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On the behavior of entropy solutions for a fractional p-Laplacian problem as t tends to infinity

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Abstract. *We prove an asymptotic behavior result of entropy solutions to fractional parabolic problems whose simplest model is*

$$(\mathcal{P})\ \begin{cases} u_t(t,x) + (-\Delta)^s_p u(x) = \mu \text{ in } Q := (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega,\ u(t,x) = 0 \text{ on } \Sigma := (0,T)\times\partial\Omega, \end{cases}$$

where, Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$, $T > 0$, $(-\Delta)^s_p u$ is the fractional p-Laplace operator ($ps < N$, $0 < s < 1$), $p > 2 - \frac{s}{N}$, $\mu \in \mathcal{M}^+(Q)$ is a nonnegative measure with bounded variation over Q and $u_0 \in L^1(\Omega)$ is a nonnegative function. We first prove some a priori estimates on the entropy solutions, we then show that, if μ does not depend on time, then the sequence of entropy solutions of such problems converge to the stationary solution of the corresponding elliptic problem as t tends to infinity.

1. Introduction

Given a parabolic cylinder $Q = (0,T)\times\Omega$ where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain and $T > 0$. We denote by $\mathcal{M}(Q)$ the vector space of all finite *Radon* measures in Q equipped with the norm $\|\mu\|_{\mathcal{M}(Q)} := |\mu|(Q)$. In this paper, we study the behavior, as t tends to infinity, of entropy solutions for a class of initial-boundary value problems of *fractional* differential type whose simplest model is

$$\begin{cases} u_t + (-\Delta)^s_p u(x) = \mu \text{ in } Q := (0,T)\times\Omega, \\ u \geq 0 \text{ in } \mathbb{R}^N,\ u = 0 \text{ in } (0,T)\times(\mathbb{R}^N\backslash\Omega), \\ u(0,x) = u_0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $p > 2 - \frac{s}{N}$, $(-\Delta)^s_p u$ is the so-called *fractional p-Laplace* operator ($ps < N$, $0 < s < 1$), which up to renormalization factors, is defined as

$$\begin{aligned} (-\Delta)^s_p u(x) &:= \textbf{P.V.} \int_\Omega \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy \\ &= \lim_{\epsilon\downarrow 0}\int_{\Omega\backslash B_\epsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}dy, \end{aligned} \tag{1.2}$$

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where $x \in \Omega \neq \mathbb{R}^N$, **P.V.** is a commonly used abbreviation for "*in the principal value sense*", in presence of a nonnegative measure with bounded variation over Q which does not charge the sets of zero fractional p-capacity (i.e., $\mu \in \mathcal{M}^+(Q)$) and a nonnegative integrable function (i.e., $u_0 \in L^1(\Omega)$).

It is well-known that the notion of *entropy* solution was introduced in order to extend the classical setting of monotone operators, see [66], and to be able to define a notion of solution to problems whose data do not belong to the dual space as, for instance, L^1 or measure data (the main interest is not to get a solution in the sense of distributions but to get a concept which allow us to obtain existence and uniqueness). The answer by *Stampacchia*, in the case where $p = 2$ and $s = 1$, in order to discover a deep relationship with duality solutions and irregular data and to get existence/uniqueness results is contained in the pioneering work [108], see also [95, 87]. For $p \neq 2$ and $s = 1$, this question was widely analyzed in [27, 28], see also [46], to study more general class of operators, in particular, *p-Laplace* or *Leray-Lions* operators; both existence and uniqueness of such a solution are proved if the datum μ belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$. This fact was proved in [45, 46, 78, 76] in the case where the right-hand side is a *Radon* measure with bounded total variation using *renormalized* solutions, we refer to [90, 85] for an exhaustive treatment of this topic. As we have seen previously, uniqueness of distributional solutions can fail even in the linear case if the regularity of the solutions is not "enough" to allow the choice of less regular test functions (the lack of regularity of the solution of the counter-example by *Serrin* [100], as modified in [95], is exactly the one which is typical of this case. However, the lack of uniqueness is avoided by using the concept of duality solution in the linear case but it is enough for the operator to be non-linear in order to "lose" the duality argument. In this case, a further condition on the solutions has been looked for in order to guarantee uniqueness and an equivalent notion, the concept of *entropy* solutions, was introduced in [12], see also [29], to overcome some of these difficulties. It should be noted that there are some obstacles when extending this notion of solution to the case of general measure data, see [92, 93, 94], because of the possibly lack of μ-measurability of the integral on the the right-hand side; however, there are cases in which this definition still makes sense outside of $L^1 + W^{-1,p'}$.

In this manuscript we are interested in finding the pointwise limit of u, entropy solution of problem (1.1), as t tends to infinity and proving that such a limit v is a solution to the "limit equation"

$$\begin{cases} (-\Delta)^s_p v = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

equipped with homogeneous Dirichlet boundary condition. The interest in studying the behavior of such solutions comes from optimal design problems (in the theory of torsion) and related geometrical problems. These equations arise also in the mathematical modeling of penetration process of an electromagnetic field into a substance with temperature dependent coefficient of electroconductivity (recall that such non-local investigations are considered as extensions of classical results obtained in [67, 87, 88, 89, 38, 2]). The asymptotic behavior of solutions, as time tends to infinity, of initial-boundary value problems are studied first in the works [53, 9, 107] under various assumptions and in different contexts (an extensive critical bibliography on the subject up to 1970 can be found in [116]). It is worthy to point out that the fractional model we have considered has intrinsic interest since it appears in a lot of applications modeling different models in continuous mechanics, phase transition phenomena, population dynamics, image processing, game theory and Lévy processes, see [8, 33, 34, 35, 81] for more details, and has challenging difficulties that must be carefully analyzed; more precisely:

- The main difficulty relies on finding a suitable and technical notion of solution which allows us to work with more general class of test functions.
- The comparison principle result can not be improved directly due to technical reasons; however, suitable choices of sub- and super-solutions with slight modifications can be used in this general setting.
- Some assumptions on p and s should be considered, these bounds are essentially used to ensure that the solution of the fractional evolution problem actually belongs to the energy space.

Since we cannot find exact (energy) solutions of the considered fractional models, particular attention should be paid to the construction of approximate solutions and to their a priori-estimates generalizing some nonlinear compactness and convergence results studied in many well-known scientific papers, books, and monographs. The detailed description of the paper is given below and more complete references and comments are given in each section.

The paper is organized as follows. In Sections 2.1–2.2 we give some auxiliary results related to fractional Sobolev spaces ans some properties of fractional (s,p)-capacity. We also present in Section 2.3 the notion of the considered solution and some intermediary results that will be used in the proofs. In Section 2.4, we consider the homogeneous case and we study the bahviour, according to the limit of the time variable t, of the sequence of solutions of problem (1.1). In Section 3, the nonhomogeneous problem, which is the more delicate case, is analyzed. Some possible extensions are proposed in the last section.

Notations. In order to make the exposition self-contained, we present here some basic notations: $\mathbb{R}^N$ is the Euclidean N-dimensional space, $x \in \mathbb{R}^N$ is denoted by $x = (x_1, \ldots, x_N)$. If $E \subset \mathbb{R}^N$, then $\overline{E}$ denotes the closure of E, $E^c = \mathbb{R}^N \backslash E$ is the complement of E in $\mathbb{R}^N$. The N-dimensional *Lebesgue* measure of E is denoted by $|E|$ (also dx will be used). The characteristic function of E is denoted by χ_E or $\chi(E)$, i.e., $|E| = \int_{\mathbb{R}^N} \chi_E dx = \int_E dx$. $B(x_0, r)$ is the open ball centered at x_0 of radius r, in other words $B(x_0, r) = \{x : |x - x_0| < r\}$. The positive and negative parts of a function f are denoted by f_+ and f_-, i.e., $f_+(x) = \max\{f(x), 0\}$ and $f_-(x) = \max\{-f(x), 0\}$. We will write $C(E)$ to denote the space of (usually real valued) continuous functions on $E \subset \mathbb{R}^N$ equipped with the topology of uniform convergence on compact subsets on E. If K is compact, $C(K)$ is usually normed with the supremum norm $\|\cdot\|_{L^\infty(K)}$. If $\Omega\subset\mathbb{R}^N$ is an open set or a domain (connected open set), then $C_0(\Omega)$ is the subset of $C(\Omega)$ consisting of functions with compact support contained in Ω. The dual of $C_0(\Omega)$ is denoted by $\mathcal{M}(\Omega)$, the *Radon* measure space on Ω, see [97, 98], and these measures are only locally when restricted to compact subsets of Ω, see [99, 61]. The cone of positive elements in $\mathcal{M}(\Omega)$ is denoted by $\mathcal{M}^+(\Omega)$, and a sequence μ_n in $\mathcal{M}(\Omega)$ is said to converge in the weak* topology to μ if

$$\lim_{n\to\infty}\int_\Omega \varphi d\mu_n=\int_\Omega \varphi d\mu \tag{1.4}$$

for all $\varphi\in C_0(\Omega)$. The class of infinity differentiable functions on Ω is denoted $C^\infty(\Omega)$, and $C_0^\infty(\Omega)$ is the subset of functions with compact support in Ω equipped with their usual topologies in distributional theory (the elements in $C_0^\infty(\Omega)$ are called test functions). The space of continuous linear functionals on $C_0^\infty(\Omega)$ is the space of *Schwartz* distributions on Ω denoted by $\mathcal{D}'(\Omega)$ and the paring between distributions and test functions is denoted $(\cdot,\cdot)$. Finally, we will use the latter C to denote various unspecified positive constants whose value can change within a sequence of inequalities.

