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Perturbation theory of evolution inclusions on real Hilbert spaces with quasi-variational structures for inner products

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Abstract. *We consider an abstract Cauchy problem of an evolution inclusion with a single-valued perturbation on a real Hilbert space. The evolution inclusion contains subdifferentials of time-dependent, proper, lower semicontinuous, convex functions which depends on a solution itself of the Cauchy problem. Moreover, the subdifferentials are taken with respect to inner products, which also depend on a solution of the Cauchy problem. Such structures are sometimes called quasi-variational structures for convex functions and inner products. The main purposes of this paper are to show the existence of strong solutions to the Cauchy problem of an evolution inclusion with a perturbation and to apply this result to a mass-conservative tumor invasion model with a degenerate cross diffusion.*

Contents

1. Introduction

Throughout this paper, a time $T > 0$ is given and fixed. The aim of this paper is to find a pair (u, v) satisfying that the first component u is a solution of a Cauchy

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problem (P) below:

$$
(\mathrm{P})\begin{cases} u'(t)+\partial_{v(t)}\phi(t,u,v(t)\,;u(t))+g(t,u(t),v(t))\ni f(t) \\ \qquad\qquad\qquad \text{in}\quad H(v(t)),\quad \text{a.a. } t\in(0,T), \\ v(t)=S(u\,;t,0)v_0 \quad \text{in}\quad A,\quad \forall\, t\in[0,T], \\ u(0)=u_0 \quad \text{in}\quad H, \end{cases}
$$

where H is a real Hilbert space with an inner product $(\cdot,\cdot)_H$ and A is a nonempty, closed subset of a real Banach space X with a norm $\|\cdot\|_X$. The evolution inclusion in (P) has three mathematically interesting characteristics. The first one is that the proper, lower semicontinuous (l.s.c., in short), convex function $\phi(t,u,v)$ on H depends on $t\in[0,T]$, $u\in C([0,T];H)$ and $v\in A$. The second one is that the real Hilbert space $H(v)=H$ also depends on v for all $v\in A$. Actually, for any $v\in A$ the inner product of H is given by $(\cdot,\cdot)_v$ instead of $(\cdot,\cdot)_H$ in our setting. The third one is that the subdifferential $\partial_v\phi(t,u,v)$ of $\phi(t,u,v)$ on H is taken with respect to the inner product $(\cdot,\cdot)_v$, that is,

$$
\xi\in\partial_v\phi(t,u,v\,;z)\Longleftrightarrow\begin{cases} z\in D(\phi(t,u,v)) \quad \text{and} \\ (\xi,y-z)_v\le\phi(t,u,v\,;y)-\phi(t,u,v\,;z), \quad \forall y\in H. \end{cases}
$$

These structures are called quasi-variational structures for the evolution inclusion in (P) and make it difficult and complicated to analyze (P) mathematically.

In [7], the evolution inclusion without a perturbation, i.e., the case $g\equiv 0$ on H, is considered and the existence of strong solutions of (P) on $[0,T]$ is shown. Moreover, the phase field model of Fix–Caginalp type with a quasi-variational boundary condition is treated as one of the typical examples of (P) without a perturbation. We entrust various results of evolution inclusions on H associated with subdifferentials of time-dependent proper l.s.c. convex functions to [1, 3, 10, 11, 13, 14] and their references. Especially, in [13, 14] the following evolution inclusion on H with a perturbation

$$
u'(t)+\partial_H\phi(t\,;u(t))+g(t,u(t))\ni f(t) \quad \text{in}\quad H,\quad \text{a.a. } t\in(0,T),
$$

is considered although in their settings there are not any quasi-variational structures.

On the other hand, most of nonlinear systems, which arise from physical, chemical, biological or economic field, usually have nonlinear perturbations. From this viewpoint, the general theory established in [7] is not sufficient to analyze such nonlinear systems. So, the main purpose of this paper is that the results obtained in [7] will be extended to evolution inclusions with a perturbation like one in (P) without changing the mathematical framework in [7] as far as possible.

Now, we give the definition of strong solutions of (P) and state main theorems in this paper.

Definition 1.1. For any time $\bar{T}\in(0,T]$ we define a strong solution of (P) as follows:

(a) A function $u\colon [0,\bar{T}] \longmapsto H$ is called a strong solution of (P) on $[0,\bar{T}]$ iff $u \in W^{1,2}(0,\bar{T};H)$ satisfies the initial condition $u(0) = u_0$ in H and the evolution inclusion

$$u'(t) + \partial_{v(t)}\phi(t,u,v(t)\,;u(t)) + g(t,u(t),v(t)) \ni f(t) \quad \text{in} \quad H(v(t)), \quad \text{a.a. } t \in [0,\bar{T}],$$

and the function $v \in W^{1,1}(0,\bar{T};X)$ satisfies the initial condition $v(0) = v_0$ and the equality

$$v(t) = S(u\,;t,0)v_0 \quad \text{in} \quad A, \quad \forall t \in [0,\bar{T}].$$

In addition, there exists a constant $C^* > 0$, which depends on $\bar{T}$, $\|u_0\|_H$, $\|v_0\|_X$, $\varphi(0,u_0,v_0\,;u_0)$ as well as all constants in (A1)–(A10) in Section 2, such that the following boundedness holds:

$$\|u'\|_{L^2(0,\bar{T};H)} + \sup_{0\le t\le \bar{T}} \|u(t)\|_H + \sup_{0\le t\le \bar{T}} |\phi(t,u,v(t)\,;u(t))| \le C^*. \tag{1.1}$$

(b) A function $u\colon [0,\bar{T}) \longmapsto H$ is called a strong solution of (P) on $[0,\bar{T})$ iff for any $T_1 \in (0,\bar{T})$ the restriction of the function u onto the interval $[0,T_1]$ is a strong solution of (P) on $[0,T_1]$.

Under suitable assumptions, which are exactly given in Section 2, we obtain Theorems 1.2 and 1.3.

Theorem 1.2. *There exist a time $T_0 \in (0,T]$ such that the Cauchy problem (P) has at least one strong solution u on $[0,T_0]$.*

Theorem 1.3. *The Cauchy problem (P) has at least one strong solution u on $[0,T]$.*

In the rest of this section, we consider a mass-conservative tumor invasion system with quasi-variational cross diffusion, denoted by (T), as one of the typical examples of (P):

$$\text{(T)} \begin{cases} u'(t) + \nabla\cdot(d_u(v)\nabla\xi - u\nabla\lambda(v)) = 0 \quad \text{a.e. in} \quad Q_T := \Omega\times(0,T), \\ \xi \in \beta(v\,;u) \quad \text{a.e. in} \quad Q_T, \\ v' = -avw \quad \text{a.e. in} \quad Q_T, \\ w' = d_w\Delta w - bw + cu \quad \text{a.e. in} \quad Q_T, \\ (d_u(v)\nabla\xi - u\nabla\lambda(v))\cdot\nu = \nabla w\cdot\nu = 0 \quad \text{a.e. on} \quad \Sigma_T := \Gamma\times(0,T), \\ u(0) = u_0, \quad v(0) = v_0, \quad w(0) = w_0 \quad \text{a.e. in} \quad \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N = 1,2,3)$ with a smooth boundary $\Gamma := \partial\Omega$. One of the most interesting points in (T) is that the first nonlinear parabolic PDE has a nonsmooth degenerate cross diffusion $d_u(v)\nabla\beta(v\,;u)$ in general. In [8] the system, in which the first PDE is replaced by

$$u'(t) + \nabla\cdot(d_u(v)\nabla\xi - u\nabla\lambda(v)) + \xi = 0 \quad \text{a.e. in} \quad Q_T,$$

is considered and the existence of strong solutions on $[0,T]$ is shown by using the general theory establish in [7], which cannot be directly applied because of the existence of the haptotaxis term $\nabla \cdot u\nabla\lambda(v)$. In other words, the haptotaxis term plays a role as a single-valued perturbation.

On the other hand, the difference between the system (T) and that in [8] is whether a mass of an unknown function u is conservative in time or not. Actually, the system (T) has the following mass-conservative property;

$$\int_\Omega u(t)dx = \int_\Omega u_0 dx, \quad \forall t \in [0,T].$$

In [4, 5, 6, 9], the following system $(\mathrm{T})_1$ with a mass-conservative property is considered:

$$(\mathrm{T})_1 \begin{cases} u'(t) + \nabla \cdot (d_u(\alpha(z),v)\nabla\xi - u\nabla\lambda(z)) = 0 \quad \text{a.e. in} \quad Q_T, \\ \xi \in \beta(v\,;u) \quad \text{a.e. in} \quad Q_T, \\ z' = d_z\Delta z + avw \quad \text{a.e. in} \quad Q_T, \\ v' = -avw \quad \text{a.e. in} \quad Q_T, \\ w' = d_w\Delta w - bw + cu \quad \text{a.e. in} \quad Q_T, \\ (d_u(\alpha(z),v)\nabla\xi - u\nabla\lambda(v)) \cdot \nu = \nabla z \cdot \nu = \nabla w \cdot \nu = 0 \quad \text{a.e. on} \quad \Sigma_T, \\ u(0) = u_0, \quad z(0) = z_0, \quad v(0) = v_0, \quad w(0) = w_0 \quad \text{a.e. in} \quad \Omega. \end{cases}$$

According to the result in [9] the system $(\mathrm{T})_1$ is expressed as an evolution inclusion with a single-valued perturbation on $V_0^*(z,v)$, where $V_0^*(z,v)$ is the dual space of the Hilbert space V_0 with an inner product $(\cdot,\cdot)_{V_0(z,v)}$ given by

$$V_0 := \left\{ \eta \in H^1(\Omega)\,;\, \int_\Omega \eta(x)dx = 0 \right\}, \tag{1.2}$$

$$(z_1,z_2)_{V_0(z,v)} := \int_\Omega d_u(\alpha(z),v)\nabla\eta_1 \cdot \nabla\eta_2 dx, \quad \forall \eta_1,\eta_2 \in V_0. \tag{1.3}$$

Using this expression, it is shown that there exists a time $T_0 \in (0,T]$ such that $(\mathrm{P})_1$ has at least one strong solution on $[0,T_0]$ for the case $N = 1,2,3$. Moreover, for the case $N = 1$ the existence of strong solutions on $[0,T]$ is shown. Hence, the purpose of Section 6 is to show the existence of strong solutions to (T) on $[0,T]$ by using the same argumentations in [8, 9] and applying the general theory obtained in Sections 4 and 5 of this paper.

2. Assumptions

All constants $C_i > 0$, which appear in the following argumentation, depend on T unless we decline in particular it. Throughout this paper, we assume that (A1)–(A10) are satisfied.

(A1) A family $\{(\cdot,\cdot)_v\,;\, v \in A\}$ of inner products of H such that the following conditions are satisfied:

(a) There exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1\|z\|_H \le \|z\|_v \le c_2\|z\|_H, \quad \forall v \in A,\ \forall z \in H,$$

where $\|z\|_v := \sqrt{(z,z)_v}$ and $\|z\|_H := \sqrt{(z,z)_H}$.

(b) There exists a constant $c_3 > 0$ such that

$$\left|\|z\|_{v_1}^2 - \|z\|_{v_2}^2\right| \le c_3\|v_1 - v_2\|_X\|z\|_{v_2}^2, \quad \forall\, v_1, v_2 \in A,\ \forall\, z \in H.$$

Hence, for any $\eta \in W^{1,1}(0,T;X)$ satisfying $\eta(t) \in A$ for all $t \in [0,T]$ we have

$$\left|\|z\|_{\eta(t)}^2 - \|z\|_{\eta(s)}^2\right| \le c_3\left|\int_s^t \|\eta'(\sigma)\|_X d\sigma\right| \|z\|_{\eta(s)}^2,$$
$$\forall z \in H, \quad \forall s,\, t \in [0,T].$$

This condition is originally proposed in [3], and revised in [7] in order to be applied to evolution inclusions with quasi-variational structures for inner products on H.

(A2) There exists a family

$$\left\{\{S(\tilde{u}\,;t,s)\,;\, 0 \le s \le t \le \tilde{T}\}\,;\, \tilde{T} \in (0,T],\ \tilde{u} \in C([0,\tilde{T}];H)\right\},$$

where $S(\tilde{u}\,;t,s)$ is the operator from A into itself for all $s,\ t$ with $0 \le s \le t \le \tilde{T}$, such that the following conditions are satisfied:

(a) Assume that a sequence $\{v_m\}_{m\in\mathbb{N}}$ and an element v in A satisfy $v_m \longrightarrow v$ in X as $m \to \infty$. Then, for any s,t with $0 \le s \le t \le \tilde{T}$ we have $S(\tilde{u}\,;t,s)v_m \longrightarrow S(\tilde{u}\,;t,s)v$ in X as $m \to \infty$

(b) Assume that a sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ in $C([0,\tilde{T}];H)$ satisfies

$$\tilde{u}_m \longrightarrow \tilde{u} \quad \text{in} \quad C([0,\tilde{T}];H) \quad \text{as} \quad m \to \infty.$$

Then, for any element $v \in A$ and $s \in [0,\tilde{T}]$ we have

$$S(\tilde{u}_m\,;\cdot\,,s)v \longrightarrow S(\tilde{u}\,;\cdot\,,s)v \quad \text{in} \quad C([s,\tilde{T}];X) \quad \text{as} \quad m \to \infty.$$

(c) $S(\tilde{u}\,;t,t)$ is the identity operator on A for all $t \in [0,\tilde{T}]$.

(d) We have $S(\tilde{u}\,;\cdot\,,0)v \in W^{1,1}(0,\tilde{T};X)$ for all $v \in A$.

(e) For any times $\tilde{T}_i \in (0,T]$ and functions $\tilde{u}_i \in C([0,\tilde{T}_i];H)$ $(i = 1,2)$ we assume that there exists $\tilde{T}_0 \in [0,\min\{\tilde{T}_1,\, \tilde{T}_2\}]$ such that $\tilde{u}_1(t) = \tilde{u}_2(t)$ in H for all $t \in [0,\tilde{T}_0]$. Then, we have

$$S(\tilde{u}_1\,;t,0) = S(\tilde{u}_2\,;t,0) \quad \text{on} \quad A, \quad 0 \le \forall t \le \tilde{T}_0.$$

(f) $S(\tilde{u}\,;t,s) = S(\tilde{u}\,;t,\tau) \circ S(\tilde{u}\,;\tau,s)$ on A for all $s,t,\tau \in [0,\tilde{T}]$ with $s \le \tau \le t$.

(g) The following equality holds for any $\tau \in [0, \tilde{T}]$:

$$S(\sigma_\tau \tilde{u}\,; t, s) = S(\tilde{u}\,; t+\tau, s+\tau) \quad \text{on} \quad A,$$
$$0 \le \forall s \le \forall t \le \tilde{T} - \tau,$$

where $\sigma_\tau \tilde{u}$ is a $\tau-$shift function of $\tilde{u}$ defined by

$$(\sigma_\tau \tilde{u})(t) := \begin{cases} \tilde{u}(t+\tau) & \text{if} \quad t \in [0, \tilde{T} - \tau], \\ \tilde{u}(\tilde{T}) & \text{if} \quad t \in (\tilde{T} - \tau, \tilde{T}]. \end{cases}$$

The conditions (f) and (g) in (A2) are used in order to show Theorem 1.3, but are not necessary to show Theorem 1.2.

(A3) A class $\mathcal{C}$, which consists of the families $\{\phi(t, \tilde{u}, \tilde{v})\,;\, 0 \le t \le T\}$ of time-dependent proper, l.s.c. convex function $\phi(t, \tilde{u}, \tilde{v})$ on H, is prescribed:

$$\mathcal{C} := \{\{\phi(t, \tilde{u}, v)\,;\, 0 \le t \le T\}\,;\, \tilde{u} \in C([0,T]; H),\ v \in A\}\,.$$

Using this class $\mathcal{C}$, we denote by $\mathcal{X}$ a set of families of time-dependent proper, l.s.c. and convex functions on H given by

$$\mathcal{X} = \{\{\varphi(t, \tilde{u}, \tilde{v})\,;\, 0 \le t \le T\}\,;\, \tilde{u} \in C([0,T]; H),\ \tilde{v} \in A\}\,,$$

where for any $\tilde{u} \in C([0,T]; H)$ the family $\{S(\tilde{u}\,; t, s)\,;\, 0 \le s \le t \le T\}$ is the same one that is given in (A2), and the functions $\varphi(t, \tilde{u}, \tilde{v})$ are defined by

$$\varphi(t, \tilde{u}, \tilde{v}) := \phi(t, \tilde{u}, S(\tilde{u}\,; t, 0)\tilde{v}), \quad \forall \tilde{u} \in C([0,T]; H),\ \forall \tilde{v} \in A,\ \forall t \in [0,T].$$

Then, the following properties are satisfied:

(a) There exists a proper l.s.c. convex function φ on H such that

$$\varphi(z) \le \varphi(t, \tilde{u}, \tilde{v}\,; z), \quad \forall \tilde{u} \in C([0,T]; H),\ \forall \tilde{v} \in A,$$
$$\forall t \in [0,T],\ \forall z \in H,$$

and for any $r \ge 0$ the level set $\{z \in H\,;\, \|z\|_H \le r,\ |\varphi(z)| \le r\}$ is relatively compact in H. In order to show Theorem 1.3 we also assume that there exists a constant $\varphi^* > 0$ such that

$$|\varphi(z)| \le \varphi^*, \quad \forall z \in D(\varphi) := \{\tilde{z} \in H\,;\, \varphi(\tilde{z}) < \infty\}, \tag{2.1}$$

which is not necessary to show Theorem 1.2.

(b) Assume that for families $\{\varphi(t, \tilde{u}_i, \tilde{v})\,;\, 0 \le t \le T\} \in \mathcal{X}$ $(i = 1, 2)$ there exists a time $\tilde{T} \in [0,T]$ such that $\tilde{u}_1(t) = \tilde{u}_2(t)$ in H for all $t \in [0, \tilde{T}]$. Then, we have

$$\varphi(t, \tilde{u}_1, \tilde{v}) = \varphi(t, \tilde{u}_2, \tilde{v}) \quad \text{on} \quad H, \quad 0 \le \forall t \le \tilde{T}.$$

(c) Assume that a sequence $\{\{\varphi(t,\tilde{u}_m,\tilde{v}_m)\,;\,0\le t\le T\}\}_{m\in\mathbb{N}}$ and $\{\varphi(t,\tilde{u},\tilde{v})\,;\,0\le t\le T\}$ in $\mathcal{X}$ satisfy the following convergence as $m\to\infty$:

$$(\tilde{u}_m,\tilde{v}_m)\longrightarrow(\tilde{u},\tilde{v})\quad\text{in}\quad C([0,T];H)\times X.$$

Then, for any $t\in[0,T]$ we have the following convergence as $m\to\infty$:

$$\varphi(t,\tilde{u}_m,\tilde{v}_m)\longrightarrow\varphi(t,\tilde{u},\tilde{v})\quad\text{on}\quad H(S(\tilde{u}\,;t,0)\tilde{v})$$
in the sense of Mosco.

That is, the following properties are satisfied:

(i) For any $z\in D(\varphi(t,\tilde{u},\tilde{v}))$ there exists a sequence $\{z_m\}_{m\in\mathbb{N}}$ in H such that

$$z_m\longrightarrow z\quad\text{in}\quad H(S(\tilde{u}\,;t,0)\tilde{v})\quad\text{as}\quad m\to\infty,$$
$$\lim_{m\to\infty}\varphi(t,\tilde{u}_m,\tilde{v}_m\,;z_m)=\varphi(t,\tilde{u},\tilde{v}\,;z).$$

(ii) For any subsequence $\{(\tilde{u}_{m_k},\tilde{v}_{m_k})\}_{k\in\mathbb{N}}$ of $\{(\tilde{u}_m,\tilde{v}_m)\}_{m\in\mathbb{N}}$ we have

$$\varphi(t,\tilde{u},\tilde{v}\,;z)\le\liminf_{k\to\infty}\varphi(t,\tilde{u}_{m_k},\tilde{v}_{m_k};z_k)$$

whenever a sequence $\{z_k\}_{k\in\mathbb{N}}$ and an element z in H satisfy

$$z_k\longrightarrow z\quad\text{weakly in}\quad H(S(\tilde{u}\,;t,0)\tilde{v})\quad\text{as}\quad k\to\infty.$$

We entrust the properties of the Mosco convergence of proper l.s.c. convex functions on a Hilbert space to [12], and omit them in this paper.

Under (A3), we have already known Lemma 2.1 below, which is obtained in [7, Lemma 2.1] and used in Sections 3, 4 and 5 repeatedly.

Lemma 2.1. *There exists a constant $C_1>0$ such that the following inequalities hold for all $\tilde{u}\in C([0,T];H)$, $\tilde{v}\in A$, $t\in[0,T]$ and $z\in H$:*

$$|\varphi(t,\tilde{u},\tilde{v}\,;z)|\le\varphi(t,\tilde{u},\tilde{v}\,;z)+C_1\left(\|z\|_{S(\tilde{u}\,;t,0)\tilde{v}}+1\right),$$

$$|\varphi(z)|\le\varphi(z)+C_1\left(\|z\|_H+1\right).$$

For a single-valued perturbation g, we assume that the following property is fulfilled.

(A4) A perturbation $g\colon[0,T]\times D(\varphi)\times A\longmapsto H$ satisfies the following properties:

(a) There exist a function $\ell\colon A\longmapsto\mathbb{R}$ and a constant $c_4>0$ such that for any $r\ge0$ a level set $\{v\in A\,;\,\ell(v)\le r\}$ is compact in X and

$$\|g(t,z,v)\|_H\le\ell(v)\sqrt{|\varphi(z)|+c_4},\qquad\forall t\in[0,T],\ \forall z\in D(\varphi),\ \forall v\in A,$$

where φ is the same function that is given in (a) in (A3).

(b) Assume that a sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ and a function $\tilde{u}$ in $C([0,T];H)$ satisfy

$$\tilde{u}_m \longrightarrow \tilde{u} \quad \text{in} \quad C([0,T];H) \quad \text{as} \quad m\to\infty.$$

Then, for any $\tilde{v}\in A$ we have

$$\begin{gathered} g(\,\cdot\,,\tilde{u}_m,S(\tilde{u}_m\,;\cdot\,,0)\tilde{v}) \longrightarrow g(\,\cdot\,,\tilde{u},S(\tilde{u}\,;\cdot\,,0)\tilde{v}) \\ \text{weakly in} \quad L^2(0,T;H) \quad \text{as} \quad m\to\infty. \end{gathered}$$

In order to make a class of the initial datum (u_0,v_0) clear, we assume the following condition:

(A5) A class of initial data D is defined by

$$D := \left\{ (u,v)\in H\times A\,; \begin{array}{c} u\in D(\varphi(0,\tilde{u},v)) \text{ for all } \tilde{u}\in C([0,T];H) \\ \text{satisfying } \tilde{u}(0)=u \end{array} \right\}.$$

Then, we assume $(u_0,v_0)\in D$. In what follows, we simply denote by $u_0\in C([0,T];H)$ the function $u_0(t)=u_0$ in H for all $t\in[0,T]$ if there is no confusion.

Next we fix a pair $(u_0,v_0)\in D$, and define three subsets $\mathcal{W}(u_0)\subset\mathcal{V}(u_0)\subset\mathcal{U}(u_0)$ of $C([0,T];H)$ by the following ways:

$$\mathcal{U}(u_0) := \left\{ \tilde{u}\in C([0,T];H)\,; \begin{array}{c} \tilde{u}(0)=u_0 \ \text{ in } \ H, \\ \displaystyle\sup_{0\le t\le T}\|\tilde{u}(t)\|_H + \int_0^T \varphi(\tilde{u}(t))dt < \infty \end{array} \right\},$$

$$\mathcal{V}(u_0) := \left\{ \tilde{u}\in\mathcal{U}(u_0)\,;\ \sup_{0\le t\le T}\|\tilde{u}(t)\|_H + \sup_{0\le t\le T}\varphi(\tilde{u}(t)) < \infty \right\},$$

$$\mathcal{W}(u_0) := \left\{ \tilde{u}\in\mathcal{U}(u_0)\,;\ \|\tilde{u}'\|_{L^2(0,T;H)} + \sup_{0\le t\le T}\|\tilde{u}(t)\|_H + \sup_{0\le t\le T}\varphi(\tilde{u}(t)) < \infty \right\}.$$

Moreover, for any $R\ge 0$ we define subsets $\mathcal{W}_R(u_0)\subset\mathcal{V}_R(u_0)$ and $\mathcal{U}_R(u_0)$ by the following ways:

$$\mathcal{U}_R(u_0) := \left\{ \tilde{u}\in\mathcal{U}(u_0)\,;\ \sup_{0\le t\le T}\|\tilde{u}(t)\|_H + \int_0^T \varphi(\tilde{u}(t))dt \le R \right\},$$

$$\mathcal{V}_R(u_0) := \left\{ \tilde{u}\in\mathcal{V}(u_0)\,;\ \sup_{0\le t\le T}\|\tilde{u}(t)\|_H + \sup_{0\le t\le T}\varphi(\tilde{u}(t)) \le R \right\},$$

$$\mathcal{W}_R(u_0) := \left\{ \tilde{u}\in\mathcal{W}(u_0)\,;\ \|\tilde{u}'\|_{L^2(0,T;H)} + \sup_{0\le t\le T}\|\tilde{u}(t)\|_H + \sup_{0\le t\le T}\varphi(\tilde{u}(t)) \le R \right\}.$$

Since we have

$$\int_0^T \varphi(\tilde{u}(t))dt \le T\sup_{0\le t\le T}\varphi(\tilde{u}(t)) \le RT, \quad \forall\tilde{u}\in\mathcal{V}_R(u_0),$$

we get

$$\sup_{0\le t\le T}\|\tilde{u}(t)\|_H+\int_0^T\varphi(\tilde{u}(t)dt\le R(T+1),\quad \forall\tilde{u}\in\mathcal{V}_R(u_0), \tag{2.2}$$

which implies $\mathcal{V}_R(u_0)\subset\mathcal{U}_{R(T+1)}(u_0)$. All subsets $\mathcal{U}(u_0)$, $\mathcal{U}_R(u_0)$, $\mathcal{V}(u_0)$, $\mathcal{V}_R(u_0)$, $\mathcal{W}(u_0)$ and $\mathcal{W}_R(u_0)$ are independent of the function v_0. But the assumptions (A6)–(A9) depend on (u_0,v_0).

(A6) For any $\tilde{u}\in\mathcal{U}(u_0)$ the family $\{\varphi(t,\tilde{u},v_0)\,;\,0\le t\le T\}\in\mathcal{X}$ satisfies the following condition: for any $r>0$ there exist nonnegative functions $\alpha_r(\tilde{u})\in L^2(0,T)$ and $\beta_r(\tilde{u})\in L^1(0,T)$ such that the following property $(\star)$ is satisfied:

$$(\star)\left\{\begin{array}{l}\text{for any } s,t\in[0,T] \text{ and } z(\tilde{u},s)\in D(\varphi(s,\tilde{u},v_0))\\ \text{with } \|z(\tilde{u},s)\|_{S(\tilde{u}\,;s,0)v_0}\le r \text{ there exists an element}\\ z(\tilde{u},s,t)\in D(\varphi(t,\tilde{u},v_0)) \text{ such that}\\ \text{(d1)}\quad \|z(\tilde{u},s,t)-z(\tilde{u},s)\|_{S(\tilde{u}\,;t,0)v_0}\\ \qquad\qquad\le\left(\sqrt{|\varphi(s,\tilde{u},v_0\,;z(\tilde{u},s))|}+1\right)\left|\displaystyle\int_s^t\alpha_r(\tilde{u}\,;\tau)d\tau\right|,\\ \text{(d2)}\quad |\varphi(t,\tilde{u},v_0\,;z(\tilde{u},s,t))-\varphi(s,\tilde{u},v_0\,;z(\tilde{u},s))|\\ \qquad\qquad\le(|\varphi(s,\tilde{u},v_0\,;z(\tilde{u},s))|+1)\left|\displaystyle\int_s^t\beta_r(\tilde{u}\,;\tau)d\tau\right|.\end{array}\right.$$

This condition was originally proposed in [11], and given in [7] which enables us to apply the evolution inclusions with quasi-variational structures for not only time-dependent subdifferentials but also inner products on H.

