boundary feedback. Dyn. Syst. **30(2)**, 241–250 (2015)

[52] Nakao, M.: On the decay of solutions of the wave equation with a local time-dependent nonlinear dissipation. Adv. Math. Sci. Appl. **7(1)**, 317–331 (1997)
[53] Nicaise, S.: Stability and controllability of an abstract evolution equation of hyperbolic type and concrete applications. Rendiconti di Matematica Serie VII **23**, 83–116 (2003)
[54] Nicaise, S.: Stability and controllability of the electromagneto-elastic system. Port. Math. **60(1)**, 37–70 (2003)
[55] Nicaise, S., Eller, M., Lagnese, J. E.: Stabilization of heterogeneous Maxwell's equations by linear or nonlinear boundary feedback. Electron. J. Differential Equations **2002**, paper No. 21 (2002)
[56] Nicaise, S., Pignotti, C.: Boundary stabilization of Maxwell's equations with space-time variable coefficients. ESAIM Control Optim. Calc. Var. **9**, 563–578 (2003)
[57] Nicaise, S., Pignotti, C.: Internal stabilization of Maxwell's equations in heterogeneous media. Abstr. Appl. Anal. **7**, 791–811 (2005)
[58] Nicaise, S., Pignotti, C.: Internal and boundary observability estimates for heterogeneous maxwell's system. Applied Math. Optim. **54**, 47–70 (2006)
[59] Pazy, A.: Semi-groups of nonlinear contractions in Hilbert space. In: Problems in non-linear analysis (Centro Internaz. Mat. Estivo (C.I.M.E.), IV Ciclo, Varenna, 1970), pages 343–430. Edizioni Cremonese, Rome (1971)
[60] Pazy, A.: Semigroups of linear operators and applications to partial differential equations, volume 44 of Applied Math. Sciences. Springer-Verlag, New York (1983)
[61] Phung, K. D.: Contrôle et stabilisation d'ondes électromagnétiques. ESAIM Control Optim. Calc. Var. **5**, 87–137 (2000)
[62] Pucci, P., Serrin, J.: Asymptotic stability for nonautonomous dissipative wave systems. Comm. Pure Appl. Math. **49(2)**, 177–216 (1996)
[63] Tcheugoué Tebou, L. R.: On the stabilization of the wave and linear elasticity equations in 2-D. PanAmer. Math. J. **6(1)**, 41–55 (1996)
[64] Triggiani, R.: Wave equation on a bounded domain with boundary dissipation: an operator approach. J. Math. Anal. Appl. **137(2)**, 438–461 (1989)
[65] Zuazua, E.: Exponential decay for the semilinear wave equation with locally distributed damping. Comm. in PDE **15**, 205–235 (1990)
[66] Zuazua, E.: Uniform stabilization of the wave equation by nonlinear boundary feedback. SIAM J. Control Optim. **28**, 466–477 (1990)

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