2. Preliminaries and functional setting

2.1. Fractional *Sobolev* spaces and *fractional* p-*Laplace* operators

We also need some basic facts about fractional *Sobolev* spaces and their properties that we will use systematically in this paper, we refer to [49, 71, 79, 109, 115] for more details. Let $\Omega\subset\mathbb{R}^N$ be an open set whose boundary $\partial\Omega$ and let $p\in[1,\infty)$, the first order *Sobolev* space defined by

$$W^{1,p}(\Omega):=\{u\in L^p(\Omega):\int_\Omega|\nabla u|^p dx<\infty\}, \tag{2.1}$$

is a *Banach* space endowed with the norm

$$\|u\|_{W^{1,p}(\Omega)}:=(\|u\|^p_{L^p(\Omega)}+\|\nabla u\|^p_{L^p(\Omega)})^{\frac{1}{p}}. \tag{2.2}$$

The *Sobolev* spaces are named for *S.L. Sobolev*, which used these spaces systematically from the mid 1930's, see [104, 105, 106]. However, the history of these spaces goes back at least to the work of *Beppo Levi* in the beginning of the century, references are given in the book by *C.B. Morrey* [82]. Denote

$$\widetilde{W}^{1,p}(\Omega)=\overline{W^{1,p}(\Omega)\cap C_c(\overline{\Omega})}^{W^{1,p}(\Omega)} \text{ and } W_0^{1,p}(\Omega)=\overline{\mathcal{D}(\Omega)}^{W^{1,p}(\Omega)}. \tag{2.3}$$

Let us recall that $\widetilde{W}^{1,p}(\Omega)$ is a proper closed subspace of $W^{1,p}(\Omega)$, see e.g., [75, 77]. Moreover, if Ω has the $W^{1,p}$-*extension property*, that is, if for every $u\in W^{1,p}(\Omega)$ there exists $w\in W^{1,p}(\mathbb{R}^N)$ such that $w_{|\Omega}=u$ then $\widetilde{W}^{1,p}(\Omega)=W^{1,p}(\Omega)$, see also [7, 6]. Now, denote by $\mathcal{D}(\Omega) = \mathbb{R}^N \times \mathbb{R}^N \backslash (\Omega^c \times \Omega^c)$ where $\Omega^c = \mathbb{R}^N \backslash \Omega$. For $s \in (0,1)$ and $p \in [1,\infty)$, the linear space of *Lebesgue measurable* functions $u\colon \mathbb{R}^N \mapsto \mathbb{R}$ such that the quantity

$$\left(\int_\Omega |u(x)|^p dx + \iint_{\mathcal{D}(\Omega)} \frac{|u(x)-u(y)|}{|x-y|^{N+ps}} dxdy\right) < \infty, \tag{2.4}$$

is denoted by $W^{s,p}(\Omega)$ (the *Sobolev space of fractional order*). It is easy to see that $W^{s,p}(\Omega)$ is not trivial since it contains *bounded* and *Lipschitz* functions, and is a *Banach* space endowed with the following norm

$$\|u\|_{W^{s,p}(\Omega)} = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_\Omega\int_\Omega |u(x)-u(y)|^p d\nu\right)^{\frac{1}{p}}. \tag{2.5}$$

Similarly, denote

$$\widetilde{W}^{s,p}(\Omega) = \overline{W^{s,p}(\Omega)\cap C_c(\overline{\Omega})}^{W^{s,p}(\Omega)} \text{ and } W_0^{s,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{W^{s,p}(\Omega)}. \tag{2.6}$$

In the same way, we define the space $W_0^{s,p}(\Omega)$ as the space of functions $u \in W^{s,p}(\Omega)$ that vanish a.e. in Ω^c. It is clear that the space $W_0^{s,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the previous norm. If Ω is a bounded regular domain, we can endow $W_0^{s,p}(\Omega)$ with the following equivalent norm

$$\|u\|_{W_0^{s,p}(\Omega)} = \left(\int_\Omega\int_\Omega |u(x)-u(y)|^p d\nu\right)^{\frac{1}{p}}.$$

For every function $u \in W_0^{s,p}(\Omega)$, it is easy to see that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^{N+ps}} = \int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{N+ps}} dxdy + 2\int_\Omega |u(x)|^p \int_{\Omega^c} \frac{1}{|x-y|^{N+ps}} dydx. \tag{2.7}$$

Recalling [49, Lemma 6.1], we have $\displaystyle\int_{\Omega^c} \frac{1}{|x-y|^{N+ps}} dy \geq C|\Omega|^{\frac{-sp}{N}}$ where $C = C(N,p,s) > 0$. A simple computation, using *Poincaré*'s inequality, gives

$$\int_\Omega |u(x)|^p dx \leq C\int_{\mathcal{D}(\Omega)} |u(x)-u(y)|^p d\nu \text{ with } d\nu = \frac{dxdy}{|x-y|^{N+ps}}, \quad \forall p \geq 1. \tag{2.8}$$

Thus, we can endow $W_0^{s,p}(\Omega)$ with the equivalent norm

$$\|u\|_{W_0^{s,p}(\Omega)} := \left(\int_{\mathcal{D}(\Omega)} |u(x)-u(y)|^p d\nu\right)^{1/p}. \tag{2.9}$$

Observe that, since $W_0^{s,p}(\Omega)$ is a reflexive *Banach* space, and as similar to $W_0^{1,p}(\Omega)$, we have $W_0^{s,p}(\Omega) = \overline{W_0^\infty(\Omega)}^{W^{s,p}(\Omega)}$. Recall that $\widetilde{W}^{s,p}(\Omega)$ contains $W_0^{s,p}(\Omega)$ as a closed subspace and, by definition, $W_0^{s,p}(\Omega)$ is the smaller closed subspace of $W^{s,p}(\Omega)$ containing $\mathcal{D}(\Omega)$ (for an overview on fractional order *Sobolev* spaces, we refer to the monographs [7, 70, 72, 73, 75, 111] and their references). In general $W^{1,p}(\Omega)$ is not a subspace of $W^{s,p}(\Omega)$, see [49, Example 9.1], but the following result holds true.

Proposition 2.1. *Let $p \in [1,\infty)$ and $s \in (0,1)$, let $\Omega \subset \mathbb{R}^N$ be an open set having the $W^{1,p}$-extension property. Then, there exists a constant $C = C(N,s,p) \geq 0$ such that for every $u \in W^{1,p}(\Omega)$*

$$\|u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}. \tag{2.10}$$

Proof. See [113, Proposition 2.3]. □

The following result is proved in [113, Lemma 2.4] under the assumption that $\varphi \in C^{0,1}(\overline{\Omega}) \cap L^\infty(\Omega)$.

Lemma 2.2. *Let $p \in [1,\infty)$ and $s \in (0,1)$, let $u \in W^{s,p}(\Omega)$ and $\varphi \in C^{0,1}(\overline{\Omega}) \cap L^\infty(\Omega)$. Then, $\varphi u \in W^{s,p}(\Omega)$ and there is a constant $C > 0$ (depending on N, p, s and $\|\varphi\|_{L^\infty(\Omega)}$) such that*

$$\|\varphi u\|_{W^{s,p}(\Omega)} \leq C\|u\|_{W^{s,p}(\Omega)}. \tag{2.11}$$

We notice that Lemma 2.2 remains true if one replace $W^{s,p}(\Omega)$ with the space $\widetilde{W}^{s,p}(\Omega)$. Now, in order to make the paper clear as possible, let us introduce the *fractional Laplace* operator $(-\Delta)^s u$: let $0 < s < 1$ and set

$$C_{N,s} = \frac{s2^{2s}\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)}, \tag{2.12}$$

where Γ denotes the usual *Gamma* function, we define the *fractional Laplacian* $(-\Delta)^s u$ by the formula

$$\begin{aligned}(-\Delta)^s u(x) &= C_{N,s}\ \mathbf{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy \\ &= \lim_{\epsilon\downarrow 0} C_{N,s} \int_{\{y\in\mathbb{R}^N:|y-x|>\epsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}}dy.\end{aligned} \tag{2.13}$$

Notice that, if $0 < s < \frac{1}{2}$ and u smooth (*Lipschitz continuous* for example), the integral in (2.13) is in fact not really singular near x. Note also that $(-\Delta)^s$ can be defined as a *pseudo-differential* operator by the *Fourier* transformation (with symbol $|\xi|^{2s}$) using the method of *bilinear Dirichlet* forms (a closed self auto-adjoint associated to a bilinear symmetric form) or by the *contraction semigroup* theory, see [24, 25, 56, 55] for more details. As concerned, we have to generalize the fractional *Laplace* operator to the case $p \neq 2$, and to study the *existence* and the *regularity* of the fractional differential equation (1.1) associated with these *nonlocal* operators $(-\Delta)^s_p$. We proceed as follows: let $w \in W^{s,p}(\mathbb{R}^N)$ be an arbitrary function, and let

$$(-\Delta)^s_p w(x) := \mathbf{P.V.} \int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+ps}} dy, \tag{2.14}$$

we restrict the *integral Kernel* of the functional *p-Laplacian* to the open set $\Omega \subset \mathbb{R}^N$, and we define the functional $\langle (-\Delta)^s_p w, \cdot \rangle$ for all $w \in W^{s,p}(\Omega)$ as

$$\langle (-\Delta)^s_p w, v \rangle = \frac{1}{2} \int_{\mathcal{D}(\Omega)} |w(x)-w(y)|^{p-2}(w(x)-w(y))(v(x)-v(y)) d\nu, \tag{2.15}$$

for all $v \in W^{s,p}(\Omega)$, also called the *regional* fractional *p-Laplacian*, see [54, 56, 55], and defined as a *pseudo-differential* operator from $W^{s,p}_0(\Omega)$ onto its dual space $W^{-s,p'}(\Omega)$. Now, for $w \in W^{s,p}(\Omega)$ we set

$$(-\Delta)^s_p w(x) = \mathbf{P.V.} \int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))}{|x-y|^{N+ps}}.$$

It is clear that for all $w, v \in W^{s,p}(\Omega)$, we have

$$\big\langle (-\Delta)^s_p w, v \big\rangle = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|w(x)-w(y)|^{p-2}(w(x)-w(y))(v(x)-v(y))}{|x-y|^{N+ps}} dxdy,$$

where $\mathcal{D}_\Omega = (\mathbb{R}^N \times \mathbb{R}^N) \backslash (C\Omega \times C\Omega)$. Now, define $L^p(0,T;W^{s,p}_0(\Omega))$, which is a *Banach* space, as the set of functions u such that $u \in L^p(Q)$ with

$$\|u\|_{L^p(0,T;W^{s,p}_0(\Omega))} = \left(\int_0^T \int_{\mathcal{D}(\Omega)} |u(t,x)-u(t,y)|^p d\nu dt \right)^{\frac{1}{p}} < \infty. \tag{2.16}$$

A simple calculation in the evolution case gives that if $w \in L^p(0,T;W^{s,p}_0(\Omega))$ then $(-\Delta)^s_p \colon L^p(0,T;W^{s,p}_0(\Omega)) \mapsto L^{p'}(0,T;W^{-s,p'}_0(\Omega))$ where $L^{p'}(0,T;W^{-s,p'}(\Omega))$ is the dual space of $L^p(0,T:W^{s,p}_0(\Omega))$.