Remark 2.2. For a time $\tilde{T}\in[0,T)$ and a function $\bar{u}\in C([0,\tilde{T}];H)$ satisfying $\bar{u}(0)=u_0$ in H we define a prolongation $\bar{u}_{\tilde{T}}\in\mathcal{U}(u_0)$ of $\bar{u}$ by

$$\bar{u}_{\tilde{T}}(t):=\begin{cases}\bar{u}(t) & \text{if}\quad t\in[0,\tilde{T}],\\ \bar{u}(\tilde{T}) & \text{if}\quad t\in(\tilde{T},T].\end{cases}$$

From (e) in (A2) and (b) in (A3) we have

$$S(\tilde{u}\,;t,0)=S(\bar{u}_{\tilde{T}}\,;t,0)=S(\bar{u}\,;t,0)\quad\text{on}\quad A,\quad\forall t\in[0,\tilde{T}],$$

$$\varphi(t,\tilde{u},v_0)=\varphi(t,\bar{u}_{\tilde{T}},v_0)\quad\text{on}\quad H,\quad\forall t\in[0,\tilde{T}],$$

whenever $\tilde{u}\in\mathcal{U}(u_0)$ satisfies $\tilde{u}(t)=\bar{u}_{\tilde{T}}(t)$ in H for all $t\in[0,\tilde{T}]$. Defining

$$\bar{\alpha}_r(\bar{u}\,;t):=\alpha_r(\bar{u}_{\tilde{T}}\,;t),\qquad\bar{\beta}_r(\bar{u}\,;t):=\beta_r(\bar{u}_{\tilde{T}}\,;t),\quad\forall t\in[0,\tilde{T}],$$

we have $\bar{\alpha}_r(\bar{u})\in L^2(0,\tilde{T})$, $\bar{\beta}_r(\bar{u})\in L^1(0,\tilde{T})$ and see that $(\star)$ in (A6) is satisfied.

(A7) A constant $R_*>0$ is given by

$$R_*:=\left(1+\frac{4}{c_1^2}\right)\{|\varphi(0,u_0,v_0\,;u_0)|+C_1\,(c_2\|u_0\|_H+1)\}+\|u_0\|_H+\frac{1}{4},$$

where $c_1 > 0$, $c_2 > 0$ and $C_1 > 0$ are the same constants that are given in (a) in (A1) and Lemma 2.1, respectively. Then, we assume that for any $R \geq R_*$ the following properties are satisfied:

(a) There exists a family $\{M_R(r)\,;\, 0 < r < \infty\}$ such that

$$\forall r > 0, \qquad \sup_{\tilde{u} \in \mathcal{U}_R(u_0)} \left(\|\alpha_r(\tilde{u})\|_{L^2(0,T)} + \|\beta_r(\tilde{u})\|_{L^1(0,T)} \right) \leq M_R(r).$$

(b) For any $r > 0$ and $\varepsilon > 0$ there exists a constant $\delta_{r,\varepsilon,R} > 0$ such that

$$\sup_{\tilde{u} \in \mathcal{U}_R(u_0)} \left\{ \sup_{0 \leq t \leq T} \int_t^{\min\{t+\delta_{r,\varepsilon,R},\, T\}} \left(|\alpha_r(\tilde{u}\,;s)|^2 + \beta_r(\tilde{u}\,;s) \right. \right.$$
$$\left. \left. + \|(S(\tilde{u}\,;s,0)v_0)'\|_X \right) ds \right\} \leq \varepsilon.$$

(A8) There exist a family $\{h(\tilde{u}) \in W^{1,2}(0,T;H)\,;\, \tilde{u} \in \mathcal{U}(u_0)\}$ and a constant $C_2 > 0$ such that

$$\sup_{\tilde{u} \in \mathcal{U}(u_0)} \left\{ \|h'(\tilde{u})\|^2_{L^2(0,T;H)} + \sup_{0 \leq t \leq T} \|h(\tilde{u}\,;t)\|_H \right.$$
$$\left. + \sup_{0 \leq t \leq T} |\varphi(t, \tilde{u}, v_0\,; h(\tilde{u}\,;t))| \right\} \leq C_2.$$

Remark 2.3. The existence of the function $h(\tilde{u})\colon [0,T] \longmapsto H$ in (A8) is guaranteed in [7]. Actually, using [7, Proposition 2.5 and Remark 2.6], we see that for any $\tilde{u} \in \mathcal{U}(u_0)$ there exists a function $h(\tilde{u})\colon [0,T] \longmapsto H$ and a constant $\tilde{C}_2(\tilde{u}) > 0$, such that

$$\sup_{0 \leq t \leq T} \|h(\tilde{u}\,;t)\|_H + \sup_{0 \leq t \leq T} |\varphi(t, \tilde{u}, v_0\,; h(\tilde{u}\,;t))| + \|h'(\tilde{u})\|^2_{L^2(0,T;H)} \leq \tilde{C}_2(\tilde{u}).$$

Hence, (A8) implies that the constants $\tilde{C}_2(\tilde{u})$ can be chosen so that they are independent of the choice of functions $\tilde{u} \in \mathcal{U}(u_0)$. Roughly speaking, there exists a constant $C_2 > 0$ such that

$$\sup_{\tilde{u} \in \mathcal{U}(u_0)} \tilde{C}_2(\tilde{u}) \leq C_2.$$

At last, we assume that the following uniform estimate is fulfilled.

(A9) There exists a constant $C_3 > 0$ such that

$$\sup_{\tilde{u} \in \mathcal{U}(u_0)} \left(\sup_{0 \leq t \leq T} \ell(S(\tilde{u}\,;t,0)v_0) + \|(S(\tilde{u}\,;t,0)v_0)'\|_{L^1(0,T;X)} \right) \leq C_3,$$

where ℓ is the same function that is given in (A4).

For a prescribed datum f we assume that the following condition is satisfied:
(A10) $f \in L^2(0,T;H)$

At the end of this section, we show Lemma 2.4 which plays a significant role in the Schauder fixed-point argumentation to show Theorem 1.2 under all assumptions (A1)–(A10).

Lemma 2.4. *For any $R \geq R^*$ the set $\mathcal{V}_R(u_0)$ is nonempty, convex and closed in $C([0,T];H)$. Moreover, $\mathcal{W}_R(u_0)$ is nonempty, convex and compact in $C([0,T];H)$.*

Proof. From (a) in (A3) and (A5) we get $u_0 \in \mathcal{W}_R(u_0) \subset \mathcal{V}_R(u_0)$. Moreover, we see from (a) in (A3) again that $\mathcal{V}_R(u_0)$ is convex and closed in $C([0,T];H)$, and $\mathcal{W}_R(u_0)$ is convex and closed in $C([0,T];H)$ and weakly closed in $W^{1,2}(0,T;H)$. Moreover, from Lemma 2.1 we get

$$\begin{aligned}\sup_{0\leq t\leq T}|\varphi(\tilde{u}(t))| &\leq \sup_{0\leq t\leq T}\varphi(\tilde{u}(t)) + C_1\left(\sup_{0\leq t\leq T}\|\tilde{u}(t)\|_H + 1\right)\\ &\leq R + C_1(R+1), \qquad \forall \tilde{u} \in \mathcal{W}_R(u_0).\end{aligned}$$

Applying the Ascoli–Arzelà theorem, we see that $\mathcal{W}_R(u_0)$ is relatively compact in $C([0,T];H)$. □

3. Auxiliary problem

In this section, for any $\tilde{u} \in \mathcal{U}(u_0)$ we consider a Cauchy problem $(\mathrm{AP})_{\tilde{u}}$ with variational structures as an auxiliary problem of (P) on $[0,T]$:

$$(\mathrm{AP})_{\tilde{u}}\begin{cases} w'(t) + \partial_{\tilde{v}(t)}\varphi(t,\tilde{u},v_0\,;w(t)) \ni f(t) - g(t,\tilde{u}(t),\tilde{v}(t)) \\ \qquad\qquad\qquad\qquad \text{in} \quad H(\tilde{v}(t)), \quad \text{a.a. } t\in[0,T], \\ \tilde{v}(t) = S(\tilde{u}\,;t,0)v_0 \quad \text{in} \quad A, \quad \forall\, t\in[0,T], \\ w(0) = u_0 \quad \text{in} \quad H. \end{cases}$$

Then, we have the following lemma, which guarantees the existence and uniqueness result of strong solutions to $(\mathrm{AP})_{\tilde{u}}$ on $[0,T]$.

Lemma 3.1. *The Cauchy problem $(AP)_{\tilde{u}}$ has a unique strong solution $w \in W^{1,2}(0,T;H) \cap \mathcal{U}(u_0)$.*

Proof. From (A4), (A9) and Lemma 2.1 we get the following inequality for all $t \in [0,T]$:

$$\begin{aligned}\|g(t,\tilde{u}(t),\tilde{v}(t))\|_H^2 &\leq \ell(\tilde{v}(t))^2\left(|\varphi(\tilde{u}(t))| + c_4\right) \\ &\leq C_3^2\left\{\varphi(\tilde{u}(t)) + C_1\|\tilde{u}(t)\|_H + C_1 + c_4\right\},\end{aligned} \tag{3.1}$$

which implies $g(\,\cdot\,,\tilde{u},\tilde{v}) \in L^2(0,T;H)$ with the following estimate:

$$\|g(\,\cdot\,,\tilde{u},\tilde{v})\|_{L^2(0,T;H)} \le C_3\Big\{\int_0^T \varphi(\tilde{u}(t))dt + T\left(C_1 \sup_{0\le t\le T}\|\tilde{u}(t)\|_H + C_1 + c_4\right)\Big\}^{\frac{1}{2}}. \tag{3.2}$$

Applying [7, Theorem 3.1], from (A10) and (3.2) we get this lemma. □

From Lemma 3.1 we can define an operator $\mathcal{S}\colon \mathcal{U}(u_0) \longmapsto W^{1,2}(0,T;H)\cap\mathcal{U}(u_0)$, which assigns the unique strong solution w of $(\mathrm{AP})_{\tilde{u}}$ on $[0,T]$ to $\tilde{u}$, by

$$\mathcal{S}\tilde{u} := w, \quad \forall \tilde{u} \in \mathcal{U}(u_0).$$

We show some estimates of the solution $\mathcal{S}\tilde{u}$ and $\{\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))\,;\,0\le t\le T\}$. In order to do this, first of all we give the first type energy inequality in Lemma 3.2, which is obtained in [7, Lemma 2.10].

Lemma 3.2. *The following first type energy inequality holds for a.a. $t\in(0,T)$:*

$$\frac{d}{dt}\|(\mathcal{S}\tilde{u})(t)-h(\tilde{u}\,;t)\|^2_{\tilde{v}(t)} - 2((\mathcal{S}\tilde{u})'(t)-h'(\tilde{u}\,;t),(\mathcal{S}\tilde{u})(t)-h(\tilde{u}\,;t))_{\tilde{v}(t)} \le c_3\|\tilde{v}'(t)\|_X\|(\mathcal{S}\tilde{u})(t)-h(\tilde{u}\,;t)\|^2_{\tilde{v}(t)},$$

where $h(\tilde{u})$ is the same function that is given in (A8).

Using the first type energy inequality in Lemma 3.2, we show Lemma 3.3.

Lemma 3.3. *There exists a constant $C_4>0$, which depends on*

$$\|u_0\|_H,\ \|v_0\|_X,\ \sup_{0\le t\le T}\|\tilde{u}(t)\|_H,\ \int_0^T \varphi(\tilde{u}(t))dt,\ T$$

such that

$$\sup_{0\le t\le T}\|(\mathcal{S}\tilde{u})(t)\|_H + \int_0^T |\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))|dt \le C_4.$$

Hence, we have $(\mathcal{S}\tilde{u})(t)\in D(\varphi(t,\tilde{u},v_0))$ for a.a. $t\in[0,T]$ and $\mathcal{S}\tilde{u}\in\mathcal{U}_{C_4}(u_0)$.

Proof. Using Lemmas 2.1 and 3.2, we see from (a) in (A1), (A8) and (3.1) that the following inequality holds for a.a. $t\in(0,T)$:

$$\begin{aligned}
&\frac{d}{dt}\|(\mathcal{S}\tilde{u})(t)-h(\tilde{u}\,;t)\|^2_{\tilde{v}(t)} + 2|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| \qquad (3.3)\\
\le\ & (c_3\|\tilde{v}'(t)\|_X+3)\,\|(\mathcal{S}\tilde{u})(t)-h(\tilde{u}\,;t)\|^2_{\tilde{v}(t)}\\
&+2C_1\left(\|(\mathcal{S}\tilde{u})(t)-h(\tilde{u}\,;t)\|_{\tilde{v}(t)}+\|h(\tilde{u}\,;t)\|_{\tilde{v}(t)}+1\right)\\
&+c_2^2\left(\|g(t,\tilde{u}(t),\tilde{v}(t))\|^2_H+\|f(t)\|^2_H+\|h'(\tilde{u}\,;t)\|^2_H\right)+2|\varphi(t,\tilde{u},v_0\,;h(\tilde{u}\,;t))|\\
\le\ & (c_3\|\tilde{v}'(t)\|_X+C_1+3)\,\|(\mathcal{S}\tilde{u})(t)-h(\tilde{u}\,;t)\|^2_{\tilde{v}(t)}+G(t),
\end{aligned}$$

where $G \in L^1(0,T)$ is a function given by

$$G(t) := c_2^2\left(\|f(t)\|_H^2 + \|h'(\tilde{u}\,;t)\|_H^2\right) + 2C_2(c_2C_1+1) + 3C_1 + (c_2C_3)^2\left\{\varphi(\tilde{u}(t)) + C_1 \sup_{0\le t\le T}\|\tilde{u}(t)\|_H + C_1 + c_4\right\}.$$

Applying the Gronwall lemma to (3.3) and using (a) in (A1) again, we see that the following inequality holds for all $t \in [0,T]$:

$$\begin{aligned} &c_1^2\|(\mathcal{S}\tilde{u})(t) - h(\tilde{u}\,;t)\|_H^2 + 2\int_0^t |\varphi(s,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(s))|ds \qquad (3.4)\\ &\le \left(\|u_0 - h(\tilde{u}\,;0)\|_{v_0}^2 + \int_0^T G(t)dt\right)\exp\left(c_3\int_0^T \|\tilde{v}'(t)\|_X dt + TC_1 + 3T\right)\\ &\le \left\{2c_2^2\left(\|u_0\|_H^2 + \|h(\tilde{u}\,;0)\|_H^2\right) + \int_0^T G(t)dt\right\}\\ &\qquad\qquad \times \exp\left(c_3\int_0^T \|\tilde{v}'(t)\|_X dt + (C_1+3)T\right). \end{aligned}$$

From (A8) and (A9) we get

$$\begin{aligned} &\left\{2c_2^2\left(\|u_0\|_H^2 + \|h(\tilde{u}\,;0)\|_H^2\right) + \int_0^T G(t)dt\right\} \qquad (3.5)\\ &\qquad \times \exp\left(c_3\int_0^T \|\tilde{v}'(t)\|_X dt + (C_1+3)T\right)\\ &\le e^{c_3C_3+(C_1+3)T}\Bigg[2c_2^2\left(\|u_0\|_H^2 + C_2^2 + \|f\|_{L^2(0,T;H)}^2 + C_2\right)\\ &\qquad + T\left\{2C_2(c_2C_1+1) + 3C_1\right\} + (c_2C_3)^2\Bigg\{\int_0^T \varphi(\tilde{u}(t))dt\\ &\qquad + T\left(C_1 \sup_{0\le t\le T}\|\tilde{u}(t)\|_H + C_1 + c_4\right)\Bigg\}\Bigg] =: \tilde{C}_4, \end{aligned}$$

We see from (3.4), (3.5) and Remark 2.3 that the following estimates holds:

$$\sup_{0\le t\le T}\|(\mathcal{S}\tilde{u})(t)\|_H + \int_0^T |\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))|dt \le \sqrt{2\left(C_2^2 + \frac{\tilde{C}_4}{c_1^2}\right)} + \frac{\tilde{C}_4}{2} =: C_4,$$

which implies that this lemma holds. □

Using [7, Lemma 2.11] and Lemma 3.3, we get the second type energy inequality in Lemma 3.4.

Lemma 3.4. *There exist constants $C_5 > 0$ and $C_6 > 0$, which also depend on*

$$\|u_0\|_H,\ \|v_0\|_X,\ \sup_{0\le t\le T}\|\tilde{u}(t)\|_H,\ \int_0^T \varphi(\tilde{u}(t))dt,\ T$$

such that the following inequality holds for a.a. $t \in (0,T)$:

$$\begin{aligned}
&\frac{d}{dt}\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) + ((\mathcal{S}\tilde{u})'(t),(\mathcal{S}\tilde{u})'(t)+g(t,\tilde{u}(t),\tilde{v}(t))-f(t))_{\tilde{v}(t)} \\
&\le C_5\|f(t)-g(t,\tilde{u}(t),\tilde{v}(t))-(\mathcal{S}\tilde{u})'(t)\|_{\tilde{v}(t)}\big(\sqrt{|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))|}+1\big) \\
&\qquad \times \alpha_{C_4}(\tilde{u}\,;t) + C_6\big(\,|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))|+1\,\big)\,(\beta_{C_4}(\tilde{u}\,;t)+\|\tilde{v}'(t)\|_X)\,,
\end{aligned}$$

where C_4 is the same constant that is obtained in Lemma 3.3.

Using Lemmas 2.1, 3.2 and 3.4, we show Lemma 3.5.

Lemma 3.5. *The following estimates are satisfied:*

(1) There exists a constant $C_7 > 0$, which depends on $C_i > 0$ $(1 \le i \le 6)$ as well as

$$\varphi(0,u_0,v_0\,;u_0),\ \|\alpha_{C_4}(\tilde{u})\|_{L^2(0,T)},\ \|\beta_{C_4}(\tilde{u})\|_{L^1(0,T)},$$

such that

$$\begin{aligned}
&\|(\mathcal{S}\tilde{u})'\|_{L^2(0,T;H)} + \sup_{0\le t\le T}\|(\mathcal{S}\tilde{u})(t)\|_H \\
&\qquad + \sup_{0\le t\le T}|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| \le C_7.
\end{aligned}$$

Hence, we have $(\mathcal{S}\tilde{u})(t) \in D(\varphi(t,\tilde{u},v_0))$ for all $t \in [0,T]$ and $\mathcal{S}\tilde{u} \in \mathcal{W}_{C_7}(u_0)$.

(2) For any $\varepsilon \in (0,1)$ there exist constants $C_8(\varepsilon) > 0$, which depends on $C_i > 0$ $(i = 5, 6)$ and $\varepsilon > 0$, and $C_9 > 0$, which depends on $C_i > 0$ $(i = 1, 4)$ but is independent of ε, such that the following inequality holds for all s, t with $0 \le s \le t \le T$:

$$\begin{aligned}
&\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) \\
&\quad + \int_s^t ((\mathcal{S}\tilde{u})'(\tau),(\mathcal{S}\tilde{u})'(\tau)+g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))-f(\tau))_{\tilde{v}(\tau)}d\tau \\
&\le \varphi(s,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) \\
&\quad + \varepsilon\int_s^t \|(\mathcal{S}\tilde{u})'(\tau)+g(\tau,\tilde{u}(\tau),\tilde{v}(\tau))-f(\tau)\|^2_{\tilde{v}(\tau)}d\tau \\
&\quad + C_8(\varepsilon)\int_s^t \{\varphi(\tau,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(\tau))+C_9\} \\
&\qquad\qquad \times \big(|\alpha_{C_4}(\tilde{u}\,;\tau)|^2+\beta_{C_4}(\tilde{u}\,;\tau)+\|\tilde{v}'(\tau)\|_X\big)\,d\tau.
\end{aligned}$$

Proof. First of all, we have the following three inequalities for a.a. $t \in (0,T)$:

$$\begin{aligned} &((\mathcal{S}\tilde{u})'(t), g(t,\tilde{u}(t),\tilde{v}(t)) - f(t))_{\tilde{v}(t)} \\ \geq\ & -\frac{1}{4}\|(\mathcal{S}\tilde{u})'(t)\|^2_{\tilde{v}(t)} - 2c_2^2\left(\|g(t,\tilde{u}(t),\tilde{v}(t))\|^2_H + \|f(t)\|^2_H\right), \end{aligned} \tag{3.6}$$

$$\begin{aligned} &C_5\|f(t) - g(t,\tilde{u}(t),\tilde{v}(t))\|_{\tilde{v}(t)}\big(\sqrt{|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))|} + 1\big)\alpha_{C_4}(\tilde{u}\,;t) \\ \leq\ & c_2^2C_5^2\left(\|f(t)\|^2_H + \|g(t,\tilde{u}(t),\tilde{v}(t))\|^2_H\right) \\ &+ \big(\,|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| + 1\,\big)|\alpha_{C_4}(\tilde{u}\,;t)|^2, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &C_5\|(\mathcal{S}\tilde{u})'(t)\|_{\tilde{v}(t)}\big(\sqrt{|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))|} + 1\big)\alpha_{C_4}(\tilde{u}\,;t) \\ \leq\ & \frac{1}{4}\|(\mathcal{S}\tilde{u})'(t)\|^2_{\tilde{v}(t)} + 2C_5^2\big(\,|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| + 1\,\big)|\alpha_{C_4}(\tilde{u}\,;t)|^2. \end{aligned} \tag{3.8}$$

Substituting (3.6), (3.7) and (3.8) into the second type inequality in Lemma 3.4, we get the following inequality for a.a. $t \in (0,T)$:

$$\begin{aligned} &\frac{1}{2}\|(\mathcal{S}\tilde{u})'(t)\|^2_{\tilde{v}(t)} + \frac{d}{dt}\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) \\ \leq\ & \left(2C_5^2 + C_6 + 1\right)\big(\,|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| + 1\,\big) \\ &\qquad \times \left(|\alpha_{C_4}(\tilde{u}\,;t)|^2 + \beta_{C_4}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\right) \\ &+ c_2^2\left(C_5^2 + 2\right)\left(\|f(t)\|^2_H + \|g(t,\tilde{u}(t),\tilde{v}(t))\|^2_H\right). \end{aligned} \tag{3.9}$$

Using (a) in (A1), (3.9), Lemmas 2.1 and 3.3, we get the following inequality for a.a. $t \in (0,T)$:

$$\begin{aligned} &\frac{c_1^2}{2}\|(\mathcal{S}\tilde{u})'(t)\|^2_H + \frac{d}{dt}\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) \\ \leq\ & C_{10}\left(|\alpha_{C_4}(\tilde{u}\,;t)|^2 + \beta_{C_4}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\right) \\ &\qquad \times \left\{\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) + C_1\left(\|(\mathcal{S}\tilde{u})(t)\|_{\tilde{v}(t)} + 1\right) + 1\right\} \\ &+ c_2^2\left(C_5^2 + 2\right)\left(\|f(t)\|^2_H + \|g(t,\tilde{u}(t),\tilde{v}(t))\|^2_H\right) \\ \leq\ & C_{10}\left(|\alpha_{C_4}(\tilde{u}\,;t)|^2 + \beta_{C_4}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\right)\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) \\ &+ C_{11}\left(\|f(t)\|^2_H + \|g(t,\tilde{u}(t),\tilde{v}(t))\|^2_H + |\alpha_{C_4}(\tilde{u}\,;t)|^2\right. \\ &\qquad\qquad \left. + \beta_{C_4}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\right), \end{aligned} \tag{3.10}$$

where the constants $C_{10} > 0$ and $C_{11} > 0$ are given by

$$C_{10} := 2C_5^2 + C_6 + 1, \qquad C_{11} := C_{10}(c_2C_1C_4 + C_1 + 1) + c_2^2\left(C_5^2 + 2\right).$$

Applying the Gronwall lemma to (3.10), we get the following inequality for all

$t \in [0,T]$:

$$\begin{aligned}
&\frac{c_1^2}{2}\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_H^2 dx + \varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) \\
&\le |\varphi(0,u_0,v_0\,;u_0)| \exp\biggl(C_{10}\int_0^T \Bigl(|\alpha_{C_4}(\tilde{u}\,;t)|^2 + \beta_{C_4}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\Bigr)dt\biggr) \\
&\quad + C_{11}\exp\biggl(C_{10}\int_0^T \left(|\alpha_{C_4}(\tilde{u}\,;t)|^2 + \beta_{C_4}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\right) ds\biggr) \\
&\qquad\qquad \times \int_0^T \left(\|f(t)\|_H^2 + \|g(t,\tilde{u}(t),\tilde{v}(t))\|_H^2 + |\alpha_{C_4}(\tilde{u}\,;t)|^2\right. \\
&\qquad\qquad\qquad\qquad \left. + \beta_{C_4}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\right) dt =: \tilde{C}_7,
\end{aligned}$$

which implies

$$\begin{aligned}
|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| &\le \varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) + C_1\left(\|(\mathcal{S}\tilde{u})(t)\|_{\tilde{v}(t)} + 1\right) \\
&\le \tilde{C}_7 + C_1(c_1C_4+1) =: \bar{C}_7,
\end{aligned} \tag{3.11}$$

hence,

$$\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_H^2 dt \le \frac{c_1^2}{2}\left(|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| + \tilde{C}_7\right) \le \frac{c_1^2(\bar{C}_7+\tilde{C}_7)}{2}. \tag{3.12}$$

From (3.11) and (3.12) we get the following estimate:

$$\|(\mathcal{S}\tilde{u})'\|_{L^2(0,T;H)} + \sup_{0\le t\le T}|\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| \le \sqrt{\frac{c_1^2(\bar{C}_7+\tilde{C}_7)}{2}} + \bar{C}_7,$$

which implies that (1) holds by using the estimate obtained in Lemma 3.3 together.
(2) We go back to the second type energy inequality in Lemma 3.4, and use Lemmas 2.1 and 3.3 again. Then we see that for any $\varepsilon > 0$ the following inequality holds for a.a. $t \in (0,T)$:

$$\begin{aligned}
&\frac{d}{dt}\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) + ((\mathcal{S}\tilde{u})'(t),(\mathcal{S}\tilde{u})'(t) + g(t,\tilde{u}(t),\tilde{v}(t)) - f(t))_{\tilde{v}(t)} \\
&\le \varepsilon\|f(t) - g(t,\tilde{u}(t),\tilde{v}(t)) - (\mathcal{S}\tilde{u})'(t)\|_{\tilde{v}(t)}^2 \\
&\quad + \left(\frac{C_5^2}{2\varepsilon} + C_6\right)\left\{\,\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t)) + c_2C_1C_4 + C_1 + 1\,\right\} \\
&\qquad\qquad \times \left(|\alpha_{C_4}(\tilde{u}\,;t)|^2 + \beta_{C_4}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\right).
\end{aligned} \tag{3.13}$$

Putting

$$C_8(\varepsilon) := \frac{C_5^2}{2\varepsilon} + C_6, \qquad C_9 := c_2C_1C_4 + C_1 + 1,$$

and integrating (3.13) on any interval $[s,t]$ with $0 \le s \le t \le T$, we see that (2) holds. $\square$

In the rest of this section, we show Lemma 3.6 which plays an important role to use the Schauder fixed-point argumentation.

Lemma 3.6. *For any $R \ge R^*$ the operator*

$$\mathcal{S}\colon \mathcal{V}_R(u_0) \longmapsto W^{1,2}([0,T];H) \cap \mathcal{U}(u_0)$$

is continuous with respect to the strong topology of $C([0,T];H)$. That is, we have

$$\mathcal{S}\tilde{u}_m \longrightarrow \mathcal{S}\tilde{u} \quad \text{in} \quad C([0,T];H) \quad \text{as} \quad m \to \infty$$

whenever a sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ and an element $\tilde{u}$ in $\mathcal{V}_R(u_0)$ satisfy

$$\tilde{u}_m \longrightarrow \tilde{u} \quad \text{in} \quad C([0,T];H) \quad \text{as} \quad m \to \infty. \tag{3.14}$$

Proof. From the definition of $\mathcal{S}$ we have the following evolution inclusion:

$$(\mathcal{S}\tilde{u}_m)'(t) + \partial_{\tilde{v}_m(t)}\varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t)) \ni f(t) - g(t,\tilde{u}_m(t),\tilde{v}_m(t)) \tag{3.15}$$
$$\text{in} \quad H(\tilde{v}_m(t)), \quad \text{a.a. } t \in (0,T),$$

with

$$\tilde{v}_m(t) = S(\tilde{u}_m\,;t,0)v_0 \quad \text{in} \quad A, \quad \forall\, t \in [0,T], \tag{3.16}$$

$$(\mathcal{S}\tilde{u}_m)(0) = u_0 \quad \text{in} \quad H. \tag{3.17}$$

From (b) in (A2), (b) in (A4), (3.14) and (3.16) we get the following convergences as $m \to \infty$:

$$\tilde{v}_m \longrightarrow S(\tilde{u}\,;\cdot\,,0)v_0 = \tilde{v} \quad \text{in} \quad C([0,T];X), \tag{3.18}$$

$$g(\,\cdot\,,\tilde{u}_m,\tilde{v}_m) \longrightarrow g(\,\cdot\,,\tilde{u},\tilde{v}) \quad \text{weakly in} \quad L^2(0,T;H). \tag{3.19}$$

First of all, we show that there exists a constant $R_1 > 0$ such that $\{\mathcal{S}\tilde{u}_m\}_{m\in\mathbb{N}} \subset \mathcal{W}_{R_1}(u_0)$. We see from Lemma 3.3 that there exists a constant $C_4(R) > 0$ such that

$$\sup_{m\in\mathbb{N}}\left(\sup_{0\le t\le T}\|(\mathcal{S}\tilde{u}_m)(t)\|_H + \int_0^T |\varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t))|dt\right) \le C_4(R),$$

hence, from (a) in (A3)

$$\sup_{m\in\mathbb{N}}\left(\sup_{0\le t\le T}\|(\mathcal{S}\tilde{u}_m)(t)\|_H + \int_0^T \varphi((\mathcal{S}\tilde{u}_m)(t))dt\right) \le C_4(R),$$

which implies $\mathcal{S}\tilde{u}_m \in \mathcal{U}_{C_4(R)}(u_0)$ for all $m \in \mathbb{N}$ because of (3.17). Using (a) in (A7) and (1) in Lemma 3.5, we see that there exists a constant $C_5(R) > 0$ such

that

$$\sup_{m\in\mathbb{N}}\Big(\|(\mathcal{S}\tilde{u}_m)'\|_{L^2(0,T;H)}+\sup_{0\le t\le T}\|(\mathcal{S}\tilde{u}_m)(t)\|_H \\ +\sup_{0\le t\le T}|\varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t))|\Big)\le C_5(R),$$

hence, from (a) in (A3) again

$$\sup_{m\in\mathbb{N}}\Big(\|(\mathcal{S}\tilde{u}_m)'\|_{L^2(0,T;H)}+\sup_{0\le t\le T}\|(\mathcal{S}\tilde{u}_m)(t)\|_H+\sup_{0\le t\le T}\varphi((\mathcal{S}\tilde{u}_m)(t))\Big)\le C_5(R),$$

which implies $\{\mathcal{S}\tilde{u}_m\}_{m\in\mathbb{N}}\subset\mathcal{W}_{C_5(R)}(u_0)$, that is, this constant $C_5(R)>0$ is a desired one as R_1.