2.2. Fractional capacity and properties

The concept of capacity is indispensable to an understanding pointwise behavior of functions in a *Sobolev* space. In a sense, capacity is a measure of size for sets which measure small sets more precisely than the usual *Lebesgue* measure (capacity theory is one of the significant aspects of potential theory). In this setting, there are two natural kinds of capacities: *Sobolev* capacity and relative capacity (both capacities have their advantages). The relative capacity is closely related to the *Wiener* criterion, thinness, fine topology and fine potential theory; see [18, 20, 64, 74] and the monographs [21, 60, 80]. In contrast, *Sobolev* capacity plays a central role when studying quasi-continuous representative and fine properties for equivalence class of *Sobolev* functions; see [22, 57, 58, 59, 65, 63, 74] and the monograph [48]. Recently, the fractional capacity has found a great number of uses, see for instance [101, 113, 114] and the references therein. We first introduce the notion of *Choquet capacity* (for more details, see [37, 50, 7], in particular [113, Section 3] and references quoted therein where more properties are presented).

Definition 2.3. A *Choquet capacity* on a topological space is defined as the mapping $\mathcal{C}\colon \mathcal{D}(T)$ (the power set of T) $\mapsto [0,\infty)$ satisfying

($\mathcal{C}_0$) $\mathcal{C}(\emptyset) = 0$,

($\mathcal{C}_1$) $A \subset B \subset \mathcal{T}$ implies $\mathcal{C}(A) \subseteq \mathcal{C}(B)$,

($\mathcal{C}_2$) $(A_n)_{n\in\mathbb{N}} \subset \mathcal{T}$ an *increasing* sequence implies $\lim\limits_{n\to\infty} \mathcal{C}(A_n) = C(\cup_{n=1}^{\infty} A_n)$,

($\mathcal{C}_3$) $(K_n)_n \subset \mathcal{T}$ a *decreasing* sequence, K_n compact, implies

$$\lim_{n\to\infty} \mathcal{C}(K_n) = C(\cap_{n=1}^{\infty} K_n).$$

Following the lines of the previous definition of *Choquet* capacities, we want to give some basic knowledge on what has been done, up to known, about the classical *Bessel capacity* of order (s,p) denoted by $\text{cap}_{(s,p)}$, see [7, 75] for details. It is defined for any *open* set $U \subset \mathbb{R}^N$ by

$$\text{cap}_{(s,p)}(U) = \inf\left\{ \|u\|^p_{W^{s,p}(\mathbb{R}^N)} :\ u \in W^{s,p}(\mathbb{R}^N),\ u \geq 1 \text{ a.e. on } U \right\}. \tag{2.17}$$

For an *arbitrary* set $E \subset \mathbb{R}^N$,

$$\text{cap}_{(s,p)}(E) = \inf\left\{ \text{cap}_{(s,p)}(U) :\ U \text{ is an open set in } \mathbb{R}^N \text{ containing } E \right\}, \tag{2.18}$$

and where, as usual, we use the convention that $\inf \emptyset = +\infty$; then one can extend this definition by regularity to any *Borel* subset of $\mathbb{R}^N$. Let us recall that a function $u \in W^{s,p}(\mathbb{R}^N)$ is said to be *$cap_{(s,p)}$-quasi-continuous* (*$cap_{(s,p)}$-q.c*) if for every $\epsilon > 0$ there exists an open set $U \subset \mathbb{R}^N$ such that $\text{cap}_{(s,p)}(U) \leq \epsilon$ and u is continuous in $\mathbb{R}^N \backslash U$. It is well known that every *Bessel* capacity $\text{cap}_{(s,p)}$ is a *Choquet* capacity, see [7, Section 2.2], and that every function $u \in W^{s,p}(\mathbb{R}^N)$ admits a unique (up to a polar set) *$cap_{(s,p)}$-q.c* function $\widetilde{u}\colon \mathbb{R}^N \mapsto \mathbb{R}$ such that $\widetilde{u} = u$ $\text{cap}_{(s,p)}$-q.e. on $\mathbb{R}^N$. Thanks to this fact it is also possible to prove the following: for any capacity set $K \subset \mathbb{R}^N$, we have

$$\text{cap}_{(s,p)}(K) = \inf\left\{ \|u\|^p_{W^{s,p}(\mathbb{R}^N)} : u \in W^{s,p}(\mathbb{R}^N) \cap C_c(\mathbb{R}^N),\ u \geq 1 \text{ on } K \right\}. \tag{2.19}$$

Moreover, if $B \subset \mathbb{R}^N$ is a Borel set, we have

$$\text{cap}_{(s,p)}(B) = \sup\left\{ \text{cap}_{(s,p)}(K) : K \subseteq B \subset \mathbb{R}^N \text{ compact} \right\}. \tag{2.20}$$

Further results on the relationship between the classical *Bessel* capacity $\text{cap}_{(s,p)}$ and the related *Hausdorff* measures can be found in [7, 75]. Now, we recall the definition of the elliptic *fractional relative* capacity (for further details on the relative $(1,p)$-capacity, we refer the reader to [10, 11, 15, 19, 17, 39, 41, 42, 112] and references therein).

Definition 2.4. Let $\mathcal{O} \subset \overline{\mathbb{R}}$ be a relative open set, i.e., open with respect to the relative topology of $\mathbb{R}$. The *relative capacity* of $\mathcal{O}$ with respect to Ω is defined by

$$\mathrm{Cap}^{\overline{\Omega}}_{(s,p)}(\mathcal{O}) := \inf\left\{\|u\|^p_{W^{s,p}(\Omega)} : u \in \widetilde{W}^{s,p}(\Omega),\ u \geq 1 \text{ a.e. on } \mathcal{O}\right\}. \tag{2.21}$$

For any set $A \subset \overline{\Omega}$,

$$\mathrm{Cap}^{\overline{\Omega}}_{(s,p)}(A) = \inf\left\{\mathrm{cap}^{\overline{\Omega}}_{(s,p)}(\mathcal{O}) : \mathcal{O} \text{ relatively open in } \overline{\Omega} \text{ containing } A\right\}. \tag{2.22}$$

If $\Omega = \mathbb{R}^N$ then $\mathrm{Cap}^{\overline{\Omega}}_{(s,p)} = \mathrm{Cap}_{(s,p))}$. By definition, it is clear that for every $A \subset \overline{\Omega}$

$$\mathrm{Cap}^{\overline{\Omega}}_{(s,p)}(A) \leq \mathrm{Cap}_{(s,p)}(A). \tag{2.23}$$

Now, let us define the notion of fractional parabolic p-capacity associated to our problem; to this aim let us denote

$$W^{s,p}(Q) = \left\{u \in L^p(0,T;W^{s,p}(\Omega)) \text{ with } u_t \in L^{p'}(0,T;W^{-s,p'}(\Omega))\right\} \tag{2.24}$$

(resp., $\widetilde{W}^{s,p}(Q)$ using $\widetilde{W}^{s,p}(\Omega)$ and $\widetilde{W}^{-s,p'}(\Omega)$ instead of $W^{s,p}(\Omega)$ and $W^{-s,p'}(\Omega)$) endowed with its natural norm

$$\|u\|_{W^{s,p}(Q)} = \|u\|_{L^p(0,T;W^{s,p}(\Omega))} + \|u_t\|_{L^{p'}(0,T;W^{-s,p'}(\Omega))}.$$

Definition 2.5. If $U \subseteq Q$ is an open set, the fractional parabolic (s,p)-capacity of U is defined as

$$\mathrm{Cap}_{(s,p)}(U) = \inf\left\{\|u\|_{W^{s,p}(Q)} :\ u \in W^{s,p}(Q),\ u \geq \chi_U \text{ a.e. in } Q\right\}, \tag{2.25}$$

where again we set $\inf \emptyset = +\infty$; then for any *Borel* set $B \subseteq Q$ we define

$$\mathrm{Cap}_{(s,p)}(B) = \inf\left\{\mathrm{cap}_{(s,p)}(U),\ U \text{ open set of } Q,\ B \subseteq U\right\}.$$

As mentioned before, this definition can be extended to the case of relatively fractional capacity.

Definition 2.6. If $U \subset \overline{Q}$ is a *relatively* open set (with respect to the relative topology of $\overline{Q}$), we define the (*relatively*) *fractional parabolic capacity* of U (with respect to Q) as

$$\mathrm{cap}^{\overline{Q}}_{(s,p)}(u) := \inf\left\{\|u\|^p_{W^{s,p}(Q)} : u \in L^p(0,T;\widetilde{W}^{s,p}(\Omega)),\ u \geq 1 \text{ a.e. on } U\right\}, \tag{2.26}$$

where as usual we set $\inf \emptyset = +\infty$, then for any *arbitrary* set $E \subset \overline{Q}$ we define

$$\mathrm{cap}^{\overline{Q}}_{(s,p)}(E) = \inf\left\{\mathrm{cap}^{\overline{Q}}_{(s,p)}(U) :\ U \text{ relatively open in } \overline{Q} \text{ containing } E\right\}. \tag{2.27}$$

Let $K \subset \overline{Q}$ be a compact set, then

$$\operatorname{cap}_{(s,p)}(K) = \inf\Big\{ \|u\|^p_{W^{s,p}(Q)} :\ u \in W^{s,p}(Q) \cap C_c(\overline{Q}),\ u \geq 1 \text{ on } K \Big\} \tag{2.28}$$

and, for any *Borel* set $B \subset \overline{Q}$ we have

$$\operatorname{cap}^{\overline{Q}}_{(s,p)}(B) = \sup\Big\{ \operatorname{cap}^{\overline{Q}}_{(s,p)}(K) :\ K \subseteq B \subset \overline{Q} \text{ compact} \Big\}$$

This second definition of capacity, that enjoys the *Choquet*-properties as well as the first we give, will turn out to be very useful since it allows to extend the notion of *Bessel* capacity.

Proposition 2.7. *Let E be an arbitrary set of $\overline{Q}$, then*

$$cap^{\overline{Q}}_{(s,p)}(E) = cap_{(s,p)}(E). \tag{2.29}$$

Next, we give some useful properties on the relative fractional capacity.

Proposition 2.8. *Some properties are in order to be given:*

(i) A set $E \subset \overline{Q}$ is called relatively polar if $cap^{\overline{Q}}_{(s,p)}(E) = 0$.

(ii) A property $\mathcal{P}(t,x)$ is said to hold on a set $F \subset \overline{Q}$ relatively quasi-everywhere (r.q.e.) if there exists a relatively polar set $E \subset F$ such that the property holds everywhere on $F \backslash E$.