We see from Lemma 2.4 that there exist a subsequence $\{\mathcal{S}\tilde{u}_{m_k}\}_{k\in\mathbb{N}}$ of $\{\mathcal{S}\tilde{u}_m\}_{m\in\mathbb{N}}$ and an element $\tilde{w}\in\mathcal{W}_{R_1}(u_0)$ such that the following convergence holds as $k\to\infty$:

$$\mathcal{S}\tilde{u}_{m_k}\longrightarrow\tilde{w}\quad\text{in}\quad C([0,T];H)\quad\text{and}\quad\text{weakly in}\quad W^{1,2}(0,T;H),\tag{3.20}$$

hence, from (3.17) and (3.20) we get

$$\tilde{w}(0)=u_0\quad\text{in}\quad H.\tag{3.21}$$

In the rest of this proof, we show $\tilde{w}=\mathcal{S}\tilde{u}$. In order to do this, for any $\tilde{u}\in\mathcal{U}(u_0)$ we consider a function space $\mathcal{L}^2(\tilde{u},v_0):=L^2(0,T;H)$ with an inner product given by

$$(\xi_1,\xi_2)_{\mathcal{L}^2(\tilde{u},v_0)}:=\int_0^T(\xi_1(t),\xi_2(t))_{\tilde{v}(t)}dt,\quad\forall\xi_1,\xi_2\in L^2(0,T;H),$$

and a proper l.s.c. convex function $\Phi(\tilde{u},v_0)\colon L^2(0,T;H)\longmapsto\mathbb{R}\cup\{\infty\}$ defined by

$$\Phi(\tilde{u},v_0\,;\eta):=\int_0^T\varphi(t,\tilde{u},v_0\,;\eta(t))dt,\quad\forall\eta\in L^2(0,T;H).$$

Then, we see from (3.15) that the following inequality holds for all $k\in\mathbb{N}$ and all $\eta\in L^2(0,T;H)$:

$$\Phi(\tilde{u}_{m_k},v_0\,;\mathcal{S}\tilde{u}_{m_k})+(f-(\mathcal{S}\tilde{u}_{m_k})'-g(\,\cdot\,,\tilde{u}_{m_k},\tilde{v}_{m_k}),\eta-\mathcal{S}\tilde{u}_{m_k})_{\mathcal{L}^2(\tilde{u}_{m_k},v_0)} \\ \le\Phi(\tilde{u}_{m_k},v_0\,;\eta).\tag{3.22}$$

We see from [7, Lemma 4.6] that $\Phi(\tilde{u}_{m_k},v_0)$ converges to $\Phi(\tilde{u},v_0)$ in the sense of Mosco in $\mathcal{L}^2(\tilde{u},v_0)$ as $k\to\infty$, that is, for any $\xi\in D(\Phi(\tilde{u},v_0))$ there exists a sequence $\{\xi_k\}_{k\in\mathbb{N}}\subset L^2(0,T;H)$ such that

$$\xi_k\longrightarrow\xi\quad\text{in}\quad\mathcal{L}^2(\tilde{u},v_0)\quad\text{as}\quad k\to\infty,\tag{3.23}$$

$$\lim_{k\to\infty}\Phi(\tilde{u}_{m_k},v_0\,;\xi_k)=\Phi(\tilde{u},v_0\,;\xi).\tag{3.24}$$

Substituting $\eta = \xi_k$ in (3.22), we get the following inequality for all $k \in \mathbb{N}$:

$$\Phi(\tilde{u}_{m_k}, v_0\,; \mathcal{S}\tilde{u}_{m_k}) + (f - (\mathcal{S}\tilde{u}_{m_k})' - g(\,\cdot\,, \tilde{u}_{m_k}, \tilde{v}_{m_k}), \xi_k - \mathcal{S}\tilde{u}_{m_k})_{\mathcal{L}^2(\tilde{u}_{m_k}, v_0)} \leq \Phi(\tilde{u}_{m_k}, v_0\,; \xi_k). \tag{3.25}$$

Using [7, Lemmas 3.4 and 4.5] and (3.14), (3.19), (3.20), (3.23), we get

$$\Phi(\tilde{u}, v_0\,; \tilde{w}) \leq \liminf_{k\to\infty} \Phi(\tilde{u}_{m_k}, v_0\,; \mathcal{S}\tilde{u}_{m_k}), \tag{3.26}$$

$$\lim_{k\to\infty} (f - (\mathcal{S}\tilde{u}_{m_k})' - g(\,\cdot\,, \tilde{u}_{m_k}, \tilde{v}_{m_k}), \xi_k - \mathcal{S}\tilde{u}_{m_k})_{\mathcal{L}^2(\tilde{u}_{m_k}, v_0)} = (f - \tilde{w}' - g(\,\cdot\,, \tilde{u}, \tilde{v}), \xi - \tilde{w})_{\mathcal{L}^2(\tilde{u}, v_0)}. \tag{3.27}$$

We take $\liminf_{k\to\infty}$ in both sides of (3.25) and use (3.24), (3.26), (3.27). Then, we get the following inequality for all $\xi \in D(\Phi(\tilde{u}))$:

$$(f - \tilde{w}' - g(\,\cdot\,, \tilde{u}, \tilde{v}), \xi - \tilde{w})_{\mathcal{L}^2(\tilde{u}, v_0)} \leq \Phi(\tilde{u}, v_0\,; \xi) - \Phi(\tilde{u}, v_0\,; \tilde{w}),$$

which implies

$$f - \tilde{w}' - g(\,\cdot\,, \tilde{u}, \tilde{v}) \in \partial_{\mathcal{L}^2(\tilde{u}, v_0)} \Phi(\tilde{u}, v_0\,; \tilde{w}), \tag{3.28}$$

where $\partial_{\mathcal{L}^2(\tilde{u}, v_0)} \Phi(\tilde{u}, v_0)$ is the subdifferential of $\Phi(\tilde{u}, v_0)$ with respect to the inner product $(\cdot, \cdot)_{\mathcal{L}^2(\tilde{u}, v_0)}$. Applying [7, Lemmas 3.5 and 3.8], we see from (3.28) that $\tilde{w}$ satisfies the following evolution inclusion:

$$\tilde{w}'(t) + \partial_{\tilde{v}(t)} \varphi(t, \tilde{u}, v_0\,; \tilde{w}(t)) \ni f(t) - g(t, \tilde{u}(t), \tilde{v}(t)) \quad \text{in} \quad H(\tilde{v}(t)) \quad \text{a.a. } t \in (0, T). \tag{3.29}$$

We see from (3.18), (3.21) and (3.29) that $\tilde{w}$ is a strong solution of $(\mathrm{AP})_{\tilde{u}}$ on $[0, T]$, that is, $\tilde{w} = \mathcal{S}\tilde{u}$. □

4. Proof of Theorem 1.2

In this section, we fix any number $\tilde{R}$ satisfying $\tilde{R} > R_*$, where R_* is the same constant in (A7). We devote this section to show Theorem 1.2 by using the Schauder fixed-point argumentation. In order to do this, for any $\bar{T} \in [0, T]$ we define a continuous operator $\Lambda(\bar{T})\colon C([0, T]; H) \longmapsto C([0, T]; H)$ by

$$(\Lambda(\bar{T})\tilde{u})(t) := \begin{cases} \tilde{u}(t) & \text{if} \quad t \in [0, \bar{T}], \\ \tilde{u}(\bar{T}) & \text{if} \quad t \in (\bar{T}, T], \end{cases}$$

and prepare Lemma 4.1 below.

Lemma 4.1. *There exists a time $T_0 \in (0, T]$ such that*

$$\big(\Lambda(T_0) \circ \mathcal{S}\big)\,(\mathcal{W}_{\tilde{R}}(u_0)) \subset \mathcal{W}_{\tilde{R}}(u_0).$$

Proof. Repeating the argumentation similar to the proof of Lemma 3.3, we see from Lemma 2.4 that there exists a constant $C_4 > 0$, which depends on $\tilde{R}$, such that

$$\sup_{\tilde{u} \in \mathcal{W}_{\tilde{R}}(u_0)} \left(\sup_{0 \le t \le T} \|(\mathcal{S}\tilde{u})(t)\|_H + \int_0^T |\varphi(t, \tilde{u}, v_0\,; (\mathcal{S}\tilde{u})(t))| dt \right) \le C_4. \tag{4.1}$$

Since from (2.2), (4.1), (a) in (A3) and Lemma 2.1 we have $\mathcal{W}_{\tilde{R}}(u_0) \subset \mathcal{V}_{\tilde{R}}(u_0) \subset \mathcal{U}_{\tilde{R}(T+1)}(u_0)$, we see from (a) in (A7) that there exists a constant $M_{\tilde{R}}(C_4) > 0$ such that

$$\begin{aligned} &\sup_{\tilde{u} \in \mathcal{W}_{\tilde{R}}(u_0)} \left(\|\alpha_{C_4}(\tilde{u})\|_{L^2(0,T)} + \|\beta_{C_4}(\tilde{u})\|_{L^1(0,T)} \right) \\ \le &\sup_{\tilde{u} \in \mathcal{U}_{\tilde{R}(T+1)}(u_0)} \left(\|\alpha_{C_4}(\tilde{u})\|_{L^2(0,T)} + \|\beta_{C_4}(\tilde{u})\|_{L^1(0,T)} \right) \le M_{\tilde{R}(T+1)}(C_4). \end{aligned} \tag{4.2}$$

Repeating the argumentation similar to the proof of (1) in Lemma 3.5 and using (4.1), (4.2), we see that there exists a constant $C_7 > 0$, which also depends on $\tilde{R}$, such that

$$\begin{aligned} \sup_{\tilde{u} \in \mathcal{W}_{\tilde{R}}(u_0)} \Bigg(&\|(\mathcal{S}\tilde{u})'\|_{L^2(0,T;H)} + \sup_{0 \le t \le T} \|(\mathcal{S}\tilde{u})(t)\|_H \\ &+ \sup_{0 \le t \le T} \varphi(t, \tilde{u}, v_0\,; (\mathcal{S}\tilde{u})(t)) \Bigg) \le C_7. \end{aligned} \tag{4.3}$$

From (3.1) we get

$$\sup_{0 \le t \le T} \|g(t, \tilde{u}(t), \tilde{v}(t))\|_H^2 \le C_3^2 \left\{ \tilde{R} + C_1(\tilde{R} + 1) + c_4 \right\}. \tag{4.4}$$

Using (2) in Lemma 3.5 as $s = 0$, we get the following inequality for all $t \in [0, T]$:

$$\begin{aligned} &\left(\frac{3}{4} - 3\varepsilon \right) \int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_{\tilde{v}(s)}^2 ds + \varphi(t, \tilde{u}, v_0\,; (\mathcal{S}\tilde{u})(t)) \\ \le &\ \varphi(0, u_0, v_0\,; u_0) + c_2^2(3\varepsilon + 2) \int_0^t \left(\|f(s)\|_H^2 + \|g(s, \tilde{u}(s), \tilde{v}(s))\|_H^2 \right) ds \\ &+ C_8(\varepsilon) \int_0^t \{\varphi(s, \tilde{u}, v_0\,; (\mathcal{S}\tilde{u})(s)) + C_9\} \\ &\qquad \times \left(|\alpha_{C_4(\tilde{R})}(\tilde{u}\,; s)|^2 + \beta_{C_4(\tilde{R})}(\tilde{u}\,; s) + \|\tilde{v}'(s)\|_X \right) ds, \end{aligned}$$

hence, from Lemma 2.1, (4.3) and (4.4)

$$\begin{aligned}
&\left(\frac{3}{4}-3\varepsilon\right)\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|^2_{\tilde{v}(s)}ds + |\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| \qquad (4.5)\\
&\le |\varphi(0,u_0,v_0\,;u_0)| + C_1\|(\mathcal{S}\tilde{u})(t)\|_{\tilde{v}(t)} + C_1\\
&\quad + c_2^2(3\varepsilon+2)\int_0^t \|f(s)\|^2_H ds + (c_2C_3)^2(3\varepsilon+2)\left\{\tilde{R}+C_1(\tilde{R}+1)+c_4\right\}t\\
&\quad + C_8(\varepsilon)\left\{C_7(\tilde{R})+C_9\right\}\\
&\qquad\qquad \times\int_0^t \left(|\alpha_{C_4(\tilde{R})}(\tilde{u}\,;s)|^2 + \beta_{C_4(\tilde{R})}(\tilde{u}\,;s) + \|\tilde{v}'(s)\|_X\right)ds.
\end{aligned}$$

Using (a) in (A1), (4.5), Lemma 2.1 and the following inequality for all $t\in[0,T]$:

$$\begin{aligned}
\|(\mathcal{S}\tilde{u})(t)\|_{\tilde{v}(t)} &\le \int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_{\tilde{v}(t)}ds + \|u_0\|_{\tilde{v}(t)} \qquad (4.6)\\
&\le \frac{c_2}{c_1}\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_{\tilde{v}(s)}ds + c_2\|u_0\|_H\\
&\le \frac{\varepsilon}{C_1}\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|^2_{\tilde{v}(s)}ds + \frac{C_1}{4\varepsilon}\left(\frac{c_2}{c_1}\right)^2 t + c_2\|u_0\|_H,
\end{aligned}$$

we get the following inequality for all $t\in[0,T]$:

$$\begin{aligned}
&\left(\frac{3}{4}-4\varepsilon\right)\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|^2_{\tilde{v}(s)}ds + |\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| \qquad (4.7)\\
&\le |\varphi(0,u_0,v_0\,;u_0)| + C_1\left(c_2\|u_0\|_H+1\right) + c_2^2(3\varepsilon+2)\int_0^t \|f(s)\|^2_H ds\\
&\quad + \left[(c_2C_3)^2(3\varepsilon+2)\left\{\tilde{R}+C_1(\tilde{R}+1)+c_4\right\} + \frac{1}{4\varepsilon}\left(\frac{c_2C_1}{c_1}\right)^2\right]t\\
&\quad + C_8(\varepsilon)\left\{C_7(\tilde{R})+C_9\right\}\\
&\qquad\qquad \times\int_0^t \left(|\alpha_{C_4(\tilde{R})}(\tilde{u}\,;s)|^2 + \beta_{C_4(\tilde{R})}(\tilde{u}\,;s) + \|\tilde{v}'(s)\|_X\right)ds.
\end{aligned}$$

Choosing $\varepsilon=\varepsilon_0=\frac{1}{16}$ and a constant $C_{12}>0$ so that

$$\begin{aligned}
C_{12} \ge \max\Bigg\{&\frac{35c_2^2}{16},\ \frac{35(c_2C_3)^2\{\tilde{R}+C_1(\tilde{R}+1)+c_4\}}{16} + 4\left(\frac{c_2C_1}{c_1}\right)^2,\\
&C_8(\varepsilon_0)\left\{C_7(\tilde{R})+C_9\right\}\Bigg\},
\end{aligned}$$

we see from (a) in (A1) and (4.7) that the following estimate holds for all $t \in [0,T]$:

$$\begin{aligned}
&\frac{c_1^2}{2}\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_H^2 ds + |\varphi(t,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(t))| \\
&\le |\varphi(0,u_0,v_0\,;u_0)| + C_1\left(c_2\|u_0\|_H+1\right) + C_{12}\Big\{t + \int_0^t \Big(\|f(s)\|_H^2 \\
&\qquad + |\alpha_{C_4(\tilde{R})}(\tilde{u}\,;s)|^2 + \beta_{C_4(\tilde{R})}(\tilde{u}\,;s) + \|\tilde{v}'(s)\|_X\Big)\,ds\Big\}.
\end{aligned} \tag{4.8}$$

Using (a) in (A3) and the following inequality (cf. See (4.6));

$$\sup_{0\le s\le t}\|(\mathcal{S}\tilde{u})(s)\|_H \le \int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_H^2 ds + \frac{t}{4} + \|u_0\|_H,$$

we see from (4.8) that the following estimate holds for all $t \in [0,T]$:

$$\begin{aligned}
&\left(\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_H^2 ds\right)^{\frac{1}{2}} + \sup_{0\le s\le t}\|(\mathcal{S}\tilde{u})(s)\|_H + \sup_{0\le s\le t}\varphi((\mathcal{S}\tilde{u})(s)) \\
&\le 2\int_0^t \|(\mathcal{S}\tilde{u})'(s)\|_H^2 ds + \sup_{0\le s\le t}|\varphi(s,\tilde{u},v_0\,;(\mathcal{S}\tilde{u})(s))| + \frac{t}{4} + \|u_0\|_H + \frac{1}{4} \\
&\le \left(1+\frac{4}{c_1^2}\right)\{|\varphi(0,u_0,v_0\,;u_0)| + C_1(c_2\|u_0\|_H+1)\} + \|u_0\|_H + \frac{1}{4} + \frac{t}{4} \\
&\quad + C_{12}\left(1+\frac{4}{c_1^2}\right)\Big\{\int_0^t \Big(\|f(s)\|_H^2 + |\alpha_{C_4(\tilde{R})}(\tilde{u}\,;s)|^2 \\
&\qquad\qquad + \beta_{C_4(\tilde{R})}(\tilde{u}\,;s) + \|\tilde{v}'(s)\|_X\Big)\,ds + t\Big\}.
\end{aligned} \tag{4.9}$$

Using (b) in (A7), we see that there exists a time $T_0 \in (0,T]$ such that

$$\begin{aligned}
&\frac{T_0}{4} + C_{12}\left(1+\frac{4}{c_2^2}\right)\Bigg[\sup_{\tilde{u}\,\in\,\mathcal{W}_{\tilde{R}}(u_0)}\Big\{\int_0^{T_0}\Big(|\alpha_{C_4(\tilde{R})}(\tilde{u}\,;t)|^2 \\
&\qquad + \beta_{C_4(\tilde{R})}(\tilde{u}\,;t) + \|\tilde{v}'(t)\|_X\Big)\,dt\Big\} + \int_0^{T_0}\|f(s)\|_H^2 ds + T_0\Bigg] \le \tilde{R} - R_*,
\end{aligned} \tag{4.10}$$

where $R_* > 0$ is the same constant that is given in (A7). From (4.9) and (4.10) we get

$$\begin{aligned}
&\|((\Lambda(T_0)\circ\mathcal{S})\tilde{u})'\|_{L^2(0,T;H)} + \sup_{0\le t\le T}\|((\Lambda(T_0)\circ\mathcal{S})\tilde{u})(t)\|_H \\
&\qquad + \sup_{0\le t\le T}\varphi(((\Lambda(T_0)\circ\mathcal{S})\tilde{u})(t)) \le \tilde{R},
\end{aligned}$$

which implies that $(\Lambda(T_0)\circ\mathcal{S})\tilde{u} \in \mathcal{W}_{\tilde{R}}(u_0)$. □

Now we are ready for showing Theorem 1.2.

Proof of Theorem 1.2. We apply the Schauder fixed-point theorem to the operator $\Lambda(T_0)\circ\mathcal{S}$ from $\mathcal{W}_{\tilde{R}}(u_0)$ into itself. Then, we see from Lemmas 2.4, 3.6 and 4.1 that the operator $\Lambda(T_0)\circ\mathcal{S}$ has at least one fixed-point u, that is, $(\Lambda(T_0)\circ\mathcal{S})u = u \in \mathcal{W}_{\tilde{R}}(u_0) \subset \mathcal{V}_{\tilde{R}}(u_0)$. Because of $((\Lambda(T_0)\circ\mathcal{S})u)(t) = (\mathcal{S}u)(t) = u(t)$ in H for all $t \in [0, T_0]$, this fixed point u satisfies

$$\begin{cases} u'(t) + \partial_{v(t)}\varphi(t, u, v_0; u(t)) + g(t, u(t), v(t)) \ni f(t) \\ \qquad\qquad \text{in} \quad H(v(t)), \quad \text{a.a. } t \in (0, T_0), \\ v(t) = S(u\,; t, 0)v_0 \quad \text{in} \quad A, \quad \forall t \in [0, T_0], \\ u(0) = u_0 \quad \text{in} \quad H. \end{cases}$$

In the rest of this proof, we show the boundedness (1.1) in Definition 1.1. Using $(\mathcal{S}u)(t) = u(t)$ in H for all $t \in [0, T_0]$ again and $(\Lambda(T_0)\circ\mathcal{S})u \in \mathcal{V}_{\tilde{R}}(u_0) \subset \mathcal{U}_{\tilde{R}(T+1)}(u_0)$, we have

$$\begin{aligned} &\sup_{0\le t\le T_0} \|(\mathcal{S}u)(t)\|_H + \sup_{0\le t\le T_0} \varphi((\mathcal{S}u)(t)) \\ &\qquad = \sup_{0\le t\le T} \|u(t)\|_H + \sup_{0\le t\le T} \varphi\left(u(t)\right) \le \tilde{R}. \end{aligned} \tag{4.11}$$

Using the second type energy inequality in Lemma 3.4, we see from (4.11) that there exists a constant $C_{13} > 0$ and $C_{14} > 0$, which depends on $\|u_0\|_H$, $\|v_0\|_X$, $\tilde{R}$ and T, such that the following inequality holds for a.a. $t \in (0, T_0)$:

$$\begin{aligned} &\frac{d}{dt}\varphi(t, u, v_0\,; u(t)) + (u'(t), u'(t) + g(t, u(t), v(t)) - f(t))_{v(t)} \\ &\le C_{13}\|f(t) - g(t, u(t), v(t)) - u'(t)\|_{v(t)}\big(\sqrt{|\varphi(t, u, v_0\,; u(t))|} + 1\big)\alpha_{\tilde{R}}(u\,; t) \\ &\qquad + C_{14}\left(|\varphi(t, u, v_0\,; u(t))| + 1\right)\left(\beta_{\tilde{R}}(u\,; t) + \|v'(t)\|_X\right). \end{aligned} \tag{4.12}$$

In order to show (1.1), we use the following estimates.
(i) From Lemma 2.1 and (A4), (A9) we get the following estimate for a.a. $t \in (0, T_0)$:

$$\begin{aligned} &C_{13}\|f(t) - g(t, u(t), v(t))\|_{v(t)}\big(\sqrt{|\varphi(t, u, v_0\,; u(t))|} + 1\big)\alpha_{\tilde{R}}(u\,; t) \\ &\le \frac{c_2^2 C_{13}^2}{2}\left(\|f(t)\|_H + \|g(t, u(t), v(t))\|_H\right)^2 \\ &\quad + \frac{1}{2}\big(\sqrt{|\varphi(t, u, v_0\,; u(t))|} + 1\big)^2|\alpha_{\tilde{R}}(u\,; t)|^2 \\ &\le c_2^2 C_{13}^2\left(\|f(t)\|_H^2 + \|g(t, u(t), v(t))\|_H^2\right) + \left(|\varphi(t, u, v_0\,; u(t))| + 1\right)|\alpha_{\tilde{R}}(u\,; t)|^2 \\ &\le \left\{\varphi(t, u, v_0\,; u(t)) + C_1\left(\|u(t)\|_{v(t)} + 1\right)\right\}|\alpha_{\tilde{R}}(u\,; t)|^2 \\ &\quad + c_2^2 C_{13}^2\|f(t)\|_H^2 + c_2^2 C_{13}^2\left\{\phi(v(t))\right\}^2\left\{\varphi(u(t)) + C_1\left(\|u(t)\|_H + 1\right)\right\} \\ &\le |\alpha_{\tilde{R}}(u\,; t)|^2\varphi(t, u, v_0\,; u(t)) + C_1\left(c_2 R_1 + 1\right)|\alpha_{\tilde{R}}(u\,; t)|^2 \\ &\quad + c_2^2 C_{13}^2\|f(t)\|_H^2 + c_2^2 C_{13}^2 C_3^2\{\tilde{R} + C_1(\tilde{R} + 1)\} \\ &\le |\alpha_{\tilde{R}}(u\,; t)|^2\varphi(t, u, v_0\,; u(t)) + C_{15}\left(|\alpha_{\tilde{R}}(u\,; t)|^2 + \|f(t)\|_H^2 + 1\right), \end{aligned}$$

where the constant $C_{15} > 0$ is given by

$$C_{15} := C_1(c_2R_1 + 1) + c_2^2C_{13}^2 + c_2^2C_{13}^2C_3^3\big\{\tilde{R} + C_1\big(\tilde{R} + 1\big)\big\}.$$

(ii) From (A4) and (A9) we get the following estimate for a.a. $t \in (0, T_0)$:

$$\begin{aligned}
&(u'(t), f(t) - g(t, u(t), v(t)))_{v(t)} \\
&\leq \frac{1}{2}\|u'(t)\|_{v(t)}^2 + \|g(t, u(t), v(t))\|_{v(t)}^2 + \|f(t)\|_{v(t)}^2 \\
&\leq \frac{1}{2}\|u'(t)\|_{v(t)}^2 + c_2^2\left\{\phi(v(t))\right\}^2\left\{\varphi(u(t)) + C_1\left(\|u(t)\|_H + 1\right)\right\} + \|f(t)\|_{v(t)}^2 \\
&\leq \frac{1}{2}\|u'(t)\|_{v(t)}^2 + c_2^2C_3^2\big\{\tilde{R} + C_1\big(\tilde{R} + 1\big)\big\} + \|f(t)\|_{v(t)}^2 \\
&\leq \frac{1}{2}\|u'(t)\|_{v(t)}^2 + C_{16}\left(\|f(t)\|_H^2 + 1\right),
\end{aligned}$$

where the constant $C_{16} > 0$ is given by

$$C_{16} := c_2^2C_3^2\big\{\tilde{R} + C_1\big(\tilde{R} + 1\big)\big\} + 1.$$

(iii) Repeating the argumentation similar to (i), we get the following inequality for a.a. $t \in (0, T_0)$:

$$\begin{aligned}
&C_{13}\|u'(t)\|_{v(t)}\big(\sqrt{|\varphi(t, u, v_0\,; u(t))|} + 1\big)\alpha_{\tilde{R}}(u\,; t) \\
&\leq \frac{1}{4}\|u'(t)\|_{v(t)}^2 + C_{13}^2\big(\sqrt{|\varphi(t, u, v_0\,; u(t))|} + 1\big)^2|\alpha_{\tilde{R}}(u\,; t)|^2 \\
&\leq \frac{1}{4}\|u'(t)\|_{v(t)}^2 + 2C_{13}^2\left(|\varphi(t, u, v_0\,; u(t))| + 1\right)|\alpha_{\tilde{R}}(u\,; t)|^2 \\
&\leq \frac{1}{4}\|u'(t)\|_{v(t)}^2 + 2C_{13}^2\left\{\varphi(t, u, v_0\,; u(t)) + C_1\left(\|u(t)\|_{v(t)} + 1\right)\right\}|\alpha_{\tilde{R}}(u\,; t)|^2 \\
&\leq \frac{1}{4}\|u'(t)\|_{v(t)}^2 + 2C_{13}^2|\alpha_{\tilde{R}}(u\,; t)|^2\varphi(t, u, v_0\,; u(t)) \\
&\quad + 2C_1C_{13}^2\big(c_2\tilde{R} + 1\big)|\alpha_{\tilde{R}}(u\,; t)|^2.
\end{aligned}$$

(iv) From Lemma 2.1 again we get the following inequality for a.a. $t \in (0, T_0)$:

$$\begin{aligned}
&C_{14}\left(|\varphi(t, u, v_0\,; u(t))| + 1\right)\left(\beta_{\tilde{R}}(u\,; t) + \|v'(t)\|_X\right) \\
&\leq C_{14}\left\{\varphi(t, u, v_0\,; u(t)) + C_1\left(\|u(t)\|_{v(t)} + 1\right)\right\}\left(\beta_{\tilde{R}}(u\,; t) + \|v'(t)\|_X\right) \\
&\leq C_{14}\left(\beta_{\tilde{R}}(u\,; t) + \|v'(t)\|_X\right)\varphi(t, u, v_0\,; u(t)) \\
&\quad + C_1C_{14}\big(c_2\tilde{R} + 1\big)\left(\beta_{\tilde{R}}(u\,; t) + \|v'(t)\|_X\right).
\end{aligned}$$

Substituting all estimates in (i)–(iv) into (4.12), we get the following inequality for a.a. $t \in (0, T_0)$:

$$\begin{aligned}
&\frac{d}{dt}\varphi(t, u, v_0\,; u(t)) + \frac{1}{4}\|u'(t)\|_{v(t)}^2 \qquad (4.13)\\
&\leq \big(2C_{13}^2 + C_{14} + 1\big)\left(|\alpha_{\tilde{R}}(u\,; t)|^2 + \beta_{\tilde{R}}(u\,; t) + \|v'(t)\|_X\right)\varphi(t, u, v_0\,; u(t)) \\
&\quad + C_{17}\left(|\alpha_{\tilde{R}}(u\,; t)|^2 + \beta_{\tilde{R}}(u\,; t) + \|v'(t)\|_X + \|f(t)\|_H^2 + 1\right),
\end{aligned}$$

where the constant $C_{17} > 0$ is given by

$$C_{17} := C_{15} + C_{16} + \big(2C_1C_{13}^2 + C_1C_{14}\big)\big(c_2\tilde{R} + 1\big).$$

We apply the Gronwall lemma to (4.13), and use (A7) and (A9). Then, we see that the following inequality holds for all $t \in [0, T_0]$:

$$\begin{aligned}
&\varphi(t, u, v_0\,; u(t)) + \frac{c_1^2}{4}\int_0^t \|u'(s)\|_H^2 ds \\
&\le \Bigg\{\varphi(0, u, v_0\,; u_0) \\
&\qquad + C_{17}\int_0^{T_0} \big(|\alpha_{\tilde{R}}(u\,;t)|^2 + \beta_{\tilde{R}}(u\,;t) + \|v'(t)\|_X + \|f(t)\|_H^2 + 1\big)\, dt\Bigg\} \\
&\quad \times \exp\Bigg((2C_{13}^2 + C_{14} + 1)\int_0^{T_0} \big(|\alpha_{\tilde{R}}(u\,;t)|^2 + \beta_{\tilde{R}}(u\,;t) + \|v'(t)\|_X\big)\, dt\Bigg) \\
&\le \Bigg\{|\varphi(0, u_0, v_0\,; u_0)| + C_{17}\Bigg(M_{R(T+1)}(\tilde{R}) + T_0 + \int_0^T \|f(t)\|_H^2 dt\Bigg)\Bigg\} \\
&\qquad \times \exp\Big((2C_{13}^2 + C_{14} + 1)\{M_{\tilde{R}(T+1)}(\tilde{R}) + C_3\}\Big) =: C_{18},
\end{aligned}$$

which implies that the following boundedness holds:

$$\sup_{0\le t\le T_0} \varphi(t, u, v_0\,; u(t)) + \int_0^{T_0} \|u'(t)\|_H^2 dt \le \left(\frac{4}{c_1^2} + 1\right) C_{18}.$$

Using Lemma 2.1, we get the boundedness (1.1), hence, u is a strong solution of (P) on $[0, T_0]$. □

Remark 4.2. As you see from the proof of Theorem 1.2, it is enough that (A7) is satisfied for some constant $\tilde{R} > R_*$ in order to show Theorem 1.2. This result is the same to the case without a perturbation, which is obtained in [7, Section 4]. Moreover, the boundedness of φ in (a) in (A3), which is given by (2.1), is not necessary for showing Theorem 1.2.

5. Proof of Theorem 1.3

In this section, we use the argumentation similar to that in [7, Section 5]. At the beginning of this section, we define a set $\mathcal{Z}$ by

$$\begin{aligned}
\mathcal{Z} &:= \big\{(\bar{u}, \bar{v}, \bar{T}) := (\bar{u}, S(\bar{u}\,;\cdot, 0)v_0, \bar{T})\,;\ \bar{u} \text{ is a strong solution of (P) on } [0, \bar{T}]\big\} \\
&\subset \bigcup_{0\le \bar{T}\le T} \big(W^{1,2}(0, \bar{T}; H) \cap \mathcal{U}(u_0)\big) \times W^{1,1}(0, \bar{T}; X) \times \{\bar{T}\},
\end{aligned}$$

and induce an order relation $\preceq$ on $\mathcal{Z}$ by

$$\begin{aligned}
&(\bar{u}_1, \bar{v}_1, \bar{T}_1) \preceq (\bar{u}_2, \bar{v}_2, \bar{T}_2) \\
&\qquad \text{iff} \quad 0 \le \bar{T}_1 \le \bar{T}_2 \le T \quad \text{and} \quad \bar{u}_1 = \bar{u}_2 \quad \text{in} \quad W^{1,2}(0, \bar{T}_1; H).
\end{aligned}$$

From (A2) we have $\bar{v}_1 = \bar{v}_2$ in $W^{1,1}(0,\bar{T}_1;X)$ whenever $(\bar{u}_1,\bar{v}_1,\bar{T}_1) \preceq (\bar{u}_2,\bar{v}_2,\bar{T}_2)$. Moreover, from Theorem 1.2 we have $\mathcal{Z} \neq \emptyset$.

Lemma 5.1. *The ordered set $(\mathcal{Z},\preceq)$ is inductively ordered. That is, any linearly ordered subset $\mathcal{Y}$ of $\mathcal{Z}$ is bounded above.*

Proof. Let $\mathcal{Y}$ be any linearly ordered subset of $\mathcal{Z}$. We define a triplet $(\hat{u},\hat{v},\hat{T})$ by

$$\hat{T} := \sup\left\{\bar{T} \in (0,T]\,;\, (\bar{u},\bar{v},\bar{T}) \in \mathcal{Y}\right\},$$

$$(\hat{u}(t),\hat{v}(t)) = (\bar{u}(t),\bar{v}(t)), \quad \forall t \in [0,\bar{T}] \quad \text{iff} \quad (\bar{u},\bar{v},\bar{T}) \in \mathcal{Y} \text{ and } \bar{T} \in [0,\hat{T}).$$

It is clear that the triplet $(\hat{u},\hat{v},\hat{T})$ is uniquely determined and $\hat{u}$ is a strong solution of (P) on $[0,\hat{T})$. From the definition of $\hat{T}$ we can take out a sequence $\{(\bar{u}_m,\bar{v}_m,\bar{T}_m)\}_{m\in\mathbb{N}} \subset \mathcal{Y}$ such that

$$\bar{T}_m \nearrow \hat{T} \quad \text{as} \quad m \to \infty,$$

$$\bar{u}_m'(t) + \partial_{\bar{v}_m(t)}\varphi(t,\bar{u}_m,v_0\,;\bar{u}_m(t)) + g(t,\bar{u}_m(t),\bar{v}_m(t)) \ni f(t) \tag{5.1}$$
$$\text{in} \quad H(\bar{v}_m(t)), \text{ a.a. } t \in (0,\bar{T}_m),$$

$$\bar{v}_m(t) = S(\bar{u}_m;t,0)v_0, \quad \forall t \in [0,\bar{T}_m], \tag{5.2}$$
$$\bar{u}_m(0) = u_0 \quad \text{in} \quad H.$$

In the following argumentation, for each $m \in \mathbb{N}$ we define a function $\tilde{u}_m \in C([0,T];H)$ by

$$\tilde{u}_m(T) := \begin{cases} \bar{u}_m(t) & \text{if} \quad t \in [0,\bar{T}_m], \\ \bar{u}_m(\bar{T}_m) & \text{if} \quad t \in (\bar{T}_m,T]. \end{cases}$$

For any $m \in \mathbb{N}$ we see from (e) in (A2) that the following equalities hold for all $t \in [0,\bar{T}_m]$:

$$\begin{cases} \bar{u}_m(t) = \tilde{u}_m(t) = (\mathcal{S}\tilde{u}_m)(t) = \hat{u}(t) \quad \text{in} \quad H, \\ \bar{v}_m(t) = \tilde{v}_m(t) = S(\tilde{u}_m\,;t,0)v_0 = \hat{v}(t) \quad \text{in} \quad A. \end{cases} \tag{5.3}$$

Using (a) in (A3), Theorem 1.2 and Definition 1.1, we see that there exists a sequence $\{C_m^*\}_{m\in\mathbb{N}}$ such that for any $m \in \mathbb{N}$ the following inequality holds:

$$\begin{aligned} &\|\tilde{u}_m'\|_{L^2(0,T;H)} + \sup_{0\le t\le T}\|\tilde{u}_m(t)\|_H + \sup_{0\le t\le T}\varphi(\tilde{u}_m(t)) \\ &\le \|\bar{u}_m'\|_{L^2(0,\bar{T}_m;H)} + \sup_{0\le t\le \bar{T}_m}\|\bar{u}_m(t)\|_H + \sup_{0\le t\le \bar{T}_m}|\varphi(t,\bar{u}_m,v_0\,;\bar{u}_m(t))| \le C_m^*, \end{aligned}$$

which implies $\tilde{u}_m \in \mathcal{W}_{C_m^*}(u_0) \subset \mathcal{U}(u_0)$. Hence, from (5.2) and (A8), (A9) we get

$$\begin{aligned} \sup_{m\in\mathbb{N}}\Bigg\{&\|h'(\tilde{u}_m)\|_{L^2(0,T;H)}^2 + \sup_{0\le t\le T}\|h(\tilde{u}_m;t)\|_H \\ &+ \sup_{0\le t\le T}|\varphi(t,\tilde{u}_m,v_0\,;h(\tilde{u}_m;t))|\Bigg\} \le C_2, \end{aligned} \tag{5.4}$$

$$\sup_{m\in\mathbb{N}}\left(\sup_{0\le t\le T}\ell(\bar{v}_m(t)) + \|\bar{v}_m'\|_{L^1(0,T;X)}\right) \le C_3. \tag{5.5}$$

In the following argumentation, we consider a sequence $\{\mathcal{S}\tilde{u}_m\}_{m\in\mathbb{N}}$ and repeat the similar argumentation to the derivation of (3.3). Then, we get the following inequality for a.a. $t \in (0,T)$:

$$\begin{aligned}
&\frac{d}{dt}\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)} + 2\varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t)) \qquad (5.6)\\
&\le c_3\|\tilde{v}_m'(t)\|_X\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)} + 2\varphi(t,\tilde{u}_m,v_0\,;h(\tilde{u}_m;t))\\
&\quad + 2(f(t) - h'(\tilde{u}_m;t) - g(t,\tilde{u}_m(t),\tilde{v}_m(t)),(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t))_{\tilde{v}_m(t)}\\
&\le \left(c_3\|\tilde{v}_m'(t)\|_X + 2\right)\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)} + 2|\varphi(t,\tilde{u}_m,v_0\,;h(\tilde{u}_m;t)|\\
&\quad + 2c_2\|g(t,\tilde{u}_m(t),\tilde{v}_m(t))\|_H\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|_{\tilde{v}_m(t)}\\
&\quad + c_2^2\left(\|f(t)\|_H^2 + \|h'(\tilde{u}_m;t)\|_H^2\right).
\end{aligned}$$

We see from (2.1) in (A3), (A4), (A8), (A9), (5.2), (5.3) and Lemma 2.1 that the following inequality holds for all $t \in [0,T]$:

$$\begin{aligned}
&2c_2\|g(t,\tilde{u}_m(t),\tilde{v}_m(t))\|_H\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|_{\tilde{v}_m(t)} \qquad (5.7)\\
&\le 2c_2\ell(\tilde{v}_m(t))\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|_{\tilde{v}_m(t)}\sqrt{|\varphi(\tilde{u}_m(t))| + c_4}\\
&\le c_2^2C_3^3\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)} + \varphi^* + c_4.
\end{aligned}$$

Substituting (5.7) into (5.6), we get the following inequality for all $m \in \mathbb{N}$ and a.a. $t \in (0,T)$:

$$\begin{aligned}
&\frac{d}{dt}\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)} + 2\varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t))\\
&\le \left(c_3\|\tilde{v}_m'(t)\|_X + c_2^2C_3^2 + 2\right)\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)}\\
&\quad + c_2^2\left(\|f(t)\|_H^2 + \|h'(\tilde{u}_m;t)\|_H^2\right) + 2C_2 + \varphi^* + c_4,
\end{aligned}$$

hence, from Lemma 2.1 again

$$\begin{aligned}
&\frac{d}{dt}\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)} + 2|\varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t))| \qquad (5.8)\\
&\le \left(c_3\|\tilde{v}_m'(t)\|_X + c_2^2C_3^2 + 2\right)\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)}\\
&\quad + 2C_1\|(\mathcal{S}\tilde{u}_m)(t)\|_{\tilde{v}_m(t)} + c_2^2\left(\|f(t)\|_H^2 + \|h'(\tilde{u}_m;t)\|_H^2\right)\\
&\quad + 2C_1 + 2C_2 + \varphi^* + c_4\\
&\le \left(c_3\|\tilde{v}_m'(t)\|_X + c_2^2C_3^2 + 3\right)\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)}\\
&\quad + 2c_2C_1\|h(\tilde{u}_m;(t)\|_H + c_2^2\left(\|f(t)\|_H^2 + \|h'(\tilde{u}_m;t)\|_H^2\right)\\
&\quad + C_1^2 + 2C_1 + 2C_2 + \varphi^* + c_4\\
&\le \left(c_3\|\tilde{v}_m'(t)\|_X + c_2^2C_3^2 + 3\right)\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)}\\
&\quad + c_2^2\left(\|f(t)\|_H^2 + \|h'(\tilde{u}_m;t)\|_H^2\right) + 2c_2C_1C_2\\
&\quad + C_1^2 + 2C_1 + 2C_2 + \varphi^* + c_4\\
&\le C_{19}\left(\|\tilde{v}_m'(t)\|_X + 1\right)\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m;t)\|^2_{\tilde{v}_m(t)}\\
&\quad + C_{20}\left(\|f(t)\|_H^2 + \|h'(\tilde{u}_m;t)\|_H^2 + 1\right),
\end{aligned}$$

where the constants $C_{19} > 0$ and $C_{20} > 0$ are given by

$$C_{19} := c_3 + c_2^2 C_3^2 + 3, \quad C_{20} := c_2^2 + 2c_2 C_1 C_2 + C_1^2 + 2C_1 + 2C_2 + \varphi^* + c_4.$$

Applying the Gronwall lemma to (5.8), we get the following inequality for all $m \in \mathbb{N}$ and all $t \in [0, T]$:

$$\begin{aligned}
&\|(\mathcal{S}\tilde{u}_m)(t) - h(\tilde{u}_m; t)\|_{\tilde{v}_m(t)}^2 + 2\int_0^t |\varphi(s, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(s))| ds \\
&\le \|u_0 - h(\tilde{u}_m; 0)\|_{v_0}^2 \exp\left(C_{19}\int_0^T (\|\tilde{v}_m'(t)\|_X + 1)\, ds\right) \\
&\quad + C_{20}\int_0^T \left(\|f(t)\|_H^2 + \|h'(\tilde{u}_m; t)\|_H^2 + 1\right) dt \\
&\qquad\qquad \times \exp\left(C_{19}\int_0^T (\|\tilde{v}_m'(t)\|_X + 1)\, ds\right),
\end{aligned}$$

hence, from (A8) and (A9) (cf. (5.4) and (5.5), respectively)

$$\begin{aligned}
&\frac{c_1^2}{2}\|(\mathcal{S}\tilde{u}_m)(t)\|_H^2 + 2\int_0^t |\varphi(s, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(s))| ds \\
&\le e^{C_{19}(C_3+T)}\Big\{c_2^2\left(\|u_0\|_H + C_2\right)^2 \\
&\qquad + C_{20}\left(\|f\|_{L^2(0,T;H)}^2 + C_2 + T\right)\Big\} + c_2^2 C_2^2 =: C_{21}.
\end{aligned}$$

Hence, we get the following uniform estimate:

$$\sup_{m\in\mathbb{N}}\left(\sup_{0\le t\le T}\|(\mathcal{S}\tilde{u}_m)(t)\|_H + \int_0^T |\varphi(t, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(t))| dt\right) \le \frac{C_{21}}{2} + \frac{\sqrt{2C_{21}}}{c_1}.$$

From (5.3) we get

$$\sup_{m\in\mathbb{N}}\left(\sup_{0\le t\le \bar{T}_m}\|\bar{u}_m(t)\|_H + \int_0^{\bar{T}_m} \varphi(\bar{u}_m(t)) dt\right) \le \frac{C_{21}}{2} + \frac{\sqrt{2C_{21}}}{c_1},$$

hence, from (a) in (A3)

$$\sup_{m\in\mathbb{N}}\left(\sup_{0\le t\le T}\|\tilde{u}_m(t)\|_H + \int_0^T \varphi(\tilde{u}_m(t)) dt\right) \le \frac{C_{21}}{2} + \frac{\sqrt{2C_{21}}}{c_1} + \varphi^* T =: R_2, \quad (5.9)$$

which implies $\{\tilde{u}_m\}_{m\in\mathbb{N}} \subset \mathcal{U}_{R_2}(u_0)$.

Next, we see from (2) in Lemma 3.5 and (5.3), (5.9) that for any $\varepsilon > 0$ there exist constants $C_{22}(\varepsilon) > 0$ and $C_{23} > 0$ such that the following inequality holds

for all $m \in \mathbb{N}$ and all s, $t \in [0,T]$ with $s \le t$:

$$\begin{aligned}
&\varphi(t, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(t)) \\
&\quad + \int_s^t ((\mathcal{S}\tilde{u}_m)'(\tau), (\mathcal{S}\tilde{u}_m)'(\tau) + g(\tau, \tilde{u}_m(\tau), \tilde{v}_m(\tau)) - f(\tau))_{\tilde{v}_m(\tau)} d\tau \\
&\le \varphi(s, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(s)) \\
&\quad + \varepsilon \int_s^t \|(\mathcal{S}\tilde{u}_m)'(\tau) + g(\tau, \tilde{u}_m(\tau), \tilde{v}_m(\tau)) - f(\tau)\|^2_{\tilde{v}_m(\tau)} d\tau \\
&\quad + C_{22}(\varepsilon) \int_s^t \{\varphi(\tau, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(\tau)) + C_{23}\} \\
&\qquad\qquad \times \left(|\alpha_{R_2}(\tilde{u}_m; \tau)|^2 + \beta_{R_2}(\tilde{u}_m; \tau) + \|\tilde{v}_m'(\tau)\|_X\right) d\tau,
\end{aligned}$$

hence, from (A9)

$$\begin{aligned}
&\varphi(t, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(t)) - \varphi(s, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(s)) \\
&+ (1 - 4\varepsilon) \int_s^t \|(\mathcal{S}\tilde{u}_m)'(\tau)\|^2_{\tilde{v}(\tau)} d\tau \\
&- C_{22}(\varepsilon) \int_s^t \{\varphi(\tau, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(\tau)) + C_{23}\} \\
&\qquad\qquad \times \left(|\alpha_{R_2}(\tilde{u}_m; \tau)|^2 + \beta_{R_2}(\tilde{u}_m; \tau) + \|\tilde{v}_m'(\tau)\|_X\right) d\tau \\
&\le c_2^2 \left(2\varepsilon + \frac{1}{4\varepsilon}\right) \int_s^t \left(\|f(\tau)\|_H^2 + \|g(\tau, \tilde{u}_m(\tau), \tilde{v}_m(\tau))\|_H^2\right) d\tau \\
&\le c_2^2 \left(2\varepsilon + \frac{1}{4\varepsilon}\right) \int_s^t \left\{\|f(\tau)\|_H^2 + \left(\sup_{0 \le t \le T} \ell(\tilde{v}_m(t))\right)^2 (|\varphi(\tilde{u}_m(\tau))| + c_4)\right\} d\tau \\
&\le c_2^2 \left(2\varepsilon + \frac{1}{4\varepsilon}\right) \int_s^t \left\{\|f(\tau)\|_H^2 + C_3^2(\varphi^* + c_4)\right\} d\tau.
\end{aligned} \tag{5.10}$$

Taking $\varepsilon = \frac{1}{8}$ and putting $\bar{C}_{22} := C_{22}(\frac{1}{8})$ in (5.10), we get the following inequality for a.a. $t \in (0,T)$:

$$\begin{aligned}
&\frac{1}{2}\|(\mathcal{S}\tilde{u}_m)'(t)\|^2_{\tilde{v}_m(t)} + \frac{d}{dt}\varphi(t, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(t)) \\
&\le \bar{C}_{22} \left(|\alpha_{R_2}(\tilde{u}_m; t)|^2 + \beta_{R_2}(\tilde{u}_m; t) + \|\tilde{v}_m'(t)\|_X\right) \varphi(t, \tilde{u}_m, v_0\,; (\mathcal{S}\tilde{u}_m)(t)) \\
&\quad + C_{24} \left(|\alpha_{R_2}(\tilde{u}_m; t)|^2 + \beta_{R_2}(\tilde{u}_m; t) + \|\tilde{v}_m'(t)\|_X + \|f(t)\|_H^2 + 1\right),
\end{aligned} \tag{5.11}$$

where $C_{24} > 0$ is given by

$$C_{24} := \bar{C}_{22} C_{23} + \frac{9c_2^2\{1 + C_3^3(\varphi^* + c_4)\}}{4}.$$

Applying the Gronwall lemma to (5.11), we see from (5.3) that the following

inequality holds for all $t \in [0,T]$ and $m \in \mathbb{N}$:

$$\begin{aligned}
&\frac{1}{2}\int_0^t \|(\mathcal{S}\tilde{u}_m)'(s)\|^2_{\bar{v}_m(s)}ds + \varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t)) \\
\leq\ & \exp\left(C_{24}\int_0^T \left(|\alpha_{R_2}(\tilde{u}_m;t)|^2 + \beta_{R_2}(\tilde{u}_m;t) + \|\tilde{v}_m'(t)\|_X\right)dt\right)\Big\{|\varphi(0,u_0,v_0\,;u_0)| \\
&+ C_{24}\left(\int_0^T \left(\|f(t)\|_H^2 + |\alpha_{R_2}(\tilde{u}_m;t)|^2 + \beta_{R_2}(\tilde{u}_m;t) + \|\tilde{v}_m'(t)\|_X + 1\right)dt\right)\Big\}.
\end{aligned}$$

Since from (A7) with (5.9) and (A9) we get the following uniform estimate:

$$\begin{aligned}
\sup_{m\in\mathbb{N}}\Big(&\|\alpha_{R_2}(\tilde{u}_m)\|^2_{L^2(0,T)} + \|\beta_{R_2}(\tilde{u}_m)\|_{L^1(0,T)} \\
&+ \|\tilde{v}_m'\|_{L^1(0,T;X)}\Big) \leq M_{R_2}(R_2) + C_3 =: C_{25},
\end{aligned}$$

we see from (5.5) and (5.9) that the following inequality holds for all $m \in \mathbb{N}$ and all $t \in [0,T]$:

$$\frac{1}{2}\int_0^t \|(\mathcal{S}\tilde{u}_m)'(s)\|^2_{\bar{v}_m(s)}ds + \varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t)) \leq C_{26}, \tag{5.12}$$

where $C_{18} > 0$ is given by

$$C_{26} := e^{C_{24}C_{25}}\left\{|\varphi(0,u_0,v_0\,;u_0)| + C_{24}\left(\|f\|^2_{L^2(0,T;H)} + C_{25} + T\right)\right\}.$$

The estimate (5.12) gives the following uniform estimates (cf. See (3.11) and (3.12)):

$$\begin{aligned}
\sup_{m\in\mathbb{N}}\Big(&\int_0^T \|(\mathcal{S}\tilde{u}_m)'(t)\|^2_{\bar{v}_m(t)}dt \\
&+ \sup_{0\leq t\leq T}|\varphi(t,\tilde{u}_m,v_0\,;(\mathcal{S}\tilde{u}_m)(t))|\Big) \leq 3\left\{C_{26} + C_1\left(c_2R_2+1\right)\right\},
\end{aligned}$$

hence, from (5.3) we get

$$\sup_{m\in\mathbb{N}}\left(\|\tilde{u}_m'\|_{L^2(0,T;H)} + \sup_{0\leq t\leq T}\|\tilde{u}_m(t)\|_H + \sup_{0\leq t\leq T}\varphi(\tilde{u}_m(t))\right) \leq R_3, \tag{5.13}$$

where $R_3 > 0$ is given by

$$R_3 := R_2 + 3\left\{C_{26} + C_1\left(c_2R_2+1\right)\right\} + \frac{\sqrt{3\left\{C_{26} + C_1\left(c_2R_2+1\right)\right\}}}{c_1}.$$

which implies $\{\tilde{u}_m\}_{m\in\mathbb{N}} \subset \mathcal{W}_{R_3}(u_0)$.

Now, we see from Lemma 2.4 that there exist a subsequence $\{\tilde{u}_{m_k}\}_{k\in\mathbb{N}}$ of $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ and an element $\tilde{u} \in \mathcal{W}_{R_3}(u_0)$ such that

$$\tilde{u}_{m_k} = (\bar{u}_{m_k})_{\bar{T}_{m_k}} \longrightarrow \tilde{u} \quad \text{in} \quad C([0,T];H) \quad \text{as} \quad k\to\infty. \tag{5.14}$$

Using (b) in (A2), (b) in (A4) and Lemma 3.6, we see from (5.14) that the following convergences hold as $k \to \infty$:

$$\tilde{v}_{m_k} \longrightarrow S(\tilde{u}\,;\cdot,0)v_0 =: \tilde{v} \quad \text{in} \quad C([0,T];X), \tag{5.15}$$

$$g(\tilde{u}_{m_k}, \tilde{v}_{m_k}) \longrightarrow g(\tilde{u}, \tilde{v}) \quad \text{weakly in} \quad L^2(0,T;H), \tag{5.16}$$

$$\mathcal{S}\tilde{u}_{m_k} \longrightarrow \mathcal{S}\tilde{u} \quad \text{in} \quad C([0,T];H). \tag{5.17}$$

From (5.3), (5.14) and (5.17) we get

$$\mathcal{S}\tilde{u}(t) = \tilde{u}(t) = \hat{u}(t) \quad \text{in} \quad H, \quad \forall t \in [0,\hat{T}), \tag{5.18}$$

$$\tilde{u}(t) = \tilde{u}(\hat{T}) \quad \text{in} \quad H, \quad \forall t \in [\hat{T},T], \tag{5.19}$$

hence,

$$\hat{u}(t) \longrightarrow \tilde{u}(\hat{T}) \quad \text{in} \quad H \quad \text{as} \quad t \nearrow \hat{T}. \tag{5.20}$$

Finally, from the definition of $\mathcal{S}$ and (5.14)–(5.20) we have

$$\begin{cases} \tilde{u}'(t) + \partial_{\tilde{v}(t)}\varphi(t,\tilde{u},v_0\,;\tilde{u}(t)) \ni f(t) - g(\tilde{u}(t),\tilde{v}(t)) \\ \qquad\qquad \text{in} \quad H(\tilde{v}(t)), \quad \text{a.a. } t \in (0,\hat{T}), \\ \tilde{v}(t) = S(\tilde{u}\,;t,0)v_0 \quad \text{in} \quad X, \quad \forall t \in [0,\hat{T}], \\ \tilde{u}(0) = u_0 \quad \text{in} \quad H, \end{cases}$$

with the following estimate, which is derived from (5.13), (5.14) and implies $\tilde{u} \in \mathcal{V}_{R_3}(u_0)$:

$$\sup_{0\le t\le \hat{T}} \|\tilde{u}(t)\|_H + \sup_{0\le t\le \hat{T}} \varphi(\tilde{u}(t)) = \sup_{0\le t\le T} \|\tilde{u}(t)\|_H + \sup_{0\le t\le T} \varphi(\tilde{u}(t)) \le R_3.$$

Repeating the argumentation similar to the proof of Lemma 3.5, we see that there exists a constant $R_4 > 0$ such that

$$\|\tilde{u}'\|_{L^2(0,\hat{T};H)} + \sup_{0\le t\le \hat{T}} \|\tilde{u}(t)\|_H + \sup_{0\le t\le \hat{T}} |\varphi(t,\tilde{u},v_0\,;\tilde{u}(t))| \le R_4.$$

Hence $\tilde{u}$ is a strong solution of (P) on $[0,\hat{T}]$, that is, the triplet $(\tilde{u},\tilde{v},\hat{T})$ is an upper bound of $\mathcal{Y}$. □

Now we are ready for giving a proof of Theorem 1.3.