(iii) A function $u\colon \overline{Q} \mapsto \mathbb{R}$ is said to be relatively quasi-continuous (r.q.c.) if for every $\epsilon > 0$ there exists a relatively open set $U \subset \overline{Q}$ such that $cap^{\overline{Q}}_{(s,p)}(U) < \epsilon$ and $u_{|\overline{Q}\backslash U}$ is continuous.

(iv) For any function in $\widetilde{W}^{s,p}(Q)$, there exists a unique (up to a relatively polar set) relatively quasi-continuous representative (r.q.c.r).

(v) Let u_n be a sequence of r.q.c. functions in $\widetilde{W}^{s,p}(Q)$ which converges to a r.q.c. function $u \in \widetilde{W}^{s,p}(Q)$. Then, there exists a subsequence which converges r.q.e. to u on $\overline{Q}$.

(vi) Assume that Q has the $W^{s,p}$-extension property, that is, for every element $w \in L^p(0,T;W^{s,p}(\Omega))$ there exists a function $U \in L^p(0,T;W^{s,p}(\mathbb{R}^N))$ such that $U_{|Q} = u$. Then, $cap_{(s,p)}$ and $cap^{\overline{Q}}_{(s,p)}$ are equivalent.

Up to minor changes, the next capacity result is similar to the classical one.

Theorem 2.9. *Let B be a Borel set in Ω, and $0 \leq t_0 \leq t_1 < T$. Then*

$$\operatorname{Cap}_{(s,p)}((0,T) \times B) = 0 \ \textit{iff} \ \operatorname{Cap}^e_{(s,p)}(B) = 0. \tag{2.30}$$

Proof. See [51, Theorem 2.16]. □

Let us denote by $\mathcal{M}_0(\Omega)$ the set of all measures not charging sets of zero elliptic (s,p)-capacity, that is, if $\mu \in \mathcal{M}_0(\Omega)$ then $\mu(E) = 0$ for all $E \subset \Omega$ such that $\text{cap}^e_{(s,p)}(E) = 0$; analogously we define $\mathcal{M}_0(Q)$ as the set of all measures not charging sets of zero parabolic (s,p)-capacity, that is, if $\mu \in \mathcal{M}_0(Q)$ then $\mu(E) = 0$ for all $E \subset Q$ such that $\text{cap}_{(s,p)}(E) = 0$. Thanks to Theorem 2.9, one can identify measures in $\mathcal{M}_0(Q)$ not depending on time with measures in $\mathcal{M}_0(\Omega)$.

Remark 2.10. Remark that:

(i) Thanks to Theorem 2.9 we deduce that $\text{cap}_{(s,p)}((0,T) \times B) = 0$ and so $\mu((0,T) \times B) = 0$, then, since μ is independent on time variable t, there exists a measure $\nu \in \mathcal{M}(\Omega)$ such that

$$0 = \mu((0,T) \times B) = T\nu(B), \tag{2.31}$$

then $\nu \in \mathcal{M}_0(\Omega)$ (i.e., we can identify μ with ν).

(ii) If $\mu \in \mathcal{M}_0(\Omega)$ then it can be decomposed, see [29], as $\mu = f - \text{div}(G)$ with $f \in L^1(\Omega)$ and $G \in L^{p'}(\Omega)^N$. Moreover, if $0 \leq \mu \in \mathcal{M}_0(\Omega)$ then f can be chosen to be nonnegative.

The following lemma of analytic nature will be useful in deriving some a priori estimates.

Lemma 2.11. *Let $G\colon \mathbb{R} \mapsto \mathbb{R}$ be a Lipschitz function such that $G(0) = 0$. Then, for every $u \in L^p(0,T; W_0^{s,p}(\mathcal{D}))$, with D is any bounded open subset of $\mathbb{R}^N$, we have $G(u) \in L^p(0,T; W_0^{s,p}(\mathcal{D}))$ and $\nabla G(u) = G'(u)\nabla u$ a.e. in $(0,T) \times \mathcal{D}$.*

Proof. Up to minor changes the proof is similar to the one in [108]. □

In order to prove some compactness/convergence results satisfied by the solutions we need some technical ingredients.

Lemma 2.12 (*Algorithmic inequality 1*)**.** *Assume that $p \geq 1$, $(a,b) \in (\mathbb{R}^+)^2$ and $\alpha > 0$, then there exist $(C_i)_{i=1}^4 > 0$ such that*

$$(a+b)^\alpha \leq C_1 a^\alpha + C_2 b^\alpha,$$

and

$$|a-b|^{p-2}(a-b)(a^\alpha - b^\alpha) \geq C_3 |a^{\frac{\alpha-1}{p}} - b^{\frac{p+\alpha-1}{p}}|^p.$$

In the case where $\alpha \geq 1$ and under the same conditions on a, b, p as above, we have

$$|a+b|^{\alpha-1}|a-b|^p \leq C_4 |a^{\frac{p+\alpha-1}{p}} - b^{\frac{p+\alpha-1}{p}}|^p.$$

Proof. See [102, Theorème 8.1]. □

We recover from Lemma 2.12 a new algebraic inequality.

Lemma 2.13 (*Algorithmic inequality 2*)**.** *There exist two constants* $(C_i)_{i=1}^2$ *(with* $C_1 < 1 < C_2$*) such that for all* $(a_i)_{i=1}^2 \in \mathbb{R}$ *and all* $(b_i)_{i=1}^2 \geq 0$*, we have*

$$|a_1-a_2|^{p-2}(a_1-a_2)(a_1b_1-a_2b_2) \geq C_1 \left|a_1 b_1^{\frac{1}{p}} - a_2 b_2^{\frac{1}{p}}\right|^p - C_2 \left(\max\{|a_1|,|a_2|\}\right)^p \left|b_1^{\frac{1}{p}} - b_2^{\frac{1}{p}}\right|^p .$$

Proof. See [3, Lemma 2.8]. □

2.3. Entropy solutions

As we said before, in the linear case the existence/uniqueness results have been proved by using duality techniques, see [108, 95, 87]; unfortunately, the duality method doesn't work in the nonlinear setting so that it was necessary to look for different techniques. For L^1-data, *H. Brezis* & *W. Strauss* are studied some problems with maximal monotone graphs, see [31, 32] (the case of *Radon* measure data and monotone operators is considered in [27, 28]); however, the uniqueness fails due to a counterexample by *Serrin*, see [100, 95] (some attempts are proved in [43] for problems with strongly monotone operators and L^1-data using approximate techniques). In [13], the authors are introduced a new notion of solution, called *entropy* solution, to deal with the question of uniqueness but this concept is meaningless when dealing with *Radon* measure data, see [27, 28]); and cannot be generalized directly to the case of singular measures. This is done by means of the concept of renormalized solution developed in [46] in the stationary case and extended in [86] for evolution problems. Thanks to the results of [29, Theorem 3.2 & Theorem 3.3], a renormalized solution turns out to coincide with an entropy solution for *Radon* measures which are zero on subsets of zero capacity (one of the tools used to prove this equivalence is a result of *G. Dal Maso*, see [44], which strongly rely on the structure decomposition of the measure (notice that entropy/renormalized solutions turn out to be also distributional solutions, see [30, 29]). Now, in order to define a notion of entropy solution of the corresponding elliptic problem of (1.1) we need the following functional space[1]

$$\mathcal{T}_0^{s,p}(\Omega) = \{v \text{ measurable s.t. } T_k(v) \in W_0^{s,p}(\Omega),\ \forall k > 0\}.$$

Notice that if u is in $\mathcal{T}_0^{s,p}(\Omega)$ and φ is in $W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ then $u - \varphi$ belongs to $\mathcal{T}_0^{s,p}(\Omega)$, see [46].

Definition 2.14. Let μ be a *Radon* measure in $\in \mathcal{M}_0(\Omega)$. A function $v \in \mathcal{T}_0^{s,p}(\Omega)$ is an *entropy* solution of

$$\begin{cases} (-\Delta)_p^s v = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.32}$$

if

$$\iint_{R_h} |v(x) - v(y)|^{p-1} d\nu \to 0 \text{ as } h \to \infty, \tag{2.33}$$

[1] Here, $T_k(\sigma) = \max\{-k, \min\{k, \sigma\}\}$.

where

$$R_h = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : h+1 \le \max\{|v(x)|,|v(y)|\} \text{ with } \min\{|v(x)|,|v(y)|\} \le h \text{ or } u(x)v(y) < 0\},$$

and for all $k > 0$ and all $\varphi \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ we have

$$\begin{aligned} &\frac{1}{2}\iint_{\mathcal{D}_\Omega} |v(x)-v(y)|^{p-2}(v(x)-v(y)) \cdot [T_k(v(x)-\varphi(x)) - T_k(v(y)-\varphi(y))]d\nu \\ &\quad \le \int_\Omega f(x)T_k(v(x)-\varphi(x))dx. \end{aligned} \tag{2.34}$$

Remark 2.15. Remark that the right-hand side of (2.34) is well defined since $T_k(u-\varphi)$ belongs to $W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$, and by choosing $\varphi = T_{h-1}(u)$, for $h > k$, one can obtain the asymptotic behavior results

$$\begin{cases} \dfrac{1}{2}\displaystyle\int_{\{h-k-1\le u(x)<u(y)\le h\}} (u(y)-u(x))^p d\nu \le k \int_{\{|u|>h-k-1\}} d\mu, \\ \dfrac{1}{2}\displaystyle\int_{\{h-k-1\le u(y)<u(x)\le h\}} (u(x)-u(y))^p d\nu \le k \int_{\{|u|>h-k-1\}} d\mu. \end{cases} \tag{2.35}$$

Now, let us consider the variational parabolic problem

$$\begin{cases} u_t + (-\Delta)_p^s u = f(t,x) \text{ in } Q, \\ u \ge 0 \text{ in } \mathbb{R}^N, \;\; u = 0 \text{ in } (0,T) \times (\mathbb{R}^N \backslash \Omega), \\ u(0,x) = u_0 \text{ in } \Omega. \end{cases} \tag{2.36}$$

If $f \in L^{p'}(0,T;W^{-s,p'}(\Omega))$ and $u_0 \in L^2(\Omega)$, we say that u is a *weak/energy* solution of problem (2.36) if $u \in C(0,T;L^2(\Omega)) \cap L^p(0,T;W_0^{s,p}(\Omega))$ with $u_t \in L^{p'}(0,T;W^{-s,p'}(\Omega))$ such that

$$\begin{aligned} &\int_0^T \langle u_t, \varphi\rangle dt + \frac{1}{2}\int_0^T \iint_{\mathcal{D}_\Omega} U(t,x,y)(\varphi(t,x)-\varphi(t,y))d\nu dt \\ &= \langle f,\varphi\rangle_{L^{p'}(0,T;W^{-s,p'}(\Omega)),L^p(0,T;W_0^{s,p}(\Omega))} \end{aligned} \tag{2.37}$$

for all $\varphi \in L^p(0,T;W_0^{s,p}(\Omega))$ and $u(\cdot,x)$ converges to u_0 strongly in $L^2(\Omega)$ as t tends to zero (here $U(t,x,y) = |u(t,x)-u(t,y)|^{p-2}(u(t,x)-u(t,y))$). Following standard arguments, one can prove existence and uniqueness of *weak/energy* solutions by using the theory of monotone operators [69], see also [68], or the nonlinear semigroup theory [79].

Because of the intrinsic interest of entropy formulations with mesure data, this notion of solution was introduced in the parabolic setting in [23, 40, 96] when $(\mu,u_0) \in L^1(Q) \times L^1(\Omega)$, and in [110] this notion of solution is proved to be equivalent to the notion of renormalized solution. The following definition is formulated in the fractional setting and is certainly closer to existing formulations in [27, 28].

Definition 2.16. Let $\mu \in \mathcal{M}_0(Q)$ and $u_0 \in L^1(\Omega)$, we say that $u \in C([0,T];L^1(\Omega))$ is an *entropy* solution of problem (1.1) if $T_k(u) \in L^p(0,T;W_0^{s,p}(\Omega))$ for all $k > 0$, and

$$\iiint_{R_h} |u(t,x) - u(t,y)|^{p-1} d\nu dt \to 0 \text{ as } h \to \infty, \tag{2.38}$$

where

$$R_h = \{(t,x,y) \in (0,T) \times \mathbb{R}^{2N} : h+1 \le \max\{|u(t,x)|, |u(t,y)|\},\\ \text{with } \min\{|u(t,x), |u(t,y)|\} \le h \text{ or } u(t,x)u(t,y) < 0\}$$

and, for all $v \in C([0,T];L^1(\Omega)) \cap L^p(0,T;W^{s,p}(\Omega)) \cap L^\infty(Q)$ such that $u_t \in L^{p'}(0,T;W^{-s,p'}(\Omega))$ we have

$$\begin{aligned}&\int_\Omega \Theta_k(u-v)dx - \int_0^T \langle v_t, T_k(u-v)\rangle dt\\&+\frac{1}{2}\int_0^T \iint_{\mathcal{D}_\Omega} U(t,x,y)[T_k(u(t,x)-\varphi(t,x)) - T_k(u(t,y)-\varphi(t,y))]d\nu dt\\&\le \int_\Omega \Theta_k(u_0(x)) - u(0,x))dx + \int_Q fT_k(u-v)dxdt + \int_Q F\cdot T_k(u-v)dxdt,\end{aligned} \tag{2.39}$$

where $\Theta_k(\sigma) = \int_0^\sigma T_k(\rho)d\rho$.

Recall that the entropy solution u of problem (1.1), with $L^1(Q)$ data, exists and is unique as shown in [110] (this result was improved in many other papers to deal with measure data problems). In [3] the authors proved the following estimate:

Theorem 2.17. *Assume that $(\mu, u_0) \in L^1(Q) \times L^1(\Omega)$, then there exists a weak/-energy solution u of problem (1.1) such that $T_k(u) \in L^p(0,T;W_0^{s,p}(\Omega))$ for all $k > 0$. Moreover, for all $q < \frac{N(p-1)+ps}{N+s}$ and all $s_1 < s$ we have*

$$\int_0^T \iint_{\Omega\times\Omega} \frac{|u(t,x)-u(t,y)|^q}{|x-y|^{N+qs}} dydxdt \le M. \tag{2.40}$$

If $p > 2 - \frac{s}{N}$, then $u \in L^p(0,T;W_0^{s_1,q}(\Omega))$ for all $1 \le q < \frac{N(p-1)+ps}{N+s}$ and all $s_1 < s$.

Finally, observe that by using the fact that

$$-\int_0^T \langle \varphi_t, T_k(v-\varphi)\rangle_{W^{-s,p'}(\Omega),W_0^{s,p}(\Omega)} dt = \int_\Omega \Theta_k(v-\varphi)(T)dx - \int_\Omega \Theta_k(v-\varphi)(0)dx, \tag{2.41}$$

an entropy solution of problem (1.1) with initial boundary value $u_0(x) = v(x)$ turns out to be an entropy solution of problem (2.32).

2.4. Comparison principle and main result (the homogeneous case)

In order to prove our asymptotic behavior result in the homogeneous case we need a comparison principle result between parabolic and elliptic entropy solutions.

Lemma 2.18. *Let $\mu \in \mathcal{M}(\Omega)$ and $u_0, v_0 \in L^1(\Omega)$ be such that $0 \le u_0 \le v_0$, and let u and v be, respectively, the entropy solutions of problems*

$$\begin{cases} u_t + (-\Delta)_p^s u = \mu \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0 \text{ in } \Omega,\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{2.42}$$

and

$$\begin{cases} v_t + (-\Delta)_p^s v = \mu \text{ in } (0,T)\times\Omega, \\ v(0,x) = v_0 \text{ in } \Omega,\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases} \tag{2.43}$$

Then

$$u \le v \text{ a.e. in } \Omega \quad \forall t \in (0,T). \tag{2.44}$$

Proof. *Step.1: The case of dual datum.* Let u and v be entropy solutions of problems (2.42) and (2.43) with $F \in W^{-s,p'}(\Omega)$ as datum, then u and v satisfy the variational formulations (in their weak sense), i.e.,

$$\int_0^T \langle u_t, \varphi\rangle dt + \frac{1}{2}\int_0^T \iint_{\mathcal{D}_\Omega} |U(t,x,y)|^{p-2}U(t,x,y)(\varphi(t,x)-\varphi(t,y))d\nu dt$$
$$= \int_0^T \int_\Omega F\varphi dx dt, \quad \forall \varphi \in L^p(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(Q),\ \varphi \ge 0 \text{ a.e. in } Q, \tag{2.45}$$

and

$$\int_0^T \langle v_t, \varphi\rangle dt + \frac{1}{2}\int_0^T \iint_{\mathcal{D}_\Omega} |V(t,x,y)|^{p-2}V(t,x,y)(\varphi(t,x)-\varphi(t,y))d\nu dt$$
$$= \int_0^T \int_\Omega F\varphi dx dt, \quad \forall \varphi \in L^p(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(Q),\ \varphi \ge 0 \text{ a.e. in } Q. \tag{2.46}$$

Hence, by using test functions $\varphi = (u-v)^+$ that belong to $L^p(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(Q)$ and subtracting the first equation from the second one, we easily obtain

$$
\begin{aligned}
0 =& \int_0^T \langle (u-v)_t, (u-v)^+ \rangle dt \\
&+ \frac{1}{2} \int_0^T \int_{\mathcal{D}_\Omega} (|U(t,x,y|^{p-2}U(t,x,y) - |V(t,x,y|^{p-2}V(t,x,y)) \\
&\qquad\qquad ((u-v)^+(t,x) - (u-v)^+(t,y)) dxdt \\
=& \frac{1}{2} \int_0^T \int_\Omega \frac{d}{dt} [(u-v)^+]^2 dxdt - \frac{1}{2} \int_\Omega [(u-v)^+]^2(t) - \frac{1}{2} \int_\Omega [(u-v)^+]^2(0) dx \\
&+ \frac{1}{2} \int_0^T \int_{\mathcal{D}_\Omega} (|U(t,x,y)|^{p-2}U(t,x,y) - |V(t,x,y)|^{p-2}V(t,x,y)) \\
&\qquad\qquad ((u-v)^+(t,x) - (u-v)^+(t,y)) dxdt.
\end{aligned}
\tag{2.47}
$$

Since the second term is zero and the last term is nonnegative, we easily obtain $(u-v)^+ = 0$ a.e. in Ω which implies that $u \le v$ a.e. in Ω for all $t \in [0,T]$.

Step.2: The case of positive measure data. Let us come back to the case where $\mu \in \mathcal{M}_0^+(\Omega)$ be such that $\mu = f - \operatorname{div}(G)$ for some $f \in L^1(\Omega)$ and $G \in L^{p'}(\Omega)^N$ and $u_0, v_0 \in L^1(\Omega)$. There are many ways to approximate these data, we will make the following choice: let $f_n \in C_0^\infty(\Omega)$ be a sequence of nonnegative functions which converges to f weakly in $L^1(\Omega)$ and $G_n \in C_0^\infty(\Omega)$ be a sequence of nonnegative functions which converges to G strongly in $L^{p'}(\Omega)^N$; moreover, let u_0^n (respectively v_0^n) be a sequence of nonnegative functions satisfying $u_0^n \le v_0^n$ and which converges to u_0 (respectively v_0) in $L^1(\Omega)$ (notice that these approximations can be easily obtained via a standard convolution argument); we also assume that $\mu_n = f_n - \operatorname{div}(G_n)$ be such that $\|\mu_n\|_{L^1(Q)} \le C\|\mu\|_{\mathcal{M}(\Omega)}$, $\|u_0^n\| \le C\|u_0\|_{L^1(\Omega)}$ and $\|v_0^n\| \le \|v_0\|_{L^1(\Omega)}$. Let us define u_n and v_n as the solutions of of problems (2.42) and (2.43) with data μ_n that exist as proved above and satisfies $u_n \le v_n$ a.e. in Ω for all $t \in [0,T]$; and let u and v be, respectively, the limits of u_n and v_n. Applying the results of [91] and recalling that u and v, limits of u_n and v_n, are entropy solutions of (2.42) and (2.43) with, respectively, μ and u_0, v_0 as data, we obtain that $u \le v$ a.e. in Ω for all $t \in [0,T]$, which concludes the proof of the comparison result of Lemma 2.18. □

Now, our aim is to prove some a priori estimates satisfied by approximate entropy solutions.