Proof of Theorem 1.3. Applying the Zorn lemma, we see from Lemma 5.1 that the inductively ordered set $(\mathcal{Z},\preceq)$ has at least one maximal element (u^*,v^*,T^*). If $T^* = T$ is shown, from the definition of $(\mathcal{Z},\preceq)$ it is clear that u^* is a strong solution of (P) on $[0,T]$. Hence, in the rest of this proof, it is enough to show $T^* = T$. For this, we assume $T^* < T$. We see from Theorem 1.2 that there exists a constant $C^* > 0$ such that

$$\|(u^*)'\|_{L^2(0,T^*;H)} + \sup_{0\le t\le T^*} \|u^*(t)\|_H + \sup_{0\le t\le T^*} |\varphi(t,u^*,v_0\,;u^*(t))| \le C^*. \tag{5.21}$$

Now, we define a subset $\mathcal{C}^*(u_0, v_0) \subset C([0, T - T^*]; H)$ by

$$\mathcal{C}^*(u_0, v_0) := \{w \in C([0, T - T^*]; H)\,;\, w(0) = u^*(T^*)\}.$$

In the following argumentation, for any $\tilde{w} \in \mathcal{C}^*(u_0, v_0)$ we define a function $\tilde{u}(\tilde{w}) \in C([0, T]; H)$ by

$$\tilde{u}(\tilde{w}\,; t) := \begin{cases} u^*(t) & \text{if} \quad t \in [0, T^*], \\ \tilde{w}(t - T^*) & \text{if} \quad t \in (T^*, T], \end{cases} \tag{5.22}$$

and a family of proper l.s.c. convex functions on H, denoted by $\{\psi(t, \tilde{w}, v_0)\,;\, 0 \leq t \leq T - T^*\}$ throughout this proof, by

$$\begin{aligned} \psi(t, \tilde{w}, v_0) &:= \varphi(t + T^*, \tilde{u}(\tilde{w}), v_0) \\ &= \phi(t + T^*, \tilde{u}(\tilde{w}), S(\tilde{u}(\tilde{w}); t + T^*, 0)v_0), \quad \forall t \in [0, T - T^*]. \end{aligned} \tag{5.23}$$

We consider a set $\mathcal{X}^*(u_0, v_0)$ given by

$$\mathcal{X}^*(u_0, v_0) := \{\{\psi(t, \tilde{w}, v_0)\,;\, 0 \leq t \leq T - T^*\}\,;\, \tilde{w} \in \mathcal{C}^*(u_0, v_0)\}\,.$$

Then, we see that (B3) is satisfied instead of (A3):

(B3) The following properties are satisfied:

(a) From (a) in (A3) we get the following inequality:

$$\varphi(z) \leq \psi(t, \tilde{w}, v_0\,; z), \quad \forall \tilde{w} \in \mathcal{C}^*(u_0, v_0), \\ \forall t \in [0, T - T^*],\ \forall z \in H.$$

(b) Assume that functions $\tilde{w}_1,\ \tilde{w}_2 \in \mathcal{C}^*(u_0, v_0)$ satisfy $\tilde{w}_1(t) = \tilde{w}_2(t)$ in H for all $t \in [0, \bar{T}]$ for some $\bar{T} \in [0, T - T^*]$. Since from (5.22) we have $\tilde{u}(\tilde{w}_1\,; t) = \tilde{u}(\tilde{w}_2\,; t)$ in H for all $t \in [0, T^* + \bar{T}]$, we see from (b) in (A3) that the following equality holds:

$$\varphi(t, \tilde{u}(\tilde{w}_1), v_0) = \varphi(t, \tilde{u}(\tilde{w}_2), v_0) \quad \text{on} \quad H, \quad 0 \leq \forall t \leq T^* + \bar{T},$$

$$\varphi(t + T^*, \tilde{u}(\tilde{w}_1), v_0) = \varphi(t + T^*, \tilde{u}(\tilde{w}_2), v_0) \quad \text{on} \quad H, \\ 0 \leq \forall t \leq \bar{T},$$

which implies

$$\psi(t, \tilde{w}_1, v_0) = \psi(t, \tilde{w}_2, v_0) \quad \text{on} \quad H, \quad 0 \leq \forall t \leq \bar{T}.$$

(c) Assume that $\{\tilde{w}_m\}_{m \in \mathbb{N}} \subset \mathcal{C}^*(u_0, v_0)$ and $\tilde{w} \in \mathcal{C}^*(u_0, v_0)$ satisfy the following convergence:

$$\tilde{w}_m \longrightarrow \tilde{w} \quad \text{in} \quad C([0, T - T^*]; H) \quad \text{as} \quad m \to \infty.$$

We see from (c) in (A3) that for any $t \in [0,T]$ the following convergence holds as $m \to \infty$:

$$\varphi(t, \tilde{u}(\tilde{w}_m), v_0) \longrightarrow \varphi(t, \tilde{u}(\tilde{w}), v_0) \quad \text{on} \quad H(S(\tilde{u}(\tilde{w}); t, 0)v_0)$$
$$\text{in the sense of Mosco,}$$

hence, for any $t \in [0, T - T^*]$

$$\psi(t, \tilde{w}_m, v_0) \longrightarrow \psi(t, \tilde{w}, v_0) \quad \text{on} \quad H(S(\tilde{u}(\tilde{w}); t + T^*, 0)v_0)$$
$$\text{in the sense of Mosco.}$$

Defining functions $f^* \colon [0, T - T^*] \longmapsto H$ and $g^* : [0, T - T^*] \times D(\varphi) \times A \longmapsto H$ by

$$f^*(t) := f(t + T^*), \quad g^*(t, z, v) := g(t + T^*, z, v),$$
$$\forall t \in [0, T - T^*],\ \forall z \in D(\varphi),\ \forall v \in A,$$

we see that (B4) and (B5) below are satisfied instead of (A4) and (A5), respectively, as well as $f^* \in L^2(0, T - T^*; H)$, which is easily obtained from (A10):

(B4) We see from (A4) that g^* satisfies the following properties:

(a) We have

$$\|g^*(t, z, v)\|_H \le \ell(v)\sqrt{|\varphi(z)| + c_4}, \quad \forall t \in [0, T - T^*],$$
$$\forall z \in D(\varphi),\ \forall v \in A.$$

(b) We have the following convergence as $m \to \infty$:

$$g^*(\,\cdot\,, \tilde{w}_m, S(\tilde{u}(\tilde{w}_m)\,;\,\cdot\, + T^*, 0)v_0)$$
$$\longrightarrow g^*(\,\cdot\,, \tilde{w}, S(\tilde{u}(\tilde{w})\,;\,\cdot\, + T^*, 0)v_0)$$
$$\text{weakly in} \quad L^2(0, T - T^*; H)$$

whenever a sequence $\{\tilde{w}_m\}_{m \in \mathbb{N}} \subset \mathcal{C}^*(u_0, v_0)$ and a function $\tilde{w} \in \mathcal{C}^*(u_0, v_0)$ satisfy

$$\tilde{w}_m \longrightarrow \tilde{w} \quad \text{in} \quad C([0, T - T^*]; H) \quad \text{as} \quad m \to \infty.$$

(B5) We define a class of initial data D^* by

$$D^* := \{w \in H\,;\, w \in D(\psi(0, \tilde{w}, v_0)) \text{ for all } \tilde{w} \in \mathcal{C}^*(u_0, v_0)\}.$$

From (5.21) and (5.22) we get

$$u(T^*) \in D(\varphi(T^*, \tilde{u}(\tilde{w}), v_0)) = D(\psi(0, \tilde{w}, v_0)), \quad \forall \tilde{w} \in \mathcal{C}(u_0, v_0),$$
$$\text{hence,} \quad u(T^*) \in D^*.$$

Next, we define subsets $\mathcal{W}^*(u_0, v_0) \subset \mathcal{V}^*(u_0, v_0) \subset \mathcal{U}^*(u_0, v_0)$ of $\mathcal{C}^*(u_0, v_0)$ by

$$\mathcal{U}^*(u_0, v_0) := \left\{ \tilde{w} \in \mathcal{C}^*(u_0, v_0)\,;\, \sup_{0 \le t \le T - T^*} \|\tilde{w}(t)\|_H + \int_0^{T - T^*} \varphi(\tilde{w}(t))dt < \infty \right\},$$

$$\mathcal{V}^*(u_0,v_0) := \left\{ \tilde{w} \in \mathcal{U}^*(u_0,v_0)\,;\ \sup_{0\le t\le T-T^*} \|\tilde{w}(t)\|_H + \sup_{0\le t\le T-T^*} \varphi(\tilde{w}(t)) < \infty \right\},$$

$$\mathcal{W}^*(u_0,v_0) := \left\{ \tilde{w} \in \mathcal{U}^*(u_0,v_0)\,;\ \begin{array}{c} \|\tilde{w}'\|_{L^2(0,T-T^*;H)} + \sup_{0\le t\le T-T^*} \|\tilde{w}(t)\|_H \\ + \sup_{0\le t\le T-T^*} \varphi(\tilde{w}(t)) < \infty \end{array} \right\},$$

and for any $R \ge 0$ we define subsets $\mathcal{W}^*_R(u_0,v_0) \subset \mathcal{V}^*_R(u_0,v_0)$ and $\mathcal{U}^*_R(u_0,v_0)$ of $\mathcal{U}(u_0,v_0)$ by

$$\mathcal{U}^*_R(u_0,v_0) := \left\{ \tilde{w} \in \mathcal{U}^*(u_0,v_0)\,;\ \sup_{0\le t\le T-T^*} \|\tilde{w}(t)\|_H + \int_0^{T-T^*} \varphi(\tilde{w}(t))dt \le R \right\},$$

$$\mathcal{V}^*_R(u_0,v_0) := \left\{ \tilde{w} \in \mathcal{V}^*(u_0,v_0)\,;\ \sup_{0\le t\le T-T^*} \|\tilde{w}(t)\|_H + \sup_{0\le t\le T-T^*} \varphi(\tilde{w}(t)) \le R \right\},$$

$$\mathcal{W}^*_R(u_0,v_0) := \left\{ \tilde{w} \in \mathcal{W}^*(u_0,v_0)\,;\ \begin{array}{c} \|\tilde{w}'\|_{L^2(0,T-T^*;H)} + \sup_{0\le t\le T-T^*} \|\tilde{w}(t)\|_H \\ + \sup_{0\le t\le T-T^*} \varphi(\tilde{w}(t)) \le R \end{array} \right\}.$$

Then, we show (B6), (B7), (B8) and (B9) instead of (A6), (A7), (A8) and (A9), respectively:

(B6) For any $\{\psi(t,\tilde{w},v_0)\,;\, 0 \le t \le T-T^*\} \in \mathcal{X}^*(u_0,v_0)$ we consider

$$\{\varphi(t,\tilde{u}(\tilde{w}),v_0)\,;\, 0 \le t \le T\} \in \mathcal{X},$$

and apply (A6) to the family $\{\varphi(t,\tilde{u}(\tilde{w}),v_0)\,;\, 0 \le t \le T\}$. Then, we see that for any $r > 0$ there exist nonnegative functions $\alpha_r(\tilde{u}(\tilde{w})) \in L^2(0,T)$ and $\beta_r(\tilde{u}(\tilde{w})) \in L^1(0,T)$ such that the condition $(\star)$ is satisfied, that is, for any $s,t \in [0,T-T^*]$ and $z(\tilde{u}(\tilde{w}),s+T^*) \in D(\varphi(s+T^*,\tilde{u}(\tilde{w}),v_0))$ with $\|z(\tilde{u}(\tilde{w}),s+T^*)\|_{S(\tilde{u}(\tilde{w})\,;\,t+T^*,0)v_0} \le r$ there exists $z(\tilde{u}(\tilde{w}),s+T^*,t+T^*) \in D(\varphi(t+T^*,\tilde{u}(\tilde{w}),v_0))$ such that

$$\begin{aligned} \text{(d1)}\quad & \|z(\tilde{u}(\tilde{w}),s+T^*,t+T^*) - z(\tilde{u}(\tilde{w}),s+T^*)\|_{S(\tilde{u}(\tilde{w})\,;\,t+T^*,0)v_0} \\ & \le (|\varphi(s+T^*,\tilde{u}(\tilde{w}),v_0\,;z(\tilde{u}(\tilde{w}),s+T^*))| + 1) \left| \int_{s+T^*}^{t+T^*} \alpha_r(\tilde{u}(\tilde{w})\,;\tau)d\tau \right| \end{aligned}$$

$$\begin{aligned} \text{(d2)}\quad & |\varphi(t,\tilde{u}(\tilde{w}),v_0\,;z(\tilde{u}(\tilde{w}),s+T^*,t+T^*) - \varphi(t,\tilde{u}(\tilde{w}),v_0\,;z(\tilde{u}(\tilde{w}),s+T^*))| \\ & \le \left(\sqrt{|\varphi(s+T^*,\tilde{u}(\tilde{w}),v_0\,;z(\tilde{u}(\tilde{w}),s+T^*))|} + 1 \right) \left| \int_{s+T^*}^{t+T^*} \beta_r(\tilde{u}(\tilde{w})\,;\tau)d\tau \right|. \end{aligned}$$

Defining the families $\{z^*(\tilde{w},s)\,;\, 0 \le s \le T-T^*\}$, $\{z^*(\tilde{w},s,t)\,;\, 0 \le t \le T-T^*\}$ $(0 \le \forall s \le T-T^*)$ and functions $\alpha^*_r(\tilde{w})$, $\beta^*_r(\tilde{w})$ by the following ways:

$$\{z^*(\tilde{w},s)\,;\, 0 \le s \le T-T^*\} := \{z(\tilde{u}(\tilde{w}),s)\,;\, 0 \le s \le T-T^*\},$$
$$\{z^*(\tilde{w},s,t)\,;\, 0 \le t \le T-T^*\} := \{z(\tilde{u}(\tilde{w}),s+T^*,t+T^*)\,;\, 0 \le t \le T-T^*\},$$
$$0 \le \forall s \le T-T^*,$$
$$\alpha^*_r(\tilde{w}\,;t) := \alpha_r(\tilde{u}(\tilde{w})\,;t+T^*), \quad \beta^*_r(\tilde{w}\,;t) := \beta_r(\tilde{u}(\tilde{w})\,;t+T^*),$$
$$\forall t \in [0,T-T^*],$$

we get $\alpha_r^*(\tilde{w}) \in L^2(0, T-T^*)$, $\beta_r^*(\tilde{w}) \in L^1(0, T-T^*)$, and the following inequalities:

$$\begin{aligned}
&(\mathrm{d1})^* \quad \|z^*(\tilde{w}, s, t) - z^*(\tilde{w}, s)\|_{S(\tilde{u}(\tilde{w})\,;\,t+T^*,0)v_0} \\
&\qquad\qquad \le (|\psi(s, \tilde{w}, v_0\,; z^*(\tilde{w}, s))| + 1) \left| \int_s^t \alpha_r^*(\tilde{w}\,;\tau) d\tau \right| \\
&(\mathrm{d2})^* \quad |\psi(t, \tilde{w}, v_0\,; z^*(\tilde{w}, s, t) - \psi(s, \tilde{w}, v_0\,; z^*(\tilde{w}, s))| \\
&\qquad\qquad \le \left(\sqrt{|\psi(s, \tilde{w}, v_0\,; z^*(\tilde{w}, s))|} + 1 \right) \left| \int_s^t \beta_r^*(\tilde{w}\,;\tau) d\tau \right|,
\end{aligned}$$

which implies that (B6) holds.

(B7) In order to do this, we define a constant $\tilde{R}^* \ge R_*$ by

$$\tilde{R}^* := \left(1 + \frac{4}{c_1^2}\right) \{C^* + C_1 (c_2 C^* + 1)\} + C^* + \frac{1}{4},$$

where $C^* > 0$ is the same constant that is obtained in (5.21). Then, we have

$$\begin{aligned}
\tilde{R}_* \ge \left(1 + \frac{2}{c_1^2}\right) &\{|\varphi(T^*, u^*, v_0\,; u^*(T^*))| + C_1 (c_2 \|u^*(T^*)\|_H + 1)\} \\
&+ \|u^*(T^*)\|_H + \frac{1}{4},
\end{aligned}$$

where u^* in $\varphi(T^*, u^*, v_0)$ denotes the prolongation of $u^* \in C([0, T^*]; H)$ given in Remark 2.2, and for any $R \ge \tilde{R}_*$ the following properties are satisfied:

(a) For any $\tilde{w} \in \mathcal{U}_R^*(u_0, v_0)$ we have

$$\begin{aligned}
&\sup_{0 \le t \le T} \|\tilde{u}(\tilde{w}\,; t)\|_H + \int_0^T \varphi(\tilde{u}(\tilde{w}\,; t)) dt \qquad (5.24) \\
&= \sup_{0 \le t \le T^*} \|u^*(t)\|_H + \sup_{0 \le t \le T-T^*} \|\tilde{w}(t)\|_H \\
&\quad + \int_0^{T^*} \varphi(u^*(t)) dt + \int_0^{T-T^*} \varphi(\tilde{w}(t)) dt \le R + C^*(T^* + 1) =: R^*,
\end{aligned}$$

which implies $\tilde{u} \in \mathcal{U}_{R^*}(u_0)$. Using (a) in (A7), we can consider a family $\{M_R^*(r)\,;\, 0 < r < \infty\}$, which is defined by

$$\forall r > 0, \quad M_R^*(r) := M_{R^*}(r) = M_{R + C^*(T^*+1)}(r).$$

Then, we see that for any $r > 0$ the following uniform estimates holds:

$$\begin{aligned}
&\sup_{\tilde{w} \in \mathcal{U}_R^*(u_0, v_0)} \left(\|\alpha_r(\tilde{u}(\tilde{w}))\|_{L^2(0,T)} + \|\beta_r(\tilde{u}(\tilde{w}))\|_{L^1(0,T)} \right) \\
&\le \sup_{\tilde{u} \in \mathcal{U}_{R^*}(u_0)} \left(\|\alpha_r(\tilde{u})\|_{L^2(0,T)} + \|\beta_r(\tilde{u})\|_{L^1(0,T)} \right) \le M_{R^*}(r),
\end{aligned}$$

where $\alpha_r(\tilde{u}(\tilde{w}))$ and $\beta_r(\tilde{u}(\tilde{w}))$ are the same functions that are given in (B6), hence,

$$\sup_{\tilde{w}\in\mathcal{U}_R^*(u_0,v_0)}\left(\|\alpha_r^*(\tilde{w})\|_{L^2(0,T-T^*)}+\|\beta_r^*(\tilde{w})\|_{L^1(0,T-T^*)}\right)\le M_R^*(r).$$

(b) We see from (5.24) and (b) in (A7) that for any $r>0$ and $\varepsilon>0$ there exists a constant $\delta_{r,\varepsilon,R}>0$ such that

$$\sup_{\tilde{u}\in\mathcal{U}_{R^*}(u_0)}\left\{\sup_{0\le t\le T}\int_t^{\min\{t+\delta_{r,\varepsilon,R},\,T\}}\left(|\alpha_r(\tilde{u}\,;s)|^2+\beta_r(\tilde{u}\,;s)\right.\right.$$
$$\left.\left.+\|(S(\tilde{u}\,;s,0)v_0)'\|_X\right)ds\right\}\le\varepsilon,$$

hence,

$$\sup_{\tilde{w}\in\mathcal{U}_R^*(u_0,v_0)}\left\{\sup_{0\le t\le T-T^*}\int_t^{\min\{t+\delta_{r,\varepsilon,R},\,T-T^*\}}\left(|\alpha_r^*(\tilde{w}\,;s)|^2+\beta_r^*(\tilde{w}\,;s)\right.\right.$$
$$\left.\left.+\|(S(\tilde{u}(\tilde{w})\,;s+T^*,0)v_0)'\|_X\right)ds\right\}\le\varepsilon,$$

(B8) We use the family $\{h(\tilde{u})\in W^{1,2}(0,T;H)\,;\,\tilde{u}\in\mathcal{U}(u_0)\}$, which is given in (A8), and for any $\tilde{w}\in\mathcal{U}(u_0,v_0)$ we define a function $h^*(\tilde{w})\colon[0,T-T^*]\longmapsto H$ by

$$h^*(\tilde{w}\,;t):=h(\tilde{u}(\tilde{w})\,;t+T^*),\quad\forall t\in[0,T-T^*].$$

From (A8) we have

$$\sup_{\tilde{w}\in\mathcal{U}^*(u_0,v_0)}\left\{\|h'(\tilde{u}(\tilde{w}))\|^2_{L^2(0,T;H)}+\sup_{0\le t\le T}\|h(\tilde{u}(\tilde{w});t)\|_H\right.$$
$$\left.+\sup_{0\le t\le T}|\varphi(t,\tilde{u}(\tilde{w}),v_0\,;h(\tilde{u}(\tilde{w});t)|\right\}\le C_2,$$

hence,

$$\sup_{\tilde{w}\in\mathcal{U}^*(u_0,v_0)}\left\{\sup_{0\le t\le T-T^*}\|h^*(\tilde{w}\,;t)\|_H+\sup_{0\le t\le T-T^*}|\psi(t,\tilde{w},v_0\,;h^*(\tilde{w}\,;t)|\right.$$
$$\left.+\|(h^*)'(\tilde{w})\|^2_{L^2(0,T-T^*;H)}\right\}\le C_2.$$

Finally, we give (B9) and its proof. From (A9) we see that the following uniform boundedness holds:

$$\sup_{\tilde{w}\in\mathcal{U}^*(u_0,v_0)}\left(\sup_{0\le t\le T-T^*}\phi(S_1(\tilde{u}(\tilde{w});t+T^*,0)v_0)\right.$$
$$\left.+\|(S_1(\tilde{u}(\tilde{w});\,\cdot\,+T^*,0)v_0)'\|_{L^1(0,T-T^*;X)}\right)\le C_3.$$

Now, we consider the Cauchy problem $(\mathrm{P})^* := \{(5.25) - (5.27)\}$:

$$w'(t) + \partial_{z(t)}\psi(t, w, v_0\,; w(t)) + g^*(t, w(t), z(t)) \ni f^*(t) \quad \text{in} \quad H(z(t)), \quad \text{a.a. } t \in (0, T - T^*), \tag{5.25}$$

$$z(t) = S(w\,; t, 0)v^*(T^*) \quad \text{in} \quad A, \quad \forall t \in [0, T - T^*], \tag{5.26}$$

$$w(0) = u^*(T^*) \quad \text{in} \quad H. \tag{5.27}$$

Applying Theorem 1.2, we see from (A1), (A2), (B3)–(B10) that there exists a time $T_1 \in (0, T - T^*]$ such that $(\mathrm{P})^*$ has at least one strong solution $w \in W^{1,2}(0, T_1; H)$, and there exists a constant $\tilde{C}^* > 0$ such that

$$\|w'\|_{L^2(0,T_1;H)} + \sup_{0 \le t \le T_1} \|w(t)\|_H + \sup_{0 \le t \le T_1} |\psi(t, w, z(t)\,; w(t))| \le \tilde{C}^*. \tag{5.28}$$

Using pairs (u^*, v^*) and (w, z), we define a pair $(u, v) \in \mathcal{U}(u_0) \times W^{1,1}(0, T_1 + T^*; X)$ by

$$(u(t), v(t)) := \begin{cases} (u^*(t), v^*(t)) & \text{if} \quad t \in [0, T^*], \\ (w(t - T^*), z(t - T^*)) & \text{if} \quad t \in (T^*, T_1 + T^*]. \end{cases}$$

Then, we easily see from (b) in (A3) that the function u satisfies not only the following evolution inclusion for a.a. $t \in (0, T_1 + T^*)$ and the initial condition:

$$u'(t) + \partial_{v(t)}\varphi(t, u, v_0\,; u(t)) + g(t, u(t), v(t)) \ni f(t) \quad \text{in} \quad H(v(t)), \tag{5.29}$$

$$u(0) = u_0 \quad \text{in} \quad H, \tag{5.30}$$

but also the equality

$$(\sigma_{T^*}u)(t) = \begin{cases} u(t + T^*) = w(t) & \text{if} \quad t \in [0, T_1], \\ u(T_1 + T^*) = w(T_1) & \text{if} \quad t \in (T_1, T_1 + T^*]. \end{cases} \tag{5.31}$$

From (e), (f) and (g) in (A2) we get the following equality for all $t \in [0, T^*]$:

$$v(t) = v^*(t) = S(u^*; t, 0)v_0 = S(u\,; t, 0)v_0, \tag{5.32}$$

and from (5.31), (5.32) for all $t \in (T^*, T_1 + T^*]$:

$$\begin{aligned} v(t) &= z(t - T^*) = S(w\,; t - T^*, 0)v^*(T^*) \\ &= (S(\sigma_{T^*}u\,; t - T^*, 0) \circ S(u\,; T^*, 0))\, v_0 \\ &= (S(u\,; t, T^*) \circ S(u\,; T^*, 0))\, v_0 = S(u\,; t, 0)v_0, \end{aligned}$$

which implies that v satisfies the following equality:

$$v(t) = S(u\,; t, 0)v_0, \quad \forall t \in [0, T_1 + T^*]. \tag{5.33}$$

Finally, (5.21), (5.28)–(5.30) and (5.33) imply that u is a strong solution of (P) on $[0, T_1 + T^*]$, which is in contradiction with the maximality of the triplet (u^*, v^*, T^*). So, $T^* = T$ must hold. □

6. Application

In this section, we investigate the possibility to deal with the Cauchy problem (T):=$\{$(6.1)–(6.8)$\}$ of a mass-conservative tumor invasion model with haptotaxis effect as one of the examples of (P).

$$u'(t) + \nabla \cdot (d_u(v)\nabla\xi - u\nabla\lambda(v)) = 0 \quad \text{a.e. in} \quad Q_T := \Omega \times (0,T), \tag{6.1}$$

$$\xi \in \big(\partial_{\mathbb{R}}\hat{\beta}(v)\big)(u) \quad \text{a.e. in} \quad Q_T, \tag{6.2}$$

$$v' = -avw \quad \text{a.e. in} \quad Q_T, \tag{6.3}$$

$$0 \le v \le \alpha \quad \text{a.e. in} \quad Q_T, \tag{6.4}$$

$$w' = d_w\Delta w - bw + cu \quad \text{a.e. in} \quad Q_T, \tag{6.5}$$

$$0 \le w \quad \text{a.e. in} \quad Q_T, \tag{6.6}$$

$$(d_u(v)\nabla\xi - u\nabla\lambda(v)) \cdot \nu = \nabla w \cdot \nu = 0 \quad \text{a.e. on} \quad \Sigma_T := \Gamma \times (0,T), \tag{6.7}$$

$$u(0) = u_0, \quad v(0) = v_0, \quad w(0) = w_0 \quad \text{a.e. in} \quad \Omega, \tag{6.8}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N = 1,2,3)$ with a smooth boundary $\Gamma := \partial\Omega$; ν is an outer unit normal vector on Γ; d and λ are smooth functions from $\mathbb{R}$ into itself; $\hat{\beta}(v)$ is a proper l.s.c. convex function on $\mathbb{R}$ and $\partial_{\mathbb{R}}\hat{\beta}(v)$ represents the subdifferential of $\hat{\beta}(v)$ on $\mathbb{R}$; a triplet (u_0, v_0, w_0) is an initial datum. The original tumor invasion model of (T) was proposed in [2], and we entrust the explanation of this model from the biological point of view to [2].

In what follows, we fix a constant $\alpha > 0$ in (6.4). First of all, for the prescribed data in (T) we assume that the following conditions are satisfied.

(T1) A function $d_u : [0,\alpha] \longmapsto (0,\infty)$ is Lipschitz continuous. Moreover, there exist constants $d_1 > 0$ and $d_2 > 0$ such that

$$d_1 \le d_u(v) \le d_2, \quad 0 \le \forall v \le \alpha.$$

(T2) A family $\{\hat{\beta}(v)\,;\, 0 \le v \le \alpha\}$ of proper, nonnegative, l.s.c., convex functions $\hat{\beta}(v)$ on $\mathbb{R}$ satisfies the following conditions:

(a) There exists a constant $u^* > 0$ such that

$$D(\hat{\beta}(v)) := \big\{\bar{r} \in \mathbb{R}\,;\, \hat{\beta}(v\,;\bar{r}) < \infty\big\} = [0,u^*], \quad 0 \le \forall v \le \alpha.$$

(b) There exists a constant $\beta_1^* > 0$ such that

$$0 \le |\hat{\beta}(v\,;r_2) - \hat{\beta}(v\,;r_1)| \le \beta_1^*|r_2 - r_1|,$$
$$\forall v \in [0,\alpha], \quad \forall r_1, r_2 \in [0,u^*].$$

(c) The family of epigraphs $\{\operatorname{epi}\hat{\beta}(v)\,;\,0 \le v \le \alpha\}$ is Lipschitz continuous in the following sense: there exists a constant $\beta_2^* > 0$ such that the following inequality holds for all $v_1, v_2 \in [0,\alpha]$:

$$\max\left\{\delta\left(\operatorname{epi}\hat{\beta}(v_1), \operatorname{epi}\hat{\beta}(v_2)\right), \delta\left(\operatorname{epi}\hat{\beta}(v_2), \operatorname{epi}\hat{\beta}(v_1)\right)\right\} \le \beta_2^*|v_1 - v_2|,$$

where for any $v \in [0,\alpha]$ the set $\operatorname{epi}\hat{\beta}(v)$ is the epigraph of $\hat{\beta}(v)$ given by

$$\operatorname{epi}\hat{\beta}(v) := \left\{(r_1, a_1) \in \mathbb{R}^2\,;\,\hat{\beta}(v\,;r_1) \le a_1\right\},$$

and for any subsets $A,\ B$ of $\mathbb{R}^2$ a nonnegative number $\delta(A,B)$ is the semidistance between A and B defined by

$$\delta(A,B) := \sup_{(r_1,a_1)\in A}\left\{\inf_{(r_2,a_2)\in B}\|(r_1,a_1)-(r_2,a_2)\|_{\mathbb{R}^2}\right\}.$$

As the typical examples, we give the following ones, where Example 6.1 is proposed in [8].