Lemma 2.19. *Let $C > 0$ and $(u_n) \subset \mathcal{T}_0^{s,p}(Q)$ be such that*

$$
\int_0^T \iint_{\mathcal{D}_\Omega} |T_k(u_n(t,x)) - T_k(u_n(t,x))|^p dxdt \le Ck. \tag{2.48}
$$

Then, if $p < N$, u_n is bounded in the Marcinkiewicz space $\mathcal{M}^{p-1+\frac{ps}{N}}(Q)$ and $|\nabla u_n|$ is bounded in the Marcinkiewicz space $\mathcal{M}^{p-\frac{N}{N+s}}$. If $p = N$, u_n is bounded in the Marcinkiewicz space $\mathcal{M}^q(Q)$ for every $q < \infty$ and $|\nabla u_n|$ is bounded in the

Marcinkiewicz space $\mathcal{M}^r(Q)$ *for every* $r < N$. *Moreover, there exists a measurable function* $u \in \mathcal{T}_0^{s,p}(Q)$ *and a subsequence, not relabeled, such that*

$$\begin{cases} u_n \to u \text{ a.e. in } Q, \\ T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } L^p(0,T;W_0^{s,p}(\Omega)) \text{ and a.e. in } Q \text{ for every } k>0, \\ \nabla u_n \to \nabla u \text{ a.e. in } Q. \end{cases} \tag{2.49}$$

Proof. First, we use a *Sobolev* type inequality to get

$$\begin{aligned}\int_0^T \left(\int_\Omega T_k(u_n)(t,x))^{p_s^*}dx\right)^{\frac{p}{p_s^*}} dt &\le \int_0^T \iint_{\mathcal{D}_\Omega} |T_k(u_n(t,x))| - T_k(u_n(t,y))|^p \, d\nu dt \\ &\le Mk,\end{aligned}$$

and so, for $1 < r = r_1 + r_2 < p_s^*$ with $r_1 = (\frac{p_s^*}{p_s^*-1})(r-1)$ and $r_2 = 1 - \frac{r_1}{p_s^*}$, we write

$$\begin{aligned} k^r \text{meas}\{|u_n| \ge k\} &\le \int_0^T \int_\Omega |T_k(u_n(t,x))|^r dxdt \\ &\le \int_0^T \int_\Omega |T_k(u_n(t,x))|^{r_1} |u_n(t,x)|^{r_2} dxdt \\ &\le \int_0^T \left(\int_\Omega |T_k(u_n(t,x))|^{p_s^*}dx\right)^{\frac{r_1}{p_s^*}} \left(\int_\Omega |u_n(t,x)|^{r_2 \frac{p_s^*-r_1}{p_s^*}} dx\right)^{1-\frac{r_1}{p_s^*}} dt \\ &\le \left(\sup_{t\in[0,T]} \int_\Omega |u_n(t,x)|dx\right) \int_0^T \left(\int_{\mathbb{R}^N} |T_k(u_n(t,x))|^{p_s^*}dx\right)^{\frac{r_1}{p_s^*}} dt \\ &\le C \int_0^T \left(\int_{\mathbb{R}^N} |T_k(u_n(t,x))|^{p_s^*}dx\right)^{\frac{r_1}{p_s^*}} dt \\ &\le CMk^{\frac{r_1}{p}} \\ &\le Ck^{\frac{r_1}{p}}, \end{aligned}$$

then

$$\text{meas}\{|u_n \ge k|\} \le \frac{C}{k^{r-\frac{r_1}{p}}} \le \frac{C}{k^{1+r_1[\frac{p_s^*(p-1)+p}{pp_s^*}]}}.$$

Letting $r_1 \to p$, we get

$$\text{meas}\{|u_n| \ge k\} \le \frac{C}{k^{\frac{p(p_s^*-1)}{p_s^*}}}.$$

Thus,

$$\text{meas}\{|u_n| \ge k\} \le CM^{\frac{p_s^*}{p}} k^{-(p-1+\frac{ps}{N})}.$$

Therefore, the sequence u_n is uniformly bounded in the *Marcinkiewicz* space $\mathcal{M}^{p-1+\frac{ps}{N}}$ that implies, since in particular $p > \frac{2N+s}{N+s}$, that u_n is uniformly bounded in the *Lebesgue* space $L^m(Q)$, for all $1 \le m < p-1+\frac{ps}{N}$.

Now, we prove that the sequence (∇u_n) is bounded in suitable fractional space; first of all, observe that

$$\text{meas}\{|\nabla u| \geq \lambda\} \leq \text{meas}\{|\nabla u| \geq k;\ |u| \leq k\} + \text{meas}\{|\nabla u| \geq \lambda;\ |u| > k\}.$$

It is clear that

$$\begin{aligned}\text{meas}\{|\nabla u| \geq \lambda; |u| \leq k\} &\leq \frac{1}{\lambda^p}\int_{\{|\nabla u|\geq\lambda;|u|\leq k\}} |\nabla u|^p dxdt \\ &= \frac{1}{\lambda^p}\int_{\{|u|\leq k\}} |\nabla u|^p dxdt = \frac{1}{\lambda^p}\int_Q |\nabla T_k(u)|^p dxdt \\ &\leq \frac{Ck}{\lambda^p};\end{aligned}$$

and

$$\text{meas}\{|\nabla u| \geq \lambda; |u| > k\} \leq \text{meas}\{|u| \geq k\} \leq \frac{\overline{C}}{k^\sigma},$$

with $\sigma = p - 1 + \frac{ps}{N}$, it holds that

$$\text{meas}\{|\nabla u_n| \geq \lambda\} \leq \frac{\overline{C}}{k^\sigma} + \frac{Ck}{\lambda^p}.$$

Thus, taking the minimum over k i.e. with the value $k = k_0 = (\frac{\sigma C}{\overline{C}})^{\frac{1}{\sigma+1}}\lambda^{\frac{p}{\sigma+1}}$, we reach that

$$\text{meas}\{|\nabla u_n| \geq \lambda\} \leq Ck^{-\gamma}$$

with $\gamma = p(\frac{\sigma}{\sigma+1}) = p - \frac{N}{N+s}$. Thus, $|\nabla u^n|$ is equibounded in the *Marcinkiewicz* space $\mathcal{M}^\gamma(Q)$ with $\gamma = p - \frac{N}{N+s}$, and, since $p > \frac{2N+s}{N+s}$, we conclude that $|\nabla u^n|$ is uniformly bounded in the *Lebesgue* space $L^\beta(Q)$ with $1 \leq \beta < p - \frac{N}{N+s}$.

Now, according to the above results, there exists $\overline{u} \in L^q(0,T;W_0^{s,q}(\Omega))$ for all $q < p - 1 + \frac{ps}{N}$ such that u_n converges to $\overline{u}$ weakly in $L^q(0,T;W_0^{s,q}(\Omega))$. Observe that, obviously we have $\overline{u} = u$ a.e. in Q and $(u_n)_t \in L^1(Q) + L^{\beta'}(0,T;W^{-s,\beta'}(\Omega))$ uniformly with respect to n where $\beta' = \frac{q}{p-1}$ for all $q < p - 1 + \frac{ps}{N}$, which imply by *Aubin-Simon* type result that u_n converges to $\overline{u}$ in $L^1(Q)$, being $T_k(s)$ is bounded and (2.48) holds, we finally obtain that $T_k(u_n)$ converges to $T_k(\overline{u})$ weakly in $L^p(0,T;W_0^{s,p}(\Omega))$ and $T_k(u_n)$ converges to $T_k(\overline{u})$ strongly in $L^p(Q)$. Now, we can follow closely [26, Theorem 3.3] to conclude that ∇u_n converges to $\nabla\overline{u}$ a.e. in Ω. □

The first asymptotic behavior result in the homogeneous case for entropy solutions is the following.

Theorem 2.20. *Assume that* $p > \frac{2N+s}{N+s}$, $\mu \in \mathcal{M}_0^+(\Omega)$ *be independent on time and* $u_0 = 0$. *If* $u(t,x)$ *is the entropy solution of problem* (1.1) *and* $v(x)$ *is the entropy solution of problem* (2.32), *then*

$$\lim_{t\to\infty} u(t,x) = v(x) \text{ in } L^1(\Omega). \tag{2.50}$$

Proof. We first introduce the approximate problems by defining $u^n(t,x)$ as the entropy solution of

$$\begin{cases} u_t^n + (-\Delta)_p^s u^n = \mu \text{ in } (0,1)\times\Omega, \\ u^n(0,x) = u(n,x) \text{ in } \Omega,\ u^n(t,x) = 0 \text{ on } (0,1)\times\partial\Omega. \end{cases} \tag{2.51}$$

Notice that if $p > 2 - \frac{s}{N}$ then $u \in L^p(0,T;W_0^{s,q}(\Omega))$ for all $1 \le q < \frac{N(p-1)+ps}{N+s} = p - \frac{N}{N+s}$ and all $s_1 < s$ (observe that $p - \frac{N}{N+s} > 1$ if and only if $p > 2 - \frac{s}{N}$ holds), and, in this case, ∇u^n belongs to $L^1(Q)$ and since $u \in C(0,T;L^1(\Omega))$ then $u(n,x)$ belongs to $L^1(\Omega)$. Now, by classical existence theorem and comparison principle result, there exists a unique nonnegative entropy solution $u(t,x)$ of problem (1.1) with $u(0,x) = 0$ as initial data (observe that for $n \ge 1$, if $u(0,x) = 0$ then $u(n,x) = u(1,x)$ a.e. in Ω) and there exists a unique entropy solution of problem (2.32) (which is also an entropy solution of problem (1.1) with $v \ge 0$ as initial data) satisfying

$$\begin{cases} u(t,x) \le v(x) \text{ a.e. in } \Omega, \quad \forall t \in [0,T], \\ u^n(t,x) \le v(x) \text{ a.e. in } \Omega, \quad \forall t \in [0,1], \end{cases} \tag{2.52}$$

and similarly, being $u(t+s,x)$ solution of the same problem (1.1) with $u(s,x)$ as initial data, we conclude that

$$u(t,x) \le u(t+s,x) \text{ a.e. in } \Omega, \quad \forall t,s \ge 0. \tag{2.53}$$

Now, let m and n be two integers ($n < m$), then by using the same reasoning as above we get $u(n,x) \le u(m,x)$, i.e., u is a monotonic nondecreasing function with respect to the time variable; therefore, we conclude that, for all $n \ge 0$ and all $t > 0$, $u^n(t,x) \le u^{n+1}(t,x)$ a.e. in Ω, and then from the monotonicity of u_n there exist a function $\widetilde{u}$ such that $u^n(t,x)$ converges to $\widetilde{u}(t,x)$ a.e. in Q as $n \to \infty$. Now, from the entropy formulation we have

$$\begin{aligned}
&\int_\Omega \Theta_k(u^n - \varphi)(1)dx && (\mathcal{A})\\
&\quad - \int_\Omega \Theta_k(u^n(0,x) - \varphi(0))dx && (\mathcal{B})\\
&\quad + \int_0^T \langle \varphi_t, T_k(u^n - \varphi)\rangle dt && (\mathcal{C})\\
&\quad + \frac{1}{2}\int_0^T \int_{\mathcal{D}_\Omega} |U^n(t,x,y)|^{p-2}U(t,x,y) \\
&\qquad\qquad \cdot [T_k(u^n(t,x) - \varphi(t,x)) - T_k(u^n(t,y) - \varphi(t,y))]d\nu dt && (\mathcal{D})\\
&\le \int_0^T \int_\Omega T_k(u^n - \varphi)d\mu, && (\mathcal{E})
\end{aligned}$$

for all $k > 0$ and all $\varphi \in C^0(0,T;L^1(\Omega)) \cap L^p(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(Q)$ with $\varphi_t \in L^{p'}(0,T;W^{-s,p'}(\Omega))$. Let us analyze term by term the limits of this inequality using in particular the convergence results of Lemma 2.19; due to the fact that $\Theta_k(u^n-\varphi)$ converges to $\Theta_k(\overline{u}-\varphi)$ weakly in $L^p(0,T;W_0^{s,p}(\Omega))$, we obtain, observing that $\Theta_k(\overline{u}-\varphi)\in L^p(0,T;W_0^{s,p}(\Omega))\cap L^\infty(Q)$, that

$$(\mathcal{A})+(\mathcal{B}))=\int_\Omega \Theta_k(\overline{u}-\varphi)(1)dx-\int_\Omega \Theta_k(\overline{u}-\varphi)(0)dx+\omega(n).$$

Since $T_k(u^n-\varphi)$ converges to $T_k(\overline{u}-\varphi)$ *-weakly in $L^\infty(Q)$ and weakly in $L^p(0,T;W_0^{s,p}(\Omega))$, we get

$$(\mathcal{C})=\int_0^1\langle\varphi_t,T_k(\overline{u})-\varphi)\rangle dt+\omega(n),$$

and

$$(\mathcal{E})=\int_Q T_k(\overline{u}-\varphi)d\mu.$$

Moreover, applying *Fatou*'s lemma and the a.e. convergence of the gradients, we obtain

$$\begin{aligned}(\mathcal{D})&=\int_0^T\langle(-\Delta)_p^s u_n-(-\Delta)_p^s\varphi,T_k(u^n-\varphi)\rangle dt+\int_0^T\langle(-\Delta)_p^s\varphi,T_k(u^n-\varphi)\rangle dt\\&\leq\liminf_{n\to\infty}\int_0^T\langle(-\Delta)_p^s u_n-(-\Delta)_p^s\varphi,T_k(u^n-\varphi)\rangle dt\\&=\int_0^t\langle(-\Delta)_p^s\varphi,T_k(\overline{u}-\varphi)\rangle dt+\omega(n).\end{aligned}$$

Now, observe that $\overline{u}$ does not depend on time; in fact

$$u^n(0,x)\leq\widetilde{u}=u(t+n,x)\leq u(n+1,x)=u^{n+1}(0,x),$$

so, being the limit of $u^n(0,x)$ and $u^{n+1}(0,x)$ the same (denoted by w) as n diverges, we deduce that $\overline{u}(x)=w(x)$ a.e. in Ω. To conclude let us prove that $\overline{u}(x)$ solves the elliptic and parabolic problems; to this aim it suffices to check that $(\mathcal{A})+(\mathcal{B})+(\mathcal{C})$ converges to zero as n tends to infinity; in fact

$$\begin{aligned}\lim_{n\to\infty}[(\mathcal{A})+(\mathcal{B})+(\mathcal{C})]&=\int_\Omega\int_0^1[\Theta_k(w(x)-\varphi)]_t dxdt+\int_0^1\langle\varphi_t,T_k(u-\varphi)\rangle dt\\&=\int_0^1\langle(w(x)-\varphi)_t,T_k(w(x)-\varphi)\rangle dt+\int_0^1\langle\varphi_t,T_k(w-\varphi)\rangle dt\\&=\int_0^1\langle w_t,T_k(w-\varphi)\rangle dt\\&=0\quad(w\text{ is independent on time}),\end{aligned}$$

and then

$$\overline{u}(x)=w(x)=v(x)$$

where $v(x)$ is the unique entropy solution of the elliptic problem (2.32). □

3. Main result and proof (the nonhomogeneous case)

In the nonhomogeneous case we need to introduce the notion of entropy sub- and super-solutions, to prove the comparison principle result and then to deal with the proof of asymptotic theorem. From now we will denote $u(t,x)$ the entropy solution of problem

$$\begin{cases} u_t + (-\Delta)_p^s u = \mu \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega,\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.1}$$

where $u_0 \in L^1(\Omega)$ is a nonnegative function, $\mu \in \mathcal{M}_0(Q)$ and $v(x)$ is the entropy solution of the corresponding elliptic problem (2.32). Let us introduce the following definition.

Definition 3.1. We say that $\overline{u}(t,x) \in C(0,T;L^1(\Omega))$ is an *entropy sub-solution* of problem (3.1) if $T_k(\overline{u}) \in L^p(0,T;W_0^{s,p}(\Omega))$ for all $k>0$, and

$$\begin{cases} \overline{u}_t(t,x) + (-\Delta)_p^s \overline{u}(t,x) \le \mu \text{ in } (0,T)\times\Omega, \\ \overline{u}(0,x) = \overline{u}_0(x) \le u_0(x) \text{ in } \Omega,\ \overline{u}(t,x) \le 0 \text{ on } (0,T)\times\Omega. \end{cases} \tag{3.2}$$

On the other hand $\underline{u}(t,x) \in C(0,T;L^1(\Omega))$ is an *entropy super-solution* of problem (3.1) if $T_k(\underline{u}) \in L^p(0,T;W_0^{s,p}(\Omega))$ for all $k>0$, and

$$\begin{cases} \underline{u}_t(t,x) + (-\Delta)_p^s \underline{u}(t,x) \ge \mu \text{ in } (0,T)\times\Omega, \\ \underline{u}(0,x) = \underline{u}_0(x) \ge u_0(x) \text{ in } \Omega,\ \overline{u}(t,x) \ge 0 \text{ on } (0,T)\times\Omega, \end{cases} \tag{3.3}$$

where both (3.2) and (3.3) are understood in their entropy sense, i.e., $\overline{u}(t,x)$ satisfies

$$\begin{aligned}
&\int_\Omega \Theta_k(\overline{u}-\varphi)^-(T)dx - \int_\Omega \Theta_k(\overline{u}(0,x)-\varphi(0))^- dx \\
&+ \int_0^T \langle \varphi_t, T_k(\overline{u}-\varphi)^- \rangle dt \\
&+ \frac{1}{2}\int_0^T \int_{\mathcal{D}_\Omega} |\overline{U}(t,x,y)|^{p-2}\overline{U}(t,x,y) \\
&\qquad \cdot [T_k(\overline{u}(t,x,y)-\varphi(t,x)) - T_k(\overline{u}(t,y)-\varphi(t,y))]d\nu dx \\
&\le \int_0^T \int_\Omega T_k(\overline{u}-\varphi)d\mu,
\end{aligned}$$

for all $\varphi \in C(0,T;L^1(\Omega)) \cap L^p(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(Q)$, $\varphi \ge 0$ a.e. in Q such that

$\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$, and $\underline{u}(t,x)$ satisfies

$$\begin{aligned}
&\int_\Omega \Theta_k(\underline{u}-\varphi)^+(T)dx - \int_\Omega \Theta_k(\underline{u}(0,x)-\varphi(0))^+dx \\
&\quad + \int_0^T \langle \varphi_t, T_k(\underline{u}-\varphi)^+\rangle dt \\
&\quad + \frac{1}{2}\int_0^T \int_{\mathcal{D}_\Omega} |\underline{U}(t,x,y)|^{p-2}\underline{U}(t,x,y) \\
&\qquad\qquad \cdot [T_k(\underline{u}(t,x,y)-\varphi(t,x)) - T_k(\underline{u}(t,y)-\varphi(t,y))]d\nu dx \\
&\geq \int_0^T \int_\Omega T_k(\underline{u}-\varphi)d\mu,
\end{aligned}$$

for all $\varphi \in C(0,T;L^1(\Omega)) \cap L^p(0,T;W_0^{s,p}(\Omega)) \cap L^\infty(Q)$, $\varphi \geq 0$ a.e. in Q, such $\varphi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega))$.

Now, we are able to state and prove the comparison principle lemma that will play the key role in the proof of our main result (these calculations are inspired from [110, Theorem 1.3]). Observe that this comparison result between sub- and super-solutions easily imply the uniqueness of solution for the corresponding problem (by observing that any solution turns out to be both a sub- and a super-solution of the same problem).

Lemma 3.2. *Let $\mu \in \mathcal{M}_0^+(Q)$ and let $\underline{u}, \overline{u}$ be, respectively, the entropy sub- and super-solution of problem (3.1), then*

$$\underline{u} \leq u \leq \overline{u}, \tag{3.4}$$

where u is the unique entropy solution of the same problem.

Proof. First, suppose that $\underline{u}_0, \overline{u}_0 \in L^1(\Omega)$ and $\mu \in \mathcal{M}_0(\Omega)$ be such that $\mu = f - \operatorname{div}(G)$ with $f \in L^1(\Omega)$ and $G \in L^{p'}(\Omega)^N$, then, by a standard approximation argument, we find a weak/energy solution of problem

$$\begin{cases} (u_n)_t + (-\Delta)_p^s u_n = \mu_n \text{ in } (0,T)\times\Omega, \\ u_n(0,x) = \widetilde{u}_{0,n} \text{ in } \Omega,\ u_n(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.5}$$

where $\widetilde{u}_{0,n} = \min\{u_{0,n}, \underline{u}_0\}$ converges to $\underline{u}_0$ in $L^1(\Omega)$ and $\mu_n = f_n - \operatorname{div}(G_n)$ converges to μ. Now, we choose $T_k(\underline{u}-u_0)^+$ as test function and we subtract the resulting inequalities satisfied by $\underline{u}$ and u_n (recalling that $\underline{u}$ is a sub-solution of problem (3.5) and $(\underline{u}-u_n)^+(0)$) to get

$$\begin{aligned}
&\int_\Omega \Theta_k(\underline{u}-u_n)^+(T)dx - \int_\Omega \Theta_k(\underline{u}_0 - \widetilde{u}_{n,0})^+dx \\
&+ \int_0^T \langle (-\Delta)_p^s \underline{u} - (-\Delta)_p^s u_n, T_k(\underline{u}-u_n)\rangle dt \\
&\leq \int_Q T_k(\underline{u}-u_n)^+ d\mu - \int_Q T_k(\underline{u}-u_n)^+ d\mu_n \leq 0.
\end{aligned}$$

Recalling that $\underline{u}_0 \leq \widetilde{u}_{n,0}$, we conclude by *Fatou*'s lemma that

$$(\underline{u} - \widetilde{u})^+ = 0 \text{ a.e. in } Q, \text{ i.e., } \underline{u} \leq \widetilde{u} \text{ a.e. in } Q$$

where $\widetilde{u}$ is an entropy solution of problem (3.1) with $\widetilde{u}(0) = \underline{u}_0$. Thus, by Lemma 2.18, we obtain that $\widetilde{u} \leq u$, i.e., $\underline{u}(t,x) \leq u(t,x)$ a.e. in Ω for all $t > 0$. Similarly, we consider $T_k(\overline{u} - u_n)^-$ as test function and we use the fact that $\overline{u}$ is a super-solution to obtain $u(t,x) \leq \overline{u}(t,x)$ a.e. in Ω for all $t > 0$, which completes the proof of Lemma 3.2. □

The second asymptotic behavior result in the nonhomogebeous case for entropy solutions is the following.