Example 6.1. For each $v \in [0,\alpha]$ we define a proper, nonnegative, l.s.c., convex function $\hat{\beta}(v)$ by

$$\hat{\beta}(v\,;r) := \begin{cases} r^{v+2}, & \text{if} \quad r \in [0,u^*], \\ \infty, & \text{if} \quad r \in (-\infty,0)\cup(u^*,\infty). \end{cases}$$

Example 6.2. Fixing a constant $\gamma_0 > 0$, for each $v \in [0,\alpha]$ we define a proper, nonnegative, l.s.c., convex function $\hat{\beta}(v)$ by

$$\hat{\beta}(v\,;r) := \begin{cases} \dfrac{r^2}{\alpha - v + \gamma_0}, & \text{if} \quad r \in [0,u^*], \\ \infty, & \text{if} \quad r \in (-\infty,0)\cup(u^*,\infty). \end{cases}$$

Then, we have (a) in (T2) and

$$0 \le |\hat{\beta}(v\,;r_2) - \hat{\beta}(v\,;r_1)| = \frac{|r_2^2 - r_1^2|}{\alpha - v + \gamma_0} \le \frac{2u^*}{\gamma_0}\cdot|r_2 - r_1|,$$
$$0 \le \forall v \le \alpha,\ \forall r_1, r_2 \in [0,u^*],$$

which implies that (b) in (T2) is satisfied. In the rest of this part, we show (c) in (T2). For this, without losing generality, we assume $0 \le v_1 \le v_2 \le \alpha$. Then, we get $\hat{\beta}(v_1\,;r) \le \hat{\beta}(v_2\,;r)$ for all $r \in \mathbb{R}$, hence, $\operatorname{epi}\hat{\beta}(v_2) \subset \operatorname{epi}\hat{\beta}(v_1)$, which implies

$$\delta\left(\operatorname{epi}\hat{\beta}(v_2), \operatorname{epi}\hat{\beta}(v_1)\right) = 0. \tag{6.9}$$

Now, for any $r \in [0, u^*]$ we define a number $r^* \in [0, r]$ by

$$r^* := r\sqrt{\frac{\alpha - v_2 + \gamma_0}{\alpha - v_1 + \gamma_0}}.$$

Using the mean-value theorem, we see that there exist numbers $\theta_1 \in [0,1]$ and $\theta_2 \in [0,1]$ such that

$$\begin{aligned} &\frac{1}{\alpha - v_2 + \gamma_0} - \frac{1}{\alpha - v_1 + \gamma_0} \\ &= \frac{v_2 - v_1}{\{\theta_1(\alpha - v_2) + (1-\theta_1)(\alpha - v_1) + \gamma_0\}^2} \le \frac{v_2 - v_1}{\gamma_0^2}, \end{aligned} \tag{6.10}$$

and

$$\begin{aligned} &\sqrt{\alpha - v_1 + \gamma_0} - \sqrt{\alpha - v_2 + \gamma_0} \\ &= \frac{v_2 - v_1}{2\sqrt{\theta_2(\alpha - v_1) + (1-\theta_2)(\alpha - v_2) + \gamma_0}} \le \frac{v_2 - v_1}{2\sqrt{\gamma_0}}, \end{aligned} \tag{6.11}$$

From (6.10) and (6.11) we get the following inequality for all $(r, \hat{\beta}(v_1\,; r))$, which are points on the boundary of $\operatorname{epi}\hat{\beta}(v_1)$:

$$\begin{aligned} &\inf_{(r_2,a_2)\in \operatorname{epi}\hat{\beta}(v_2)} \|(r, \hat{\beta}(v_1\,; r)) - (r_2, a_2)\|_{\mathbb{R}^2} \\ &\le \min\left\{\hat{\beta}(v_2\,; r) - \hat{\beta}(v_1\,; r),\ r - r^*\right\} \\ &= \min\left\{\left(\frac{1}{\alpha - v_2 + \gamma_0} - \frac{1}{\alpha - v_1 + \gamma_0}\right) r^2,\ \frac{\sqrt{\alpha - v_1 + \gamma_0} - \sqrt{\alpha - v_2 + \gamma_0}}{\sqrt{\alpha - v_1 + \gamma_0}} \cdot r\right\} \\ &\le \min\left\{\frac{\alpha^2}{\gamma_0^2},\ \frac{\alpha}{2\gamma_0}\right\} \cdot (v_2 - v_1), \end{aligned}$$

which implies

$$\delta\left(\operatorname{epi}\hat{\beta}(v_1), \operatorname{epi}\hat{\beta}(v_2)\right) \le \min\left\{\frac{\alpha^2}{\gamma_0^2},\ \frac{\alpha}{2\gamma_0}\right\} \cdot (v_2 - v_1). \tag{6.12}$$

We see from (6.9) and (6.12) that (c) in (T2) is satisfied.

On the other hand, unfortunately, for the singular case $\gamma_0 = 0$; that is, for each $v \in [0, \alpha)$

$$\hat{\beta}(v\,; r) := \begin{cases} \dfrac{r^2}{\alpha - v}, & \text{if} \quad r \in [0, u^*], \\ \infty, & \text{if} \quad r \in (-\infty, 0) \cup (u^*, \infty). \end{cases}$$

and for $v = \alpha$

$$\hat{\beta}(\alpha\,; r) := \begin{cases} 0, & \text{if} \quad r = 0, \\ \infty, & \text{if} \quad r \in \mathbb{R} \setminus \{0\}. \end{cases}$$

it is clear that (T2) is not satisfied.

Moreover, we assume that the following conditions are satisfied:

(T3) A function $\lambda\colon \mathbb{R} \longmapsto \mathbb{R}$ is in C^2-function. We define a constant $C_\lambda > 0$ by

$$C_\lambda := \max_{0\le r\le \alpha} |\lambda'(r)| + \max_{0\le r\le \alpha} |\lambda''(r)| \le C_\lambda.$$

(T4) $a > 0,\ b > 0,\ c > 0$ and $d_w > 0$ are constants.

(T5) We define a subset $A_v \subset W^{1,\infty}(\Omega)$ by

$$A_v := \left\{\tilde{v} \in W^{1,\infty}(\Omega)\,;\, 0 \le \tilde{v} \le \alpha \quad \text{a.e. in} \quad \Omega\right\} \subset C(\overline{\Omega}),$$

and a subset $D \subset L^\infty(\Omega) \times A_v$ by

$$D := \left\{(\tilde{u}, \tilde{v}) \in L^\infty(\Omega) \times A_v\,;\, \int_\Omega \hat{\beta}(\tilde{v}\,;\tilde{u})dx < \infty\right\}.$$

Then, for any $(\tilde{u}, \tilde{v}) \in D$ we have

$$0 \le u \le \alpha \quad \text{a.e. in} \quad \Omega.$$

Moreover, we define a subset $A_w \subset W^{1,\infty}(\Omega)$ by

$$A_w := \left\{\tilde{w} \in W^{1,\infty}(\Omega)\,;\, 0 \le \tilde{w} \quad \text{a.e. in} \quad \Omega\right\} \subset C(\overline{\Omega}).$$

We assume $(u_0, v_0, w_0) \in D \times A_w$.

6.1. Real Hilbert spaces V_0^* and V^*

In order to treat the system (T) as an evolution inclusion with quasi-variational structures, it is essential that a real Hilbert space $V := H^1(\Omega)$ with an inner product $(\cdot,\cdot)_V$ given by

$$(z_1, z_2)_V := \int_\Omega \nabla z_1 \cdot \nabla z_2 dx + \left(\int_\Omega z_1 dx\right)\left(\int_\Omega z_2 dx\right), \quad \forall z_1, z_2 \in V,$$

is considered. Actually, because of the result in [15, Appendix], we see that the inner product $(\cdot,\cdot)_V$ is equivalent to the usual inner product $(\cdot,\cdot)_{H^1(\Omega)}$, which is given by

$$(z_1, z_2)_{H^1(\Omega)} := \int_\Omega \nabla z_1 \cdot \nabla z_2 dx + \int_\Omega z_1 z_2 dx, \quad \forall z_1, z_2 \in H^1(\Omega).$$

Moreover, we consider a closed subspace V_0 of V defined by (1.2) whose inner product is given by

$$(z_{0,1}, z_{0,2})_{V_0} := (z_{0,1}, z_{0,2})_V = \int_\Omega \nabla z_{0,1} \cdot \nabla z_{0,2} dx, \quad \forall z_{0,1}, z_{0,2} \in V_0.$$

We denote by V^* the dual space of V, V_0^* the dual space of V_0, $\langle\cdot,\cdot\rangle_{V^*,V}$ the duality pair between V^* and V, and $\langle\cdot,\cdot\rangle_{V_0^*,V_0}$ the duality pair between V_0^* and

V_0. Owing to the Gelfand triplets $V \subset L^2(\Omega) \subset V^*$ and $V_0^* \subset (L^2(\Omega))_0 \subset V_0^*$, we have the following equalities:

$$\langle z^*, z\rangle_{V^*,V} = (z^*, z)_{L^2(\Omega)}, \quad \forall z^* \in L^2(\Omega),\ \forall z \in V, \tag{6.13}$$

$$\langle z_0^*, z_0\rangle_{V_0^*,V_0} = (z_0^*, z_0)_{(L^2(\Omega))_0}, \quad \forall z_0^* \in (L^2(\Omega))_0,\ \forall z \in V_0, \tag{6.14}$$

where $(L^2(\Omega))_0$ is a closed subspace of $L^2(\Omega)$ given by

$$(L^2(\Omega))_0 := \left\{ z^* \in L^2(\Omega)\,;\, \int_\Omega z^* dx = 0 \right\},$$

$$(z_{0,1}^*, z_{0,2}^*)_{(L^2(\Omega))_0} = (z_{0,1}^*, z_{0,2}^*)_{L^2(\Omega)}, \quad \forall z_{0,1}^*, z_{0,2}^* \in (L^2(\Omega))_0.$$

Moreover, we consider a projection operator $P : L^2(\Omega) \longmapsto (L^2(\Omega))_0$ defined by

$$Pz := z - \frac{1}{|\Omega|}\int_\Omega z\, dx, \quad \forall z \in L^2(\Omega). \tag{6.15}$$

In this subsection, we prepare some properties of real Hilbert spaces V_0^* and V^* in order to treat a mass conservative property of (T) under the framework of evolution inclusions on real Hilbert spaces.

First of all, we define a function $\zeta : V^* \longmapsto \mathbb{R}$ by

$$\zeta(z^*) = \langle z^*, 1\rangle_{V^*,V}, \quad \forall z^* \in V^*,$$

which is surjective, i.e., $\zeta(V^*) = \mathbb{R}$. Using the function ζ, we consider an equivalence relation $\sim$ on V^*, which is defined by

$$z_1^* \sim z_2^* \quad \text{if and only if} \quad \zeta(z_1^*) = \zeta(z_2^*), \quad \text{i.e.,} \quad \langle z_1^*, 1\rangle_{V^*,V} = \langle z_2^*, 1\rangle_{V^*,V}.$$

For any $c \in \mathbb{R}$ we put $W^*(c) := \{z^* \in V^*\,;\, \langle z^*, 1\rangle_{V^*,V} = c\}$. Using the Gelfand triplet (6.13), we see that Lemma 6.3 holds.

Lemma 6.3. *We have $W^*(c_1) \cap W^*(c_2) = \emptyset$ whenever $c_1 \neq c_2$ and*

$$V^* = \bigcup_{c\in\mathbb{R}} W^*(c).$$

Moreover, for any $c \in \mathbb{R}$ the equivalent class $W^*(c)$ is convex and weakly closed in V^*, and satisfies the following relation:

$$W^*(c) = \left\{ z_0^* + \frac{c}{|\Omega|} \in V^*\,;\, z_0^* \in W^*(0) \right\}. \tag{6.16}$$

Proof. Since it is easy to show that $W^*(c)$ is convex and weakly closed in V^*, we omit their proofs and only show (6.16). For any $z_0^* \in W^*(0)$ we consider $z_0^* + \frac{c}{|\Omega|} \in V^*$. Using (6.13) as $\frac{c}{|\Omega|} \in L^2(\Omega)$ and $z = 1 \in V$, we get

$$\left\langle z_0^* + \frac{c}{|\Omega|}, 1 \right\rangle_{V^*,V} = \langle z_0^*, 1\rangle_{V^*,V} + \left\langle \frac{c}{|\Omega|}, 1 \right\rangle_{V^*,V} = c,$$

which implies $z_0^* + \frac{c}{|\Omega|} \in W^*(c)$, hence,

$$\left\{ z_0^* + \frac{c}{|\Omega|} \in V^* \,;\, z_0^* \in W^*(0) \right\} \subset W^*(c).$$

Next, we use (6.13) again. Then, for any $z^* \in W^*(c)$ we get

$$\left\langle z^* - \frac{c}{|\Omega|}, 1 \right\rangle_{V^*,V} = \langle z^*, 1 \rangle_{V^*,V} - \left\langle \frac{c}{|\Omega|}, 1 \right\rangle_{V^*,V} = 0,$$

which implies $z^* - \frac{c}{|\Omega|} \in W^*(0)$. So, we have

$$z^* = \left(z^* - \frac{c}{|\Omega|} \right) + \frac{c}{|\Omega|} \in \left\{ z_0^* + \frac{c}{|\Omega|} \in V^* \,;\, z_0^* \in W^*(0) \right\},$$

that is,

$$W^*(0) \subset \left\{ z_0^* + \frac{c}{|\Omega|} \in V^* \,;\, z_0^* \right\},$$

Hence, we see that (6.16) holds. □

Next, for any $z^* \in W^*(0)$ we define a function $\tilde{z}_0^* : V_0 \longmapsto \mathbb{R}$ by

$$\tilde{z}_0^*(z_0) := \langle z^*, z_0 \rangle_{V^*,V}, \quad \forall z_0 \in V_0. \tag{6.17}$$

Then, we have Lemma 6.4

Lemma 6.4. *For any $z^* \in W^*(0)$ we have $\tilde{z}_0^* \in V_0^*$ and*

$$\|\tilde{z}_0^*\|_{V_0^*} = \|z^*\|_{V^*}, \quad \forall z^* \in W^*(0). \tag{6.18}$$

Moreover, the operator $\pi : W^*(0) \longmapsto V_0^*$, which is defined by $\pi z^* := \tilde{z}_0^*$, is injective.

Proof. It is clear that $\tilde{z}_0^*$ is linear on V_0. Because of $\|z_0\|_{V_0} = \|z_0\|_V$ for all $z_0 \in V_0$, we get

$$|\tilde{z}_0^*(z_0)| \le \|z_0^*\|_{V^*} \|z_0\|_{V_0}, \quad \forall z_0 \in V_0,$$

which implies that $\tilde{z}_0^*$ is bounded on V_0, that is, $\tilde{z}_0^* \in V_0^*$, and the following inequality holds:

$$\|\tilde{z}_0^*\|_{V_0^*} \le \|z^*\|_{V^*}, \quad \forall z^* \in W^*(0). \tag{6.19}$$

Moreover, from (6.17) we get the following equality:

$$\langle \pi z^*, z_0 \rangle_{V_0^*,V_0} = \langle z^*, z_0 \rangle_{V^*,V}, \quad \forall z_0 \in V_0. \tag{6.20}$$

We see from (6.15) and (6.20) that the following equality holds for all $z \in V$:

$$\begin{aligned} |\langle z^*, z \rangle_{V^*,V}| &= \left| \langle z^*, Pz \rangle_{V^*,V} + \left(\frac{1}{|\Omega|} \int_\Omega z \, dx \right) \langle z^*, 1 \rangle_{V^*,V} \right| \\ &= \left| \langle \pi z_1^*, Pz \rangle_{V_0^*,V_0} \right| \le \|\pi z^*\|_{V_0^*} \|Pz\|_{V_0} \le \|\pi z^*\|_{V_0^*} \|z\|_V, \end{aligned} \tag{6.21}$$

which implies that the following inequality holds:

$$\|z^*\|_V \le \|\pi z^*\|_{V_0^*}, \quad \forall z^* \in W^*(0). \tag{6.22}$$

Hence, we see from (6.19) and (6.22) that (6.18) holds.

Next, for z_1^*, $z_2^* \in W^*(0)$ we assume $\pi z_1^* = \pi z_2^*$ on V_0^*. Going back to (6.21), we see that the following equality holds for all $z \in V$:

$$\begin{aligned}\langle z_1^*, z\rangle_{V^*,V} &= \langle z_1^*, Pz\rangle_{V^*,V} + \left(\frac{1}{|\Omega|}\int_\Omega z\,dx\right)\langle z_1^*, 1\rangle_{V^*,V} \\ &= \langle \pi z_1^*, Pz\rangle_{V_0^*,V_0} = \langle \pi z_2^*, Pz\rangle_{V_0^*,V_0} \\ &= \langle \pi z_2^*, Pz\rangle_{V_0^*,V_0} + \left(\frac{1}{|\Omega|}\int_\Omega z\,dx\right)\langle z_2^*, 1\rangle_{V^*,V} \\ &= \langle z_1^*, Pz\rangle_{V^*,V} + \left(\frac{1}{|\Omega|}\int_\Omega z\,dx\right)\langle z_1^*, 1\rangle_{V^*,V} = \langle z_2^*, z\rangle_{V^*,V},\end{aligned} \tag{6.23}$$

which implies $z_1^* = z_2^*$ on V^*. □

From Lemma 6.4 we can identify $W^*(0)(\subset V^*)$ with $\pi(W^*(0))(\subset V_0^*)$, and get

$$W^*(0) \cap L^2(\Omega) = (L^2(\Omega))_0, \tag{6.24}$$

$$\langle z_0^*, z_0\rangle_{V_0^*,V_0} = \langle z_0^*, z_0\rangle_{V^*,V}, \quad \forall z_0^* \in W^*(0),\ \forall z_0 \in V_0. \tag{6.25}$$

For any $c \in \mathbb{R}$ we define an operator $P_c^* : W^*(c) \longmapsto V^*$ by

$$P_c^* z^* := z^* - \frac{c}{|\Omega|}, \quad \forall z^* \in W^*(c).$$

Then, we have Lemma 6.5, which gives us some properties of the operator P_c^*.

Lemma 6.5. *For any $c \in \mathbb{R}$ the operator P_c^* is injective and $P_c^*(W^*(c)) = W^*(0) \subset V_0^*$. Moreover, the operator $P_c^*\colon W^*(c) \longmapsto W^*(0)$ and its inverse $(P_c^*)^{-1}\colon W^*(0) \longmapsto W^*(c)$ are continuous with respect to the strong topologies of V_0^* and V^*, respectively.*

Proof. As the direct consequence of Lemmas 6.3 and 6.4, we get $P_c^*(W^*(c)) = W^*(0) \subset V_0^*$. At first, we show that the operator P_c^* is injective. In order to do this, we assume $P_c^* z_1^* = P_c^* z_2^*$ on V_0^*. From (6.13), (6.24) and (6.25) we get

$$\begin{aligned}\langle z_1^*, z_0\rangle_{V^*,V} &= \left\langle P_c^* z_1^* + \frac{c}{|\Omega|}, z_0\right\rangle_{V^*,V} = \langle P_c^* z_1^*, z_0\rangle_{V^*,V} \\ &= \langle P_c^* z_1^*, z_0\rangle_{V_0^*,V_0} = \langle P_c^* z_1^*, z_0\rangle_{V_0^*,V_0} = \langle P_c^* z_2^*, z_0\rangle_{V^*,V} \\ &= \left\langle P_c^* z_2^* + \frac{c}{|\Omega|}, z_0\right\rangle_{V^*,V} = \langle z_2^*, z_0\rangle_{V^*,V}, \quad \forall z_0 \in V_0.\end{aligned} \tag{6.26}$$

Using the equalities $\langle z_1^*, 1\rangle_{V^*,V} = \langle z_2^*, 1\rangle_{V^*,V} = c$, we see from (6.23) and (6.26) that the following equality holds for all $z \in V$:

$$\begin{aligned}\langle z_1^*, z\rangle_{V^*,V} &= \langle z_1^*, Pz\rangle_{V^*,V} + \left(\frac{1}{|\Omega|}\int_\Omega z\,dx\right)\langle z_1^*, 1\rangle_{V^*,V}\\ &= \langle z_2^*, Pz\rangle_{V^*,V} + \left(\frac{1}{|\Omega|}\int_\Omega z\,dx\right)\langle z_2^*, 1\rangle_{V^*,V} = \langle z_2^*, z\rangle_{V^*,V},\end{aligned}$$

which implies $z_1^* = z_2^*$ in V^*. Hence the operator P_c^* is injective.

Next, we show that the operator $P_c^*\colon W^*(c) \longmapsto W^*(0)$ is continuous with respect to the strong topology of V^*. In order to do this, we consider a sequence $\{z_m^*\}_{m\in\mathbb{N}} \subset W^*(c)$ and a function $z^* \in W^*(c)$ satisfying

$$z_m^* \longrightarrow z^* \quad \text{in} \quad V^* \quad \text{as} \quad m \to \infty.$$

Since we have $z_m^* - z^* \in W^*(0)$, we see from Lemma 6.4 that the following equality holds:

$$\|P_c^* z_m^* - P_c^* z^*\|_{V_0^*} = \|z_m^* - z^*\|_{V_0^*} = \|z_m^* - z^*\|_{V^*}, \quad \forall n \in \mathbb{N}, \tag{6.27}$$

which implies $P_c^* z_m^* \longrightarrow P_c^* z^*$ in V_0^* as $m \to \infty$.

Finally, we show that the operator $(P_c^*)^{-1}\colon W^*(0) \longmapsto W^*(c)$ is continuous with respect to the strong topology of V_0^*. We consider a sequence $\{z_{0,m}^*\}_{m\in\mathbb{N}} \subset W^*(0)$ and a function $z_0^* \in W^*(0)$ satisfying

$$z_{0,m}^* \longrightarrow z_0^* \quad \text{in} \quad V_0^* \quad \text{as} \quad m \to \infty.$$

Since we have $(P_c^*)^{-1} z_{0,m}^* - (P_c^*)^{-1} z_0^* \in W^*(0)$, we see from (6.27) that the following equality holds:

$$\begin{aligned}\|(P_c^*)^{-1} z_{0,m}^* - (P_c^*)^{-1} z_0^*\|_{V^*} &= \left\|\left(z_{0,m}^* + \frac{c}{|\Omega|}\right) - \left(z_0^* + \frac{c}{|\Omega|}\right)\right\|_{V^*}\\ &= \|z_{0,m}^* - z_0^*\|_{V_0^*}, \quad \forall n \in \mathbb{N},\end{aligned}$$

which implies $(P_c^*)^{-1} z_{0,m}^* \longrightarrow (P_c^*)^{-1} z_0^*$ in V^* as $m \to \infty$. □

From Lemmas 6.4 and 6.5 (cf. (6.24) and (6.25)) we have

$$P_c^*\left(W^*(c) \cap L^2(\Omega)\right) = (L^2(\Omega))_0, \quad \forall c \in \mathbb{R},$$

$$\langle P_c^* z^*, z_0\rangle_{V_0^*,V_0} = \langle z^*, z_0\rangle_{V^*,V}, \quad \forall c \in \mathbb{R},\ \forall z^* \in W^*(c),\ \forall z_0 \in V_0.$$

From Lemmas 6.3–6.5, we get Proposition 6.6, which is obtained in [9, Lemma 1.1].

Proposition 6.6. *For any $z^* \in V^*$ there exists a constant $c \in \mathbb{R}$, which is uniquely determined, such that $z^* \in W^*(c)$ and the following equality holds for all $z \in V$:*

$$\begin{aligned}\langle z^*, z\rangle_{V^*,V} &= \langle P_c^* z^*, Pz\rangle_{V_0^*,V_0} + \frac{c}{|\Omega|}\int_\Omega z\,dx \\ &= \langle P_c^* z^*, Pz\rangle_{V_0^*,V_0} + \left(\frac{1}{|\Omega|}\int_\Omega z\,dx\right)\langle z^*, 1\rangle_{V^*,V}.\end{aligned}$$

6.2. Quasi-variational inner products on V_0^*

In order to induce a quasi-variational structure of inner products of V_0^*, for any $v \in A_v$ we denote by $V_0(v)$ a real Hilbert space V_0 with an inner product

$$(z_{0,1}, z_{0,2})_{V_0(v)} := \int_\Omega d_u(v)\nabla z_{0,1}\cdot\nabla z_{0,2}dx, \quad \forall z_{0,1}, z_{0,2} \in V_0,$$

and by $V_0^*(v)$ a real Hilbert space V_0^* whose inner product is given by

$$(z_{0,1}^*, z_{0,2}^*)_{V_0^*(v)} := \langle z_{0,1}^*, F_0(v)^{-1} z_{0,2}^*\rangle_{V_0^*,V_0}, \quad \forall z_{0,1}^*, z_{0,2}^* \in V_0^*,$$

where $F_0(v)\colon V_0(v) \longmapsto V_0^*$ is the duality map. Using (T1), we get Lemma 6.7 which implies that (A1) is satisfied. We entrust its proof to [9, Lemmas 3.1 and 3.2] and omit it in this paper.

Lemma 6.7. *The following properties are satisfied:*

(a) There exist constants $\tilde{c}_1 > 0$ and $\tilde{c}_2 > 0$ such that

$$\tilde{c}_1\|z_0^*\|_{V_0^*} \le \|z_0^*\|_{V_0^*(v)} \le \tilde{c}_2\|z_0^*\|_{V_0^*}, \quad \forall v \in A_v,\ \forall z_0^* \in V_0^*.$$

(b) There exists a constant $\tilde{c}_3 > 0$ such that the following inequality holds:

$$\begin{gathered}\left|\|z_0^*\|^2_{V_0^*(v_1)} - \|z_0^*\|^2_{V_0^*(v_2)}\right| \le \tilde{c}_3\|v_1 - v_2\|_{C(\overline{\Omega})}\cdot\|z_0^*\|^2_{V_0^*(v_2)}, \\ \forall v_1,\ v_2 \in A_v,\ \forall z_0^* \in V_0^*.\end{gathered}$$

Using the family $\{(\cdot,\cdot)_{V_0^*(v)}\,;\, v \in A_v\}$ and a function $\tilde{v} \in W^{1,1}(0,T;C(\overline{\Omega}))$, we consider a family of quasi-variational inner product of V_0^*, which is given by $\{(\cdot,\cdot)_{V_0^*(\tilde{v}(t))}\,;\, 0 \le t \le T\}$, and see from (b) in Lemma 6.7 that the following inequality holds for all $s,\ t \in [0,T]$:

$$\left|\|z_0^*\|^2_{V_0^*(\tilde{v}(t))} - \|z_0^*\|^2_{V_0^*(\tilde{v}(s))}\right| \le \tilde{c}_3\left|\int_s^t \|\tilde{v}'(\sigma)\|_{C(\overline{\Omega})}d\sigma\right|\|z_0^*\|^2_{V_0^*(\tilde{v}(s))}.$$

At the end of this subsection, we give Lemma 6.8, which gives the relation between the duality maps $F(\tilde{v})$ and $F_0(\tilde{v})$.