Theorem 3.3. *Assume that* $p > \frac{2N+s}{N+s}$, $\mu \in \mathcal{M}_0(\Omega)$ *be independent on time and* $u_0 \in L^1(\Omega)$. *If* $u(t,x)$ *is the entropy solution of problem* (3.1) *and* $v(x)$ *is the entropy solution of problem* (2.32), *then*

$$\lim_{t\to\infty} u(t,x) = v(x) \text{ in } L^1(\Omega). \tag{3.6}$$

Proof. Our aim is to extend the ideas of Theorem 2.20 and to avoid the nonhomogeneuous technicalities, the asymptotic behavior result of entropy solutions is obtained by following few steps:

Step.1: The case $0 \leq u_0 \leq v$. Let v be the entropy solution of stationary problem (2.32) which is also an entropy solution of problem (3.1) with $u_0 = v$ as initial datum, and let $\widetilde{u}(t,x)$ be the entropy solution of parabolic problem (3.1) with $u_0 = 0$ as initial datum which converges, by Theorem 2.20, to v in $L^1(\Omega)$ as t tends to infinity. Observe that v is a super-solution of problem (3.1) with u_0 as initial datum, so by comparison Lemma 3.2 and for any $t \in [0,T]$ we have $u(t,x) \leq v(x)$ a.e. in Ω where $u(t,x)$ is an entropy solution of problem (3.1) with initial datum $u(0,x) = v_0(x) \leq v(x)$ a.e. in Ω. Moreover, by the comparison Lemma 2.18 we get $\widetilde{u}(t,x) \leq u(t,x)$ a.e. in Ω for any $t \in [0,T]$, and again by the same comparison lemma we deduce that $u(t,x)$ converges to v in $L^1(\Omega)$ as t tends to infinity.

Step.2: The case $0 \leq u_0 \leq v^\tau$. Let $\widehat{u}$ be the entropy solution of the following problem

$$\begin{cases} u_t + (-\Delta)_p^s u = \mu_\tau \text{ in } (0,T) \times \Omega, \\ u(0,x) = v^\tau \text{ in } \Omega, \;\; u(t,x) = 0 \text{ in } (0,T) \times \partial\Omega, \end{cases} \tag{3.7}$$

where[2] $\mu_\tau \geq \mu$ for some $\tau > 1$ and v_τ be the solution of (2.32) with μ_τ as datum. Observe that v^τ is a super-solution of (3.1) with $u_0(0,x) = v^\tau(x)$ as initial datum and v is a sub-solution of (3.1) with $v(0,x) = v(x) \leq v^\tau(x)$ as initial datum, see [84]. So, by the comparison Lemma 3.2, we easily get

$$v(x) \leq \widehat{u}(t,x) \leq v^\tau(x) \text{ a.e. in } \Omega, \quad \forall t \in [0,T]. \tag{3.8}$$

[2]As an example of μ_τ one can take $\mu_\tau = \tau f - \operatorname{div} G$ if $f \neq 0$ and $\mu_\tau = \tau\mu$ if $f = 0$.

Again, using the comparison principle result between $\widehat{u}(t+s,x)$ (with $s>0$), solution of problem (3.7) with $u_0=(s,x)$ as initial datum, and $\widehat{u}(t,x)$ solution of problem (3.7) with $u_0=v^\tau$ as initial datum, we obtain

$$\widehat{u}(t+s,x)\le \widehat{u}(t,x)\le v(x) \text{ a.e. in } \Omega. \tag{3.9}$$

Now, thanks to Lemma 3.2 we know that $\widetilde{u}$, u and $\widehat{u}$, the entropy solutions of problem (3.1) with respectively 0, u_0 and v^τ as initial data satisfy the following inequality (according to Theorem 2.20 and the monotonicity result (3.9))

$$\widetilde{u}(t,x)\le u(t,x)\le \widehat{u}(t,x), \tag{3.10}$$

and since both entropy solutions $\widetilde{u}(t,x)$ and $\widehat{u}(t,x)$ converge, as t tends to infinity, to v in $L^1(\Omega)$ we finally conclude that $u(t,x)$ converges to v in $L^1(\Omega)$ as t tends to infinity.

Step.3: The case $\mu\neq 0$ and $u_0\in L^1_+(\Omega)$. Let $u(t,x)$ be the entropy solution of problem (1.1) with $u_0\in L^1(\Omega)$, and u_τ be the entropy solution of the following problem

$$\begin{cases} u_t+(-\Delta)^s_p u=\mu \text{ in } (0,T)\times\Omega,\\ u(0,x)=u_{0,\tau}(x)=\min(u_0,v^\tau),\;\; u(t,x)=0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.11}$$

for every $\tau>1$, then we have (see [96])

$$\begin{aligned} &\|u(t,x)-v(x)\|_{L^1(\Omega)}\le \|u(t,x)-u_\tau(t,x)\|_{L^1(\Omega)}+\|u_\tau(t,x)-v(x)\|_{L^1(\Omega)}\\ &\le \|u_0(x)-u_{0,\tau}(x)\|_{L^1(\Omega)}+\|u_\tau(t,x)-v(x)\|_{L^1(\Omega)}, \end{aligned}$$

by using *Lebesgue*'s theorem and the result of [88, Lemma 3.4], we easily obtain that $u_{0,\tau}$ converges to u_0 in $L^1(\Omega)$ and since $u_\tau(t,x)$ converges to v a.e. in Ω as t tends to infinity, we conclude that $u(t,x)$ converges to $v(x)$ in $L^1(\Omega)$ a.e. in Ω as t tends to infinity.

Step.4: The case $\mu=0$ and $u_0\in L^1(\Omega)$. Let $u(t,x)$ be the entropy solution of problem (1.1) with initial datum $u_0\in L^1(\Omega)$ and u_ϵ be the entropy solution of the following problem

$$\begin{cases} (u_\epsilon)_t+(-\Delta)^s_p u_\epsilon=\epsilon \text{ in } (0,T)\times\Omega,\\ u_\epsilon(0,x)=u_0 \text{ in } \Omega,\;\; u_\epsilon(t,x)=0 \text{ on } (0,T)\times\Omega, \end{cases} \tag{3.12}$$

for every $\epsilon>0$; we know, by virtue of the previous result, that $u_\epsilon(t,x)$ converges, as t tends to infinity, to $v_\epsilon(x)$ the entropy solution of the associated elliptic problem, then by virtue of Lemma 3.2, we know that $u(t,x)\le u^\epsilon(t,x)$ a.e. in Ω which implies, since v_ϵ is strongly compact in $L^1(\Omega)$ and v_ϵ converges to zero as ϵ tends to zero, that

$$0\le \lim_{\epsilon\to 0}\limsup_{t\to\infty} u(t,x)\le \lim_{\epsilon\to 0} u_\epsilon(t,x)\le \lim_{\epsilon\to 0} v_\epsilon(x)=0,$$

which concludes the proof of Theorem 3.3. □

Remark 3.4. *(i)* In order to to avoid some technicalities, we limit ourselves to the case of nonnegative measure data (the sign assumption on the data is rather technical since it allows us to work with the trivial sub-solution $u \equiv 0$). Indeed, the asymptotic behavior results of entropy solutions obtained in Theorem 2.20 and Theorem 3.3 can be extended to nonpositive data or general sign data. With slight modifications of the proofs by splitting both μ and u_0 in their positive and negative parts and using suitable sub- and super-solutions the convergence in norm to the stationary solution can be improved, see [89] for more details.

(ii) Motivated by the results mentioned above, an interesting question would be whether or not similar results can be achieved for fractional parabolic problem

$$\begin{cases} u_t + (-\Delta)_p^s u + g(u)|\nabla u|^p = \mu \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) \text{ in } \Omega,\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.13}$$

where $u_0 \in L^1(\Omega)$ is nonnegative, $g\colon \mathbb{R} \mapsto \mathbb{R}$ is a real function in $C^1(\mathbb{R})$ satisfying

$$g(s)s \geq 0,\ g'(s) > 0 \quad \forall s \in \mathbb{R},$$

while $\mu \in \mathcal{M}(Q)$ is a nonnegative measure data. This kind of problems has been largely studied in different contexts, see [90, 85, 2]; in particular, for $g = 1$ or for any power-like nonlinearity with respect to $|\nabla u|$ (the so-called *Viscous Hamilton-Jacobi* equation) like

$$\begin{cases} u_t + (-\Delta)_p^s u + |\nabla u|^q = \mu \text{ in } (0,T)\times\Omega, \\ u(0,x) = u_0 \text{ in } \Omega,\ u(t,x) = 0 \text{ on } (0,T)\times\partial\Omega, \end{cases} \tag{3.14}$$

with $q > 1$, $u_0 \in L^1(\Omega) \cap W^{1,\infty}(\Omega)$ is a nonnegative initial datum. This problem is quite different from the first one due to the fact that the asymptotic behavior of solutions depends on the power q, see [14] for more details, and, in presence of the nonlinear term g in (3.14), the absorption term $g(u)|\nabla u|^q$ becomes dominant yielding a concentration phenomenon (notice that the comparison principle result of such a type of problems is a hard task to achieve and can not improved directly due to technical difficulties except if we consider a regularizing zero order term).

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