Lemma 6.8. *For any $\tilde{v} \in A_v$ the following equality holds for all $z_1,\ z_2 \in V$:*

$$\langle F(\tilde{v})z_1, z_2\rangle_{V^*,V} = \langle (F_0(\tilde{v})\circ P)z_1, Pz_2\rangle_{V_0^*,V_0} + \left(\int_\Omega z_1\,dx\right)\left(\int_\Omega z_2\,dx\right).$$

Proof. For any $z_1,\, z_2 \in V$ we have

$$\begin{aligned}\langle F(\tilde{v})z_1, z_2\rangle_{V^*,V} &= \int_\Omega d_u(\tilde{v})\nabla z_1 \cdot \nabla z_2\, dx + \left(\int_\Omega z_1\, dx\right)\left(\int_\Omega z_2\, dx\right)\\ &= \int_\Omega d_u(\tilde{v})\nabla P z_1 \cdot \nabla P z_2\, dx + \left(\int_\Omega z_1\, dx\right)\left(\int_\Omega z_2\, dx\right)\\ &= \langle (F_0(\tilde{v}) \circ P)z_1, Pz_2\rangle_{V_0^*,V_0} + \left(\int_\Omega z_1\, dx\right)\left(\int_\Omega z_2\, dx\right).\end{aligned}$$

Hence, we see that this lemma holds. □

6.3. Evolution system on $A_v \times A_w$

First of all, we denote by $\hat{\beta} : \mathbb{R} \longmapsto \mathbb{R}$ the indicator function on the compact interval $[0, \alpha]$, that is,

$$\hat{\beta}(r) := \begin{cases} 0 & \text{if} \quad r \in [0, \alpha], \\ \infty & \text{if} \quad r \in (-\infty, 0) \cup (\alpha, \infty), \end{cases}$$

which is proper, nonnegative, l.s.c. and convex on $\mathbb{R}$. From (a) in (T2) we have the following inequality:

$$r^2 - \alpha^2 \le \hat{\beta}(r) \le \hat{\beta}(v\,; r), \quad \forall r \in \mathbb{R},\ \forall v \in [0, \alpha]. \tag{6.28}$$

Using the function $\hat{\beta}$, we define a function $\varphi : V^* \longmapsto \mathbb{R} \cup \{\infty\}$ by

$$\varphi(z^*) := \begin{cases} \displaystyle\int_\Omega \hat{\beta}(z^*)dx = 0, \\ \qquad\qquad \text{if} \quad z^* \in D(\varphi) := \left\{\tilde{z}^* \in L^\infty(\Omega)\,;\, \hat{\beta}(\tilde{z}^*) \in L^1(\Omega)\right\}, \\ \infty, \quad \text{if} \quad z^* \in V^* \setminus D(\varphi), \end{cases} \tag{6.29}$$

In what follows, we denote by $c_0 > 0$ the constant

$$c_0 := \int_\Omega u_0\, dx. \tag{6.30}$$

From (T5) we have

$$0 \le \frac{c_0}{|\Omega|} \le \alpha. \tag{6.31}$$

For any $\tilde{T} \in (0, T]$ we define a nonempty, closed and convex subset $\mathcal{V}(c_0, \tilde{T})$ of $C([0, \tilde{T}]; V^*)$ by

$$\mathcal{V}(c_0, \tilde{T}) := \left\{\tilde{u} \in C([0, \tilde{T}]; V^*)\,;\ \begin{array}{l} \tilde{u}(t) \in W^*(c_0), \\ \varphi\left(\tilde{u}(t)\right) = 0 \quad \text{for all } t \in [0, \tilde{T}] \end{array}\right\},$$

and for any $\tilde{u} \in \mathcal{V}(c_0, \tilde{T})$ we define a family $\{S(\tilde{u}\,; t, s)\,;\, 0 \le s \le t \le \tilde{T}\}$ by the following way:

$$S(\tilde{u}\,; t, s)(\tilde{v}, \tilde{w}) := \left(S_1(\tilde{u}, \tilde{w}\,; t, s)\tilde{v}, S_2(\tilde{u}\,; t, s)\tilde{w}\right), \quad \forall (\tilde{v}, \tilde{w}) \in A_v \times A_w, \tag{6.32}$$

where $S_1(\tilde{u},\tilde{w}\,;t,s)\tilde{v}$ and $S_2(\tilde{u}\,;t,s)\tilde{w}$ are defined by

$$S_1(\tilde{u},\tilde{w}\,;t,s)\tilde{v} := \tilde{v}\exp\left(-a\int_s^t S_2(\tilde{u}\,;t,\sigma)\tilde{w}d\sigma\right), \tag{6.33}$$

$$S_2(\tilde{u}\,;t,s)\tilde{w} := e^{(t-s)(d_w\Delta-b)}\tilde{w} + c\int_s^t e^{(t-\sigma)(d_v\Delta-b)}\tilde{u}(\sigma)d\sigma. \tag{6.34}$$

Remark 6.9. We see that the pair $(\bar{v}(t),\bar{w}(t)) := S(\tilde{u}\,;t,s)(\tilde{v},\tilde{w})$ is a unique strong solution to the following Cauchy problem on $[s,\tilde{T}]$, which is a system describing the dynamics of MDE and ECM in tumor invasion model:

$$\begin{cases} \bar{v}_t = a\bar{v}\bar{w}, & \text{a.e. in} \quad \Omega\times(s,\tilde{T}),\\ \bar{w}_t = d_w\Delta\bar{w} - b\bar{w} + c\tilde{u}, & \text{a.e. in} \quad \Omega\times(s,\tilde{T}),\\ \nabla\bar{w}\cdot\nu = 0, & \text{a.e. on} \quad \Gamma\times(s,\tilde{T}),\\ \bar{v}(s) = \tilde{v}, & \text{a.e. in} \quad \Omega,\\ \bar{w}(s) = \tilde{w}, & \text{a.e. in} \quad \Omega. \end{cases}$$

Moreover, we have

$$0 \le \bar{v}(x,t) \le \tilde{v}(x), \quad 0 \le \bar{w}(x), \quad \forall (x,t)\in\overline{\Omega}\times[s,\tilde{T}].$$

Then, we obtain Lemmas 6.10 and 6.11 in [8, Section 2], which implies that (A2) holds.

Lemma 6.10. *There exist constants $K_1 > 0$, which depends on $\|\tilde{w}\|_{W^{1,\infty}(\Omega)}$, and $K_2 > 0$, which depends on $\|\tilde{v}\|_{W^{1,\infty}(\Omega)}$ and $\|\tilde{w}\|_{W^{1,\infty}(\Omega)}$, such that the following uniform estimates hold:*

$$\begin{cases} \displaystyle\sup_{\tilde{u}\in\mathcal{V}(c_0,\tilde{T})}\left\{\sup_{0\le s\le\tilde{T}}\left(\sup_{s\le t\le\tilde{T}}\|S_2(\tilde{u}\,;t,s)\tilde{w}\|_{W^{1,\infty}(\Omega)}\right)\right\} \le K_1,\\ \displaystyle\sup_{\tilde{u}\in\mathcal{V}(c_0,\tilde{T})}\left\{\sup_{0\le s\le\tilde{T}}\left(\sup_{s\le t\le\tilde{T}}\|\nabla S_1(\tilde{u},\tilde{w}\,;t,s)\tilde{v}\|_{(L^\infty(\Omega))^N}\right)\right\} \le K_2(T+1),\\ \displaystyle\sup_{\tilde{u}\in\mathcal{V}(c_0,\tilde{T})}\left\{\sup_{0\le s\le\tilde{T}}\left(\sup_{s\le t\le\tilde{T}}\|(S_1(\tilde{u},\tilde{w}\,;t,s)\tilde{v})'\|_{C(\overline{\Omega})}\right)\right\} \le a\alpha K_1. \end{cases} \tag{6.35}$$

We see from Remark 6.9 and Lemma 6.10 that for any $\tilde{T}\in(0,T]$ and $\tilde{u}\in\mathcal{V}(c_0,\tilde{T})$ the operator $S(\tilde{u}\,;t,s)\colon A_v\times A_w \longmapsto A_v\times A_w$ is well-defined for all $s,\ t$ with $0\le s\le t\le\tilde{T}$.

Lemma 6.11. *We consider a family*

$$\left\{\{S(\tilde{u}\,;t,s)\,;\,0\le s\le t\le\tilde{T}\}\,;\,\tilde{T}\in(0,T],\ \tilde{u}\in\mathcal{V}(c_0,\tilde{T})\right\},$$

where $S(\tilde{u}\,;t,s)$ $(0\le s\le t\le\tilde{T})$ is the same operator that is defined by (6.32)–(6.34). Then, the following properties (a)–(g) are satisfied:

(a) Assume that a sequence $\{(\tilde{u}_m, \tilde{v}_m, \tilde{w}_m)\}_{m\in\mathbb{N}}$ *and a triplet* $(\tilde{u}, \tilde{v}, \tilde{w})$ *in* $\mathcal{V}(c_0, \tilde{T}) \times A_v \times A_w$ *satisfy*

$$(\tilde{u}_m, \tilde{v}_m, \tilde{w}_m) \longrightarrow (\tilde{u}, \tilde{v}, \tilde{w}) \quad \text{in} \quad C([0, \tilde{T}; V^*) \times C(\overline{\Omega}) \times L^2(\Omega) \quad \text{as} \quad m \to \infty.$$

Then, for any $s \in [0, \tilde{T})$ *we have*

$$S(\tilde{u}_m\,; \cdot, s)(\tilde{v}_m, \tilde{w}_m) \longrightarrow S(\tilde{u}\,; \cdot, s)(\tilde{v}, \tilde{w}) \quad \text{in} \quad C([s, \tilde{T}]; C(\overline{\Omega})) \times \big(C([s, \tilde{T}]; L^2(\Omega)) \cap L^2(s, \tilde{T}; H^1(\Omega))\big).$$

(b) Assume that a sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ *and a function* $\tilde{u}$ *in* $\mathcal{V}(c_0, \tilde{T})$ *satisfy*

$$\tilde{u}_m \longrightarrow \tilde{u} \quad \text{in} \quad C([0, \tilde{T}]; V^*) \quad \text{as} \quad m \to \infty.$$

Then, for any $s \in [0, \tilde{T}]$ *we have*

$$S(\tilde{u}_m\,; \cdot, s)(\tilde{v}, \tilde{w}) \longrightarrow S(\tilde{u}\,; \cdot, s)(\tilde{v}, \tilde{w}) \quad \text{in} \quad C([s, \tilde{T}]; C(\overline{\Omega}) \cap H^1(\Omega)) \times \big(C([s, \tilde{T}]; L^2(\Omega)) \cap L^2(s, \tilde{T}; H^1(\Omega))\big).$$

(c) $S(\tilde{u}\,; t, t)$ *is the identity operator on* $A_v \times A_w$ *for all* $t \in [0, \tilde{T}]$.

(d) $S_1(\tilde{u}, \tilde{w}\,; \cdot, 0)\tilde{v} \in W^{1,1}(0, \tilde{T}; C(\overline{\Omega}))$ *for all* $(\tilde{v}, \tilde{w}) \in A_v \times A_w$.

(e) For any times $\tilde{T}_i \in (0, T]$ *and functions* $\tilde{u}_i \in \mathcal{V}(c_0, \tilde{T}_i)$ $(i = 1, 2)$ *we assume that there exists a time* $\tilde{T}_0 \in [0, \min\{\tilde{T}_1, \tilde{T}_2\}]$ *such that* $\tilde{u}_1(t) = \tilde{u}_2(t)$ *in* V^* *for all* $t \in [0, \tilde{T}_0]$. *Then, we have* $S(\tilde{u}_1\,; t, 0) = S(\tilde{u}_2\,; t, 0)$ *on* $A_v \times A_w$ *for all* $t \in [0, \tilde{T}_0]$.

(f) $S(\tilde{u}\,; t, s) = S(\tilde{u}\,; t, \tau) \circ S(\tilde{u}\,; \tau, s)$ *on* $A_v \times A_w$ *for all* s, t, τ *with* $0 \le s \le \tau \le t \le \tilde{T}$.

(g) The following equality holds for any $\tau \in [-\tilde{T}, \tilde{T}]$:

$$S(\sigma_\tau \tilde{u}\,; t, s) = S(\tilde{u}\,; t + \tau, s + \tau) \quad \text{on} \quad A_v \times A_w, \quad 0 \le \forall s \le \forall t \le \tilde{T} - \tau.$$

Remark 6.12. Strictly speaking, Lemma 6.5 is not the same to (A2). Actually, a function $\tilde{u}$ in Lemma 6.5 is in $\mathcal{V}(c_0, \tilde{T})$, not in $C([0, \bar{T}]; V^*) \setminus \mathcal{V}(c_0, \tilde{T})$ in (A2). This difference arises from the boundedness

$$\sup_{0 \le t \le \bar{T}} \|S_2(\tilde{u}\,; t, 0)\tilde{w}\|_{C(\overline{\Omega})} < \infty,$$

which is only obtained for the case $\tilde{u} \in \mathcal{V}(c_0, \tilde{T})$, not the case $\tilde{u} \in C([0, \bar{T}]; V^*) \setminus \mathcal{V}(c_0, \tilde{T})$.

Moreover, we get Lemma 6.13, which is a direct consequence of (6.33) and Remark 6.9.

Lemma 6.13. *For any* $(\tilde{u}, \tilde{v}, \tilde{w}) \in \mathcal{V}(c_0, \tilde{T}) \times A_v \times A_w$ *we have*

$$0 \le (S_1(\tilde{u}, \tilde{w}\,; t_2, s)\tilde{v})(x) \le (S_1(\tilde{u}, \tilde{w}\,; t_1, s)\tilde{v})(x) \le \alpha,$$
$$\forall x \in \Omega,\ 0 \le \forall s \le \forall t_1 \le \forall t_2 \le \tilde{T}.$$

6.4. Quasi-variational convex functions on V_0^*

Using the function φ given by (6.29), we define a function $\varphi_{c_0} \colon V_0^* \longmapsto \mathbb{R} \cup \{\infty\}$ by

$$\varphi_{c_0}(z_0^*) := \begin{cases} \varphi\left(z_0^* + \dfrac{c_0}{|\Omega|}\right) = 0, & \text{if} \quad z_0^* \in D(\varphi_{c_0}), \\ \infty, & \text{if} \quad z_0^* \in V_0^* \setminus D(\varphi_{c_0}), \end{cases} \tag{6.36}$$

where the effective domain $D(\varphi_{c_0})$ of φ_{c_0} is given by

$$D(\varphi_{c_0}) := \left\{ \tilde{z}_0^* \in (L^2(\Omega))_0 \cap L^\infty(\Omega)\,;\, 0 \le \tilde{z}_0^*(x) + \frac{c_0}{|\Omega|} \le \alpha \quad \text{a.e. } x \in \Omega \right\}.$$

Moreover, for each $(\tilde{u}, \tilde{v}, \tilde{w}) \in \mathcal{V}(c_0, T) \times A_v \times A_w$ satisfying $(\tilde{u}(0), \tilde{v}) \in D$, where D is given in (T5), we define a function $\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}) := \varphi_{c_0}(\tilde{u}\,; S_1(\tilde{u}, \tilde{w}\,; t, 0)\tilde{v}) : V_0^* \longmapsto \mathbb{R} \cup \{\infty\}$ by

$$\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}\,; z_0^*) := \begin{cases} \displaystyle\int_\Omega \hat{\beta}\left(S_1(\tilde{u}, \tilde{w}\,; t, 0)\tilde{v}\,; z_0^* + \frac{c_0}{|\Omega|}\right) dx, \\ \qquad \text{if} \quad z^* \in D(\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w})), \\ \infty, \quad \text{if} \quad z^* \in V^* \setminus D(\varphi(t, \tilde{u}, \tilde{v}, \tilde{w})), \end{cases} \tag{6.37}$$

where the effective domain $D(\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}))$ of $\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w})$ is defined by

$$D(\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w})) := \Bigg\{ \tilde{z}^* \in (L^2(\Omega))_0 \cap L^\infty(\Omega)\,; \\ \hat{\beta}\left(S_1(\tilde{u}, \tilde{w}\,; t, 0)\tilde{v}\,; \tilde{z}_0^* + \frac{c_0}{|\Omega|}\right) \in L^1(\Omega) \Bigg\}.$$

Next, we define a set $\mathcal{X}$ by

$$\mathcal{X} := \left\{ \{\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w})\,;\, 0 \le t \le T\}\ ; \quad \begin{array}{l} \tilde{u} \in \mathcal{V}(c_0, T),\ (\tilde{v}, \tilde{w}) \in A_v \times A_w, \\ (\tilde{u}(0), \tilde{v}) \in D \end{array} \right\}.$$

For the functions φ_{c_0} and $\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w})$ we have Lemma 6.14, which is obtained in [9, Lemma 3.3] and implies that (a) and (b) in (A3) are satisfied.

Lemma 6.14. *The following properties hold:*

(a) The function φ_{c_0} *is proper, nonnegative, l.s.c. and convex on* V_0^*.

(b) For any family $\{\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})\,;\,0\le t\le T\}\in\mathcal{X}$ *the function* $\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})$ *is proper, nonnegative, l.s.c. and convex on* $V_0^*(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})$.

(c) For any family $\{\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})\,;\,0\le t\le T\}\in\mathcal{X}$ *we have*

$$\varphi_{c_0}(z_0^*)\le\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z_0^*),\quad 0\le\forall t\le T, \tag{6.38}$$

and $D(\varphi_{c_0}(t,\tilde{u},\tilde{w},\tilde{v}))=D(\varphi_{c_0})$ *for all* $t\in[0,T]$.

(d) For any $r\ge 0$ *the level set* $\{z_0^*\in V_0^*\,;\,\|z_0^*\|_{V_0^*}\le r,\ \varphi_{c_0}(z_0^*)\le r\}$ *is compact in* V_0^*.

(e) Assume that for families $\{\varphi_{c_0}(t,\tilde{u}_i,\tilde{v},\tilde{w})\,;\,0\le t\le T\}\in\mathcal{X}$ $(i=1,2)$ *there exists a time* $\tilde{T}\in[0,T]$ *such that* $\tilde{u}_1(t)=\tilde{u}_2(t)$ *on* V^* *for all* $t\in[0,\bar{T}]$. *Then, we have*

$$\varphi_{c_0}(t,\tilde{u}_1,\tilde{v},\tilde{w})=\varphi_{c_0}(t,\tilde{u}_2,\tilde{v},\tilde{w})\quad on\quad V_0^*,\quad 0\le\forall t\le\bar{T}.$$

Proof. (a) Using the argumentation similar to that of (b), we can show (a). Hence, we omit this proof and entrust it to (b).
(b) First of all, we see from (T2) that $\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})$ is nonnegative and convex on $V_0^*(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})$. Using (a) in (T2) and (T5), we have

$$0\le\tilde{u}(0)\le\alpha\quad\text{a.e. in}\quad\Omega,$$

hence, from (b) in (T2)

$$0\le\hat{\beta}(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v}\,;\tilde{u}(0))\le\beta_1^*\tilde{u}(0)\le\alpha\beta_1^*\quad\text{a.e. in}\quad\Omega,$$

which implies

$$\tilde{u}(0)-\frac{c_0}{|\Omega|}\in D(\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})),\quad 0\le\forall t\le T.$$

So we see that $\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})$ is proper. Moreover, in order to show that $\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})$ is l.s.c. on $V_0^*(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})$, we show that for any $a\ge 0$ a level set below

$$K(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v}\,;a):=\{z_0^*\in V_0^*(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})\,;\,\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z_0^*)\le a\},$$

is closed in $V_0^*(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})$. For this, we consider a proper, nonnegative, weakly sequentially l.s.c. and convex function $\psi(t,\tilde{u},\tilde{v},\tilde{w}):L^2(\Omega)\longmapsto\mathbb{R}\cup\{\infty\}$, which is defined by

$$\psi(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*):=\begin{cases}\displaystyle\int_\Omega\hat{\beta}(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v}\,;z^*)dx,\\ \qquad\text{if}\quad z^*\in D(\psi(t,\tilde{u},\tilde{v},\tilde{w})),\\ \infty,\quad\text{if}\quad z^*\in L^2(\Omega)\setminus D(\psi(t,\tilde{u},\tilde{v},\tilde{w})),\end{cases}$$

where the effective domain $D(\psi(t,\tilde{u},\tilde{v},\tilde{w}))$ of $\psi(t,\tilde{u},\tilde{v},\tilde{w})$ is given by

$$D(\psi(t,\tilde{u},\tilde{v},\tilde{w})):=\left\{\tilde{z}^*\in L^2(\Omega)\,;\,\hat{\beta}(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v}\,;\tilde{z}^*)\in L^1(\Omega)\right\}.$$

First of all, we note that the following equality holds:

$$\psi(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*)=\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;P^*_{c_0}z^*),\quad \forall z^*\in W^*(c_0)\cap L^2(\Omega). \tag{6.39}$$

We consider a sequence $\{z^*_{0,n}\}_{n\in\mathbb{N}}$ and a function z^*_0 in $V^*_0(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})$ satisfying

$$z^*_{0,n}\longrightarrow z^*_0\quad\text{in}\quad V^*_0(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})\quad\text{as}\quad n\to\infty.$$

Since $\{z^*_{0,n}\}_{n\in\mathbb{N}}$ is bounded in $(L^2(\Omega))_0$, without losing generality we may assume that the following convergence holds:

$$z^*_{0,n}\longrightarrow z^*_0\quad\text{weakly in}\quad (L^2(\Omega))_0\quad\text{as}\quad n\to\infty,$$

hence,

$$z^*_{0,n}+\frac{c_0}{|\Omega|}\longrightarrow z^*_0+\frac{c_0}{|\Omega|}\quad\text{weakly in}\quad L^2(\Omega)\quad\text{as}\quad n\to\infty. \tag{6.40}$$

Using (6.39) and (6.40), we have

$$\begin{aligned}
&\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_0)=\varphi_{c_0}\left(t,\tilde{u},\tilde{v},\tilde{w}\,;P^*_{c_0}\left(z^*_0+\frac{c_0}{|\Omega|}\right)\right)\\
&=\psi\left(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_0+\frac{c_0}{|\Omega|}\right)\le\liminf_{n\to\infty}\psi\left(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_{0,n}+\frac{c_0}{|\Omega|}\right)\\
&=\liminf_{n\to\infty}\varphi_{c_0}\left(t,\tilde{u},\tilde{v},\tilde{w}\,;P^*_{c_0}\left(z^*_{0,n}+\frac{c_0}{|\Omega|}\right)\right)=\liminf_{n\to\infty}\varphi_{c_0}\left(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_{0,n}\right)\le a,
\end{aligned}$$

which implies $z^*_0\in K(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v}\,;a)$, that is, $K(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v}\,;a)$ is closed in $V^*_0(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})$. Hence, $\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})$ is l.s.c. on $V^*_0(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})$.
(c) We see from (6.28), (6.29), (6.36) and (6.37) that (6.38) holds. Moreover, (6.38) implies

$$D(\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}))\subset D(\varphi_{c_0}),\quad 0\le\forall t\le T.$$

Conversely, since for any $z^*_0\in D(\varphi_{c_0})$ we have

$$0\le z^*_0(x)+\frac{c_0}{|\Omega|}\le\alpha,\quad\text{a.e. } x\in\Omega,$$

we see from (a) and (b) in (T2) that the following inequality holds for all $t\in[0,T]$:

$$\begin{aligned}
0&\le\hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_0(x)+\frac{c_0}{|\Omega|}\right)\\
&\le\beta^*_1\left(z^*_0(x)+\frac{c_0}{|\Omega|}\right)\le\alpha\beta^*_1,\quad\text{a.e. } x\in\Omega,
\end{aligned}$$

hence, we have $\varphi_{c_0}(t,\tilde{u},\tilde{w},\tilde{v})\le\alpha\beta^*_1|\Omega|$ for all $t\in[0,T]$. So we get

$$D(\varphi_{c_0})\subset D(\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})),\quad 0\le\forall t\le T.$$

(d) Since φ_{c_0} is l.s.c. on V_0^*, we see that for any $r \geq 0$ the level set $\{z_0^* \in V_0^* \,;\, \|z_0^*\|_{V_0^*} \leq r,\ \varphi_{c_0}(z_0^*) \leq r\}$ is closed in V_0^* as well as bounded in $(L^2(\Omega))_0$ because of the following inequality, which is derived from (6.28):

$$\|z_0^*\|_{(L^2(\Omega))_0} \leq \sqrt{r + \alpha^2|\Omega|}.$$

Hence the level set $\{z_0^* \in V_0^* \,;\, \|z_0^*\|_{V_0^*} \leq r,\ \varphi_{c_0}(z_0^*) \leq r\}$ is compact in V_0^*.
(e) This property is a direct consequence of (e) in Lemma 6.10 and omit its proof here. □

Remark 6.15. From (c) in Lemma 6.14 the family $\{D(\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}))\,;\, 0 \leq t \leq T\}$ of the effective domains of $\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w})$ is independent of $t \in [0, T]$ and a choice of $\{\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{v})\,;\, 0 \leq t \leq T\} \in \mathcal{X}$. Roughly speaking, in our setting the effective domains do not change in time. Although this condition may decrease the mathematical interest, the mass conservative property does not allow us to move the effective domains in time.

Next, we show Lemma 6.16, which implies that the condition (c) in (A3) is satisfied.

Lemma 6.16. *Assume that a sequence* $\{\varphi_{c_0}(t, \tilde{u}_m, \tilde{v}_m, \tilde{w}_m)\,;\, 0 \leq t \leq T\}_{m\in\mathbb{N}}$ *and* $\{\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w})\,;\, 0 \leq t \leq T\}$ *in* $\mathcal{X}$ *satisfy the following convergence as* $m \to \infty$*:*

$$(\tilde{u}_m, \tilde{v}_m, \tilde{w}_m) \longrightarrow (\tilde{u}, \tilde{v}, \tilde{w}) \quad \text{in} \quad C([0,T]; V^*) \times C(\overline{\Omega}) \times L^2(\Omega).$$

Then, for any $t \in [0, T]$ *we have the following convergence as* $m \to \infty$*:*

$$\varphi_{c_0}(t, \tilde{u}_m, \tilde{v}_m, \tilde{w}_m) \longrightarrow \varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}) \quad \text{on} \quad V_0^*(S_1(\tilde{u}, \tilde{w}\,; t, 0)\tilde{v})$$
in the strong sense of Mosco.

That is, the following properties are satisfied:

(i) For any $z_0^* \in D(\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}))$ *there exists a sequence* $\{z_{0,m}^*\}_{m\in\mathbb{N}}$ *in* $L^\infty(\Omega)$ *with*

$$z_{0,m}^* \longrightarrow z_0^* \quad \text{in} \quad L^\infty(\Omega) \quad \text{as} \quad m \to \infty,$$

$$\lim_{m\to\infty} \varphi_{c_0}(t, \tilde{u}_m, \tilde{v}_m, \tilde{w}_m\,; z_{0,m}^*) = \varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}\,; z_0^*).$$

(ii) For any subsequence $\{(\tilde{u}_{m_k}, \tilde{v}_{m_k}, \tilde{w}_{m_k})\}_{k\in\mathbb{N}}$ *of* $\{(\tilde{u}_m, \tilde{v}_m, \tilde{w}_m)\}_{m\in\mathbb{N}}$ *we have*

$$\varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}\,; z_0^*) \leq \liminf_{k\to\infty} \varphi_{c_0}(t, \tilde{u}_{m_k}, \tilde{v}_{m_k}, \tilde{w}_{m_k}\,; z_{0,k}^*)$$

whenever a sequence $\{z_{0,k}^*\}_{k\in\mathbb{N}}$ *and a function* z_0^* *in* V_0^* *satisfy*

$$z_{0,k}^* \longrightarrow z_0^* \quad \text{weakly in} \quad V_0^*(S_1(\tilde{u}, \tilde{w}\,; t, 0)\tilde{v}) \quad \text{as} \quad k \to \infty.$$

Proof. We use the argumentation similar to that of [8, Proposition 4.2] and [9, Lemma 4.1].
(i) First of all, we note that the following property holds:

$$\left\{ \tilde{z}_0^* + \frac{c_0}{|\Omega|} \in L^\infty(\Omega)\,;\, \tilde{z}_0^* \in D(\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})) \right\} \subset D(\psi(t,\tilde{u},\tilde{v},\tilde{w})).$$

We see that for any $z_0^* \in D(\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}))$ there exists a sequence $\{z_m^*\}_{m\in\mathbb{N}} \subset L^\infty(\Omega)$ such that

$$z_m^* \longrightarrow z_0^* + \frac{c_0}{|\Omega|} \quad \text{in} \quad L^\infty(\Omega) \quad \text{as} \quad m \to \infty, \tag{6.41}$$

$$\lim_{m\to\infty} \psi(t,\tilde{u}_m,\tilde{v}_m,\tilde{v}_m\,;z_m^*) = \varphi_{c_0}\left(t,\tilde{u},\tilde{v},\tilde{w}\,;z_0^*\right). \tag{6.42}$$

Then, we see from (6.41) that the following convergence holds:

$$\lim_{m\to\infty} c_m = c_0, \quad \text{where} \quad c_m := \int_\Omega z_m^* dx, \quad \forall m \in \mathbb{N}. \tag{6.43}$$

Using the following inequality:

$$\left|(P_{c_m}^* z_m^*)(x) - z_0^*(x)\right| \le \left| z_m^* - \left(z_0^* + \frac{c_0}{|\Omega|} \right) \right| + \frac{|c_m - c_0|}{|\Omega|},$$

we see from (6.41) and (6.43) that the following convergence holds:

$$P_{c_m}^* z_m^* \longrightarrow z_0^* \quad \text{in} \quad L^\infty(\Omega) \quad \text{as} \quad m \to \infty. \tag{6.44}$$

Moreover, from (T2) we get the following inequality:

$$\begin{aligned}
&\left|\varphi_{c_0}(t,\tilde{u}_m,\tilde{v}_m,\tilde{w}_m\,;P_{c_m}^* z_m^*) - \varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z_0^*)\right| \\
\le\ & \left|\psi\left(t,\tilde{u}_m,\tilde{v}_m,\tilde{w}_m\,;P_{c_m}^* z_m^* + \frac{c_0}{|\Omega|}\right) - \psi(t,\tilde{u}_m,\tilde{v}_m,\tilde{w}_m\,;z_m^*)\right| \\
&\qquad + |\psi(t,\tilde{u}_m,\tilde{v}_m,\tilde{w}_m\,;z_m^*) - \varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z_0^*)| \\
\le\ & \int_\Omega \left| \hat{\beta}\left(S_1(\tilde{u}_m,\tilde{w}_m\,;t,0)\tilde{v}\,;z_m^* - \frac{c_m}{|\Omega|} + \frac{c_0}{|\Omega|} \right) \right. \\
&\qquad \left. - \hat{\beta}(S_1(\tilde{u}_m,\tilde{w}_m\,;t,0)\tilde{v}_m\,;z_m^*) \right| dx \\
&\quad + |\psi(t,\tilde{u}_m,\tilde{v}_m,\tilde{w}_m\,;z_m^*) - \varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z_0^*)| \\
\le\ & \beta_1^*|c_m - c_0| + |\psi(t,\tilde{u}_m,\tilde{v}_m,\tilde{w}_m\,;z_m^*) - \varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z_0^*)|.
\end{aligned} \tag{6.45}$$

Using (6.42), (6.43) and (6.45), we get

$$\lim_{n\to\infty} \varphi_{c_0}(t,\tilde{u}_m,\tilde{v}_m,\tilde{w}_m\,;P_{c_m}^* z_m^*) = \varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z_0^*). \tag{6.46}$$

Putting $z^*_{0,m} := P^*_{c_m} z^*_m$, we see from (6.44) and (6.46) that the sequence $\{z^*_{0,m}\}_{m\in\mathbb{N}}$ is a desired one.
(ii) Since $\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w})$ is weakly sequentially l.s.c. on $V_0^*(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})$, we have

$$\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_0) \le \liminf_{k\to\infty} \varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_{0,k}). \tag{6.47}$$

For the case $K_3 := \liminf_{k\to\infty} \varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_{0,k}) < \infty$ we can take out a subsequence of $\{z^*_{0,k}\}_{k\in\mathbb{N}}$, which is denoted by the same notation here, such that

$$\lim_{k\to\infty} \varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_{0,k}) = K_3. \tag{6.48}$$

For any $k\in\mathbb{N}$ we define a subset $\Omega_k(t)\subset\Omega$ by

$$\Omega_k(t) := \left\{ x\in\Omega\,; \begin{array}{r} \hat{\beta}\left((S_1(\tilde{u}_{m_k},\tilde{w}_{m_k}\,;t,0)\tilde{v}_{m_k})(x,t)\,;z^*_{0,k}(x)+\dfrac{c_0}{|\Omega|}\right) \\ < \hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_{0,k}(x)+\dfrac{c_0}{|\Omega|}\right) \end{array} \right\}.$$

Using (a) in (T2), we see that there exists a sequence $\{z^*_k\}_{k\in\mathbb{N}}\subset L^\infty(\Omega)$ such that the following inequalities hold for all $k\in\mathbb{N}$ and $x\in\Omega_k(t)$:

- $z^*_k(x) < z^*_{0,k}(x) + \dfrac{c_0}{|\Omega|}$,
- $\hat{\beta}((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_k(x)) < \hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_{0,k}(x)+\dfrac{c_0}{|\Omega|}\right)$,
- $\left(z^*_{0,k}(x)+\dfrac{c_0}{|\Omega|}\right) - z^*_k(x)$
$$\le \beta^*_2\left|(S_1(\tilde{u}_{m_k},\tilde{w}_{m_k};t,0)\tilde{v}_{m_k})(x,t) - (S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\right|,$$
- $\left|\hat{\beta}\left((S_1(\tilde{u}_{m_k},\tilde{w}_{m_k}\,;t,0)\tilde{v}_{m_k})(x,t)\,;z^*_{0,k}(x)+\dfrac{c_0}{|\Omega|}\right) - \hat{\beta}((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_k(x))\right|$
$$\le \beta^*_2\left|(S_1(\tilde{u}_{m_k},\tilde{w}_{m_k};t,0)\tilde{v}_{m_k})(x,t) - (S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\right|.$$

Hence, from (b) in (T2) we get the following inequality for all $x\in\Omega_k(t)$:

$$\begin{aligned} 0 &< \hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right) \\ &\quad - \hat{\beta}\left((S_1(\tilde{u}_{m_k},\tilde{w}_{m_k}\,;t,0)\tilde{v}_{m_k})(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right) \\ &\le \left|\hat{\beta}\left((S_1(\tilde{u}_{m_k},\tilde{w}_{m_k}\,;t,0)\tilde{v}_{m_k})(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)\right. \\ &\qquad\qquad \left. - \hat{\beta}((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_k(x))\right| \end{aligned} \tag{6.49}$$

$$
\begin{aligned}
&+\hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})\,(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)\\
&-\hat{\beta}((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_k(x))\\
\le\ &\beta^*_2\left|(S_1(\tilde{u}_{m_k},\tilde{w}_{m_k};t,0)\tilde{v}_{m_k})\,(x,t)-(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})\,(x,t)\right|\\
&+\beta^*_1\left|\left(z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)-z^*_k(x)\right|\\
\le\ &(\beta^*_1+\beta^*_2)\left|(S_1(\tilde{u}_{m_k},\tilde{w}_{m_k};t,0)\tilde{v}_{m_k})\,(x,t)-(S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})\,(x,t)\right|.
\end{aligned}
$$

Since from (a) in Lemma 6.10 we have the convergence

$$
S_1(\tilde{u}_{m_k},\tilde{w}_{m_k}\,;t,0)\tilde{v}_m \longrightarrow S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v} \quad \text{in} \quad C(\overline{\Omega}) \quad \text{as} \quad m\to\infty,
$$

the inequality (6.49) implies the following convergence:

$$
\begin{aligned}
\lim_{k\to\infty}\int_{\Omega_k(t)}&\left\{\hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)\right.\\
&\left.-\hat{\beta}\left((S_1(\tilde{u}_{m_k},\tilde{w}_{m_k}\,;t,0)\tilde{v}_{m_k})(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)\right\}dx=0.
\end{aligned}
\tag{6.50}
$$

Moreover, we have

$$
\begin{aligned}
&\varphi_{c_0}\left(S_1(\tilde{u}_{m_k},w_{m_k};t,0)\tilde{v}_{m_k}\,;z^*_{0,k}\right)\\
\ge\ &\int_{\Omega_k(t)}\hat{\beta}\left((S_1(\tilde{u}_{m_k},\tilde{w}_{m_k}\,;t,0)\tilde{v}_{m_k})\,(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)dx\\
&+\int_{\Omega\setminus\Omega_k(t)}\hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})\,(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)dx\\
=\ &\int_{\Omega}\hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})\,(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)dx\\
&-\int_{\Omega_k(t)}\left\{\hat{\beta}\left((S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v})\,(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)\right.\\
&\qquad\left.-\hat{\beta}\left((S_1(\tilde{u}_{m_k},\tilde{w}_{m_k}\,;t,0)\tilde{v}_{m_k})\,(x,t)\,;z^*_{0,k}(x)+\frac{c_0}{|\Omega|}\right)\right\}dx.
\end{aligned}
\tag{6.51}
$$

Taking $\liminf_{k\to\infty}$ in both sides of (6.51) and using (6.47), (6.48), (6.50), we get

$$
\liminf_{k\to\infty}\varphi_{c_0}(t,\tilde{u}_{m_k},\tilde{v}_{m_k},\tilde{w}_{m_k};z^*_{0,k})\ge K_3\ge\varphi_{c_0}(t,\tilde{u},\tilde{v},\tilde{w}\,;z^*_0).
$$

Next, we consider the case $\liminf_{k\to\infty}\varphi_{c_0}(t,\tilde{u},\tilde{w},\tilde{w}\,;z^*_{0,k})=\infty$. For this case, we assume

$$
K_4:=\liminf_{k\to\infty}\varphi_{c_0}(t,\tilde{u}_{m_k},\tilde{v}_{m_k},\tilde{w}_{m_k};z^*_{0,k})<\infty.
$$

Then, we take out a subsequence of $\{(\tilde{u}_{m_k},w_{m_k},v_{m_k},z^*_{0,k})\}_{k\in\mathbb{N}}$, which is denoted by the same notation here, such that

$$
K_4=\lim_{k\to\infty}\varphi_{c_0}(t,\tilde{u}_{m_k},\tilde{v}_{m_k},\tilde{w}_{m_k}\,;z^*_{0,k}).
$$

Repeating the argumentation similar to the case $\liminf_{k\to\infty} \varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}\,; z^*_{0,k}) < \infty$, we get (6.51). So, we take $\liminf_{k\to\infty}$ in both sides of (6.51), and get the following inequality:

$$\liminf_{k\to\infty} \varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}\,; z^*_{0,k}) \le K_4,$$

which contradicts with $\liminf_{k\to\infty} \varphi_{c_0}(t, \tilde{u}, \tilde{v}, \tilde{w}\,; z^*_{0,k}) = \infty$. Hence, (ii) is completely shown. □

Next, for the initial datum (u_0, v_0, w_0) in (T5), we define subsets $\mathcal{V}(u_0)$ and $\mathcal{W}(u_0)$ of $\mathcal{V}(c_0, T)$ by

$$\mathcal{V}(u_0) := \{\tilde{u} \in \mathcal{V}(c_0, T)\,;\, \tilde{u}(0) = u_0\}\,,$$

$$\mathcal{W}(u_0) := \left\{\tilde{u} \in \mathcal{V}(u_0)\,;\, \|\tilde{u}'\|_{L^2(0,T;V^*)} < \infty\right\}.$$

Then, we have Lemma 6.17 which implies that (A6) is satisfied and is obtained in [9, Lemma 4.2].

Lemma 6.17. *The property* $(\star)$ *holds for all* $\tilde{u} \in \mathcal{V}(u_0)$ *and* $\{\varphi_{c_0}(t, \tilde{u}, v_0, w_0)\,;\, 0 \le t \le T\} \in \mathcal{X}$*:*

$$(\star)\quad \left(\begin{array}{l} \textit{there exists a constant } K_5 > 0 \textit{ such that the following property} \\ \textit{is satisfied: for any } s \in [0,T] \textit{ and } z^*_0(\tilde{u}, s) \in D(\varphi_{c_0}(s, \tilde{u}, v_0, w_0)) \\ \textit{we have} \\ \quad |\varphi_{c_0}(t, \tilde{u}, v_0, w_0\,; z^*_0(\tilde{u}, s)) - \varphi_{c_0}(s, \tilde{u}, v_0, w_0\,; z^*_0(\tilde{u}, s))| \\ \qquad \le K_5 \|S_1(\tilde{u}, w_0\,; t, 0)v_0 - S_1(\tilde{u}, w_0\,; s, 0)v_0\|_{C(\overline{\Omega})}, \quad 0 \le \forall t \le T. \end{array}\right)$$

Proof. In the following proof, for the simplicity we put $\tilde{v}(t) := S_1(\tilde{u}, w_0\,; t, 0)v_0$ in $W^{1,\infty}(\Omega)$ for all $t \in [0,T]$. Using the argumentation similar to the proof of [8, Proposition 4.2], we show this lemma. For any $z^*_0(\tilde{u}, s) \in D(\varphi_{c_0}(s, \tilde{u}, v_0, w_0))$ we define a subset $\Omega_1(s,t)$ of Ω by

$$\Omega_1(s,t) := \left\{ x \in \Omega\,;\, \begin{array}{r} \hat{\beta}\left(\tilde{v}(x,s)\,; z^*_0(\tilde{u}, s\,; x) + \dfrac{c_0}{|\Omega|}\right) \\ \ge \hat{\beta}\left(\tilde{v}(x,t)\,; z^*_0(\tilde{u}, s\,; x) + \dfrac{c_0}{|\Omega|}\right) \end{array} \right\}.$$

For any $x \in \Omega_1(s,t)$ we have

$$\left(z^*_0(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}, \hat{\beta}\left(\tilde{v}(s)\,; z^*_0(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right)\right) \in \operatorname{epi} \hat{\beta}(\tilde{v}(x,t)).$$

Using [8, Lemma 7.3] and (T2), we see that there exists a point

$$(\bar{z}^*(\tilde{u}, s, t\,; x), \hat{\beta}(\tilde{v}(x,s)\,; \bar{z}^*(\tilde{u}, s, t\,; x))) \in \partial\left(\operatorname{epi} \hat{\beta}(\tilde{v}(x,s))\right),$$

which is uniquely determined, such that

- $\bar{z}^*(\tilde{u}, s, t\,; x) \le z_0^*(\tilde{u}, s\,; x) + \dfrac{c_0}{|\Omega|}$,
- $\hat{\beta}(\tilde{v}(x,s)\,; \bar{z}^*(\tilde{u}, s, t\,; x)) \le \hat{\beta}\left(\tilde{v}(x,s)\,; z_0^*(\tilde{u}, s\,; x) + \dfrac{c_0}{|\Omega|}\right)$,

- $$\begin{aligned} &\left\| \left(z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}, \hat{\beta}\left(\tilde{v}(x,t)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right)\right) \right. \\ &\qquad \left. - \left(\bar{z}^*(\tilde{u}, s, t\,; x), \hat{\beta}(\tilde{v}(x,s)\,; \bar{z}^*(\tilde{u}, s, t\,; x))\right)\right\|_{\mathbb{R}^2} \\ &= \inf_{(r_1, a_1) \in \operatorname{epi} \hat{\beta}(\tilde{v}(x,s))} \left\| \left(z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}, \right.\right. \\ &\qquad \left.\left. \hat{\beta}\left(\tilde{v}(x,t)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right)\right) - (r_1, a_1) \right\|_{\mathbb{R}^2} \\ &\le \delta\left(\operatorname{epi} \hat{\beta}(\tilde{v}(x,t)), \operatorname{epi} \hat{\beta}(\tilde{v}(x,s))\right) \le \beta_2^* |\tilde{v}(x,t) - \tilde{v}(x,s)|. \end{aligned} \tag{6.52}$$

From (T2) and (6.52) we get

$$\begin{aligned} &\hat{\beta}\left(\tilde{v}(x,s)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right) - \hat{\beta}\left(\tilde{v}(x,t)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right) \\ &\le \left\| \left(z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}, \hat{\beta}\left(\tilde{v}(x,t)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right)\right) \right. \\ &\qquad \left. - \left(\bar{z}^*(\tilde{u}, s, t\,; x), \hat{\beta}(\tilde{v}(x,s)\,; \bar{z}^*(\tilde{u}, s, t\,; x))\right)\right\|_{\mathbb{R}^2} \\ &\qquad + \left\| \left(z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}, \hat{\beta}\left(\tilde{v}(x,s)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right)\right) \right. \\ &\qquad\qquad \left. - \left(\bar{z}^*(\tilde{u}, s, t\,; x), \hat{\beta}(\tilde{v}(x,s)\,; \bar{z}^*(\tilde{u}, s, t\,; x))\right)\right\|_{\mathbb{R}^2} \\ &\le \beta_2^* |\tilde{v}(x,t) - \tilde{v}(x,s)| + \left\{ \left(z_0^* + \frac{c_0}{|\Omega|}\right) - \bar{z}^*(\tilde{u}, s, t\,; x) \right\} \\ &\qquad + \left| \hat{\beta}\left(\tilde{v}(x,s)\,; z_0^*(\tilde{u}\,; s\,; x) + \frac{c_0}{|\Omega|}\right) - \hat{\beta}\left(\tilde{v}(x,s)\,; \bar{z}^*(\tilde{u}, s, t\,; x)\right) \right| \\ &\le \beta_2^* |\tilde{v}(x,t) - \tilde{v}(x,s)| + (\beta_1^* + 1) \left\{ \left(z_0^* + \frac{c_0}{|\Omega|}\right) - \bar{z}^*(\tilde{u}, s, t\,; x) \right\} \\ &\le (\beta_1^* + \beta_2^* + 1) |\tilde{v}(x,t) - \tilde{v}(x,s)|. \end{aligned} \tag{6.53}$$

Next, we consider the case $x \in \Omega \setminus \Omega_1(s,t)$. Then, we have

$$\hat{\beta}\left(\tilde{v}(x,s)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right) < \hat{\beta}\left(\tilde{v}(x,t)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right).$$

Using [8, Lemma 7.3] again, we see that there exists a point

$$(z^*(\tilde{u}, s, t\,; x), \hat{\beta}(\tilde{v}(x,t)\,; z^*(\tilde{u}, s, t\,; x))) \in \partial\left(\operatorname{epi}\hat{\beta}(\tilde{v}(x,t))\right),$$

which is uniquely determined, such that

- $z^*(\tilde{u}, s, t\,; x) < z_0^*(\tilde{u}, s\,; x) + \dfrac{c_0}{|\Omega|}$,
- $\hat{\beta}(\tilde{v}(x,t)\,; z^*(\tilde{u}, s, t\,; x)) < \hat{\beta}\left(\tilde{v}(x,t)\,; z_0^*(\tilde{u}, s\,; x) + \dfrac{c_0}{|\Omega|}\right)$,
- $$\begin{aligned}&\left\|\left(z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}, \hat{\beta}\left(\tilde{v}(x,s)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right)\right)\right.\\ &\qquad\left. - \left(z^*(\tilde{u}, s, t\,; x), \hat{\beta}(\tilde{v}(x,t)\,; z^*(\tilde{u}, s, t\,; x))\right)\right\|_{\mathbb{R}^2}\\ &= \inf_{(r_2,a_2)\in\operatorname{epi}\hat{\beta}(\tilde{v}(x,t))}\left\|\left(z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|},\right.\right.\\ &\qquad\left.\left.\hat{\beta}\left(\tilde{v}(x,s)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right)\right) - (r_2, a_2)\right\|_{\mathbb{R}^2}\\ &\le \delta\left(\operatorname{epi}\hat{\beta}(\tilde{v}(x,s)),\ \operatorname{epi}\hat{\beta}(\tilde{v}(x,t))\right) \le \beta_2^*|\tilde{v}(x,t) - \tilde{v}(x,s)|.\end{aligned}$$

Repeating the argumentation similar to that of the case $\Omega_1(s,t)$, we get

$$\begin{aligned}&\hat{\beta}\left(\tilde{v}(x,t)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right) - \hat{\beta}\left(\tilde{v}(x,s)\,; z_0^*(\tilde{u}, s\,; x) + \frac{c_0}{|\Omega|}\right)\\ &\qquad\le (\beta_1^* + \beta_2^* + 1)|\tilde{v}(x,t) - \tilde{v}(x,s)|.\end{aligned} \tag{6.54}$$

Hence, we see from (6.53) and (6.54) that the condition $(\star)$ holds. □

At the end of this subsection, we give Lemma 6.18, which implies that (A9) is satisfied.

Lemma 6.18. *There exists a constant $K_6 > 0$ such that*

$$\sup_{\tilde{u}\in\mathcal{V}(u_0)}\left(\sup_{0\le t\le T}\varphi_{c_0}(t, \tilde{u}, v_0, w_0\,; 0)\right) \le K_6.$$

Proof. For any $\tilde{u} \in \mathcal{V}(u_0)$, from (T2) and (6.31) we get

$$0 \le \hat{\beta}\left((S_1(\tilde{u}, w_0\,; t_1, 0)v_0)(x,t)\,; \frac{c_0}{|\Omega|}\right) \le \frac{c_0\beta_1^*}{|\Omega|}, \quad \text{a.a. } x\in\Omega, \quad 0 \le \forall t \le T,$$

which implies

$$\psi\left(t, \tilde{u}, v_0, w_0\,; \frac{c_0}{|\Omega|}\right) = \varphi_{c_0}(t, \tilde{u}, v_0, w_0\,; 0) \le c_0\beta_1^*, \quad 0 \le \forall t \le T.$$

Hence, we see that this lemma holds. □

6.5. Haptotaxis term as perturbation on V_0^*

For any $(z_0^*, v) \in D(\varphi_{c_0}) \times A_v$ we define a perturbation $g(z_0^*, v)\colon V_0 \longmapsto \mathbb{R}$ by

$$(g(z_0^*, v))(z_0) := \int_\Omega \left(z_0^* + \frac{c_0}{|\Omega|} \right) \nabla\lambda(v) \cdot \nabla z dx, \quad \forall z_0 \in V_0.$$

Then, we get Lemma 6.19, which implies that (a) in (A4) and (A9) are satisfied.

Lemma 6.19. *For any $(z_0^*, v) \in D(\varphi_{c_0}) \times A_v$ we have $g(z_0^*, v) \in V_0^*$ and there exist a nonnegative function ℓ on A_v and a constant $K_7 > 0$ such that for any $r \geq 0$ the level set $\{v \in A_v\,;\, \ell(v) \leq r\}$ is compact in $C(\overline{\Omega})$ and the following inequality holds:*

$$\|g(z_0^*, v)\|_{V_0^*} \leq \ell(v)\sqrt{\varphi_{c_0}(z_0^*) + K_7}, \quad \forall (z_0^*, v) \in D(\varphi_{c_0}) \times A_v.$$

Moreover, there exists a constant $K_8 > 0$, which depends on $\|v_0\|_{W^{1,\infty}(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$, such that

$$\sup_{\tilde{u} \in \mathcal{V}(u_0)} \left(\sup_{0 \leq t \leq T} \ell(S_1(\tilde{u}, w_0, t, 0)v_0) + \int_0^T \|(S(\tilde{u}, w_0\,; t, 0)v_0)'\|_{C(\overline{\Omega}))} dt \right) \leq K_8.$$

Proof. Since the perturbation $g(z_0^*, v)$ is linear, we only show that $g(z_0^*, v)$ is bounded on V_0. We see from (T3) that the following inequality holds for all $z_0 \in V_0$:

$$|(g(z_0^*, v))(z_0)| \leq C_\lambda \|\nabla v\|_{(L^\infty(\Omega))^N} \left\| z_0^* + \frac{c_0}{|\Omega|} \right\|_{L^2(\Omega)} \|z_0\|_{V_0},$$

which implies $g(z_0^*, v) \in V_0^*$ and

$$\|g(z_0^*, v)\|_{V_0^*} \leq C_\lambda \|v\|_{W^{1,\infty}(\Omega)} \left\| z_0^* + \frac{c_0}{|\Omega|} \right\|_{L^2(\Omega)}. \tag{6.55}$$

Moreover, using the following inequality:

$$r^2 \leq \hat{\beta}(r) + \alpha^2, \quad \forall r \in \mathbb{R},$$

we get

$$\left\| z_0^* + \frac{c_0}{|\Omega|} \right\|_{L^2(\Omega)}^2 \leq \varphi_{c_0}(z_0^*) + \alpha^2|\Omega|, \quad \forall z_0^* \in D(\varphi_{c_0}). \tag{6.56}$$

From (6.55) and (6.56) we get

$$\|g(z_0^*, v)\|_{V_0^*} \leq C_\lambda \|v\|_{W^{1,\infty}(\Omega)} \sqrt{\varphi_{c_0}(z_0^*) + \alpha^2|\Omega|}.$$

We see that the function $\ell(v) := C_\lambda \|v\|_{W^{1,\infty}(\Omega)}$ and the constant $K_7 := \alpha^2|\Omega|$ are desired ones. □

Next, we show Lemma 6.20, which implies that (b) in (A4) holds.

Lemma 6.20. *Assume that a sequence* $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ *and a function* $\tilde{u}$ *in* $\mathcal{V}(u_0)$ *satisfy*

$$\begin{aligned} \tilde{u}_m \longrightarrow \tilde{u} \quad & in \quad C([0,T];V^*) \\ & and \quad *-weakly\ in \quad L^\infty(\Omega\times(0,T)) \quad as \quad m\to\infty. \end{aligned} \tag{6.57}$$

Then, for any $(\tilde{v},\tilde{w})\in A_v\times A_w$ *we have the following convergence as* $m\to\infty$*:*

$$\begin{aligned} \left(P^*_{c_0}\tilde{u}_m, S_1(\tilde{u}_m,\tilde{w}\,;\cdot,0)\tilde{v}\right) \longrightarrow g\left(P^*_{c_0}\tilde{u}, S_1(\tilde{u},\tilde{w}\,;\cdot,0)\tilde{v}\right) \\ *-weakly\ in \quad L^\infty(0,T;V_0^*) \end{aligned} \tag{6.58}$$

Proof. We assume that (6.57) is satisfied, and throughout this proof we put $\tilde{v}_m(t) := S_1(\tilde{u}_m,\tilde{w}\,;t,0)\tilde{v}$ and $\tilde{v}(t) := S_1(\tilde{u},\tilde{w}\,;t,0)\tilde{v}$ for all $t\in[0,T]$ and $m\in\mathbb{N}$. For any $\xi\in L^1(0,T;V_0)$ we have

$$\left|\int_0^T\left(\int_\Omega\{\tilde{u}_m(t)\nabla\lambda(\tilde{v}_m(t))-\tilde{u}(t)\nabla\lambda(\tilde{v}(t))\}\cdot\nabla\xi(t)dx\right)dt\right| \le \sum_{i=1}^3\Psi_{m,i}, \tag{6.59}$$

where $\Psi_{m,i}$ $(i=1,2,3)$ are given by

$$\begin{aligned} \Psi_{m,1} &:= \left|\int_0^T\left(\int_\Omega\{\lambda'(\tilde{v}_m(t))-\lambda'(\tilde{v}(t))\}\,\tilde{u}_m(t)\nabla\tilde{v}(t)\cdot\nabla\xi(t)dx\right)dt\right|, \\ \Psi_{m,2} &:= \left|\int_0^T\left(\int_\Omega\lambda'(\tilde{v}(t))\tilde{u}_m(t)\nabla\{\tilde{v}(t)-\tilde{v}(t)\}\cdot\nabla\xi(t)dx\right)dt\right|, \\ \Psi_{m,3} &:= \left|\int_0^T\left(\int_\Omega\{\tilde{u}_m(t)-\tilde{u}(t)\}\lambda'(\tilde{v}(t))\nabla\tilde{v}(t)\cdot\nabla\xi(t)dx\right)dt\right|. \end{aligned}$$

Substituting the following estimates into (6.59), which arise from (6.35);

$$\begin{aligned} \Psi_{m,1} &\le \alpha C_\lambda K_2(T+1)\left\{\int_0^T\left(\int_\Omega|\nabla\xi(t)|dx\right)dt\right\} \\ &\qquad\times\left(\sup_{0\le t\le T}\|\tilde{v}_m(t)-\tilde{v}(t)\|_{C(\overline{\Omega})}\right), \\ \Psi_{m,2} &\le \alpha C_\lambda\left(\int_0^T\|\xi(t)\|_{V_0}dt\right)\left(\sup_{0\le t\le T}\|\tilde{v}_m(t)-\tilde{v}(t)\|_{H^1(\Omega)}\right), \\ \Psi_{m,3} &= \left|\langle\tilde{u}_m-\tilde{u},\lambda'(\tilde{v}(t))\nabla\tilde{v}(t)\cdot\nabla\xi\rangle_{L^\infty(\Omega\times(0,T)),\,L^1(\Omega\times(0,T))}\right|, \end{aligned}$$

and using the convergences in (a) and (b) in Lemma 6.11, we get (6.58). □

6.6. Evolution inclusion with quasi-variational structures on V_0^*

In this subsection, we consider the Cauchy problem (E):=$\{$(6.60)–(6.63)$\}$ of an evolution inclusion with quasi-variational structures as one of the examples of (P):

$$u(t)\in W^*(c_0)\cap L^\infty(\Omega), \quad \forall t\in[0,T], \tag{6.60}$$

$$\left(P_{c_0}^* u\right)'(t) + \partial_{V_0^*(v(t))}\varphi_{c_0}\left(t, u, w_0, v_0\,; P_{c_0}^* u(t)\right) + g\left(P_{c_0}^* u(t), v(t)\right) \ni 0 \quad (6.61)$$
$$\text{in} \quad V_0^*(v(t)), \text{ a.a. } t \in (0,T),$$

$$(v(t), w(t)) = S(u\,; t, 0)(v_0, w_0) \quad \text{in} \quad A_v \times A_w, \quad \forall t \in [0,T], \quad (6.62)$$

$$u(0) = u_0 \quad \text{in} \quad V^*. \quad (6.63)$$

First of all, for the Cauchy problem (E) we show Theorem 6.22. In order to show Theorem 6.22, for any $R \geq 0$ satisfying

$$R^2 > R_* := \frac{2\varphi_{c_0}\left(u_0, w_0, v_0\,; P_{c_0}^* u_0\right)}{\tilde{c}_1^2},$$

where $\tilde{c}_1 > 0$ is the same constant that is obtained in Lemma 6.7, we prepare subsets $\mathcal{V}_R(u_0)$ and $\mathcal{W}_R(u_0)$ of $\mathcal{V}(u_0)$ defined by

$$\mathcal{V}_R(u_0) := \left\{ \tilde{u} \in \mathcal{V}(u_0)\,;\ \sup_{0 \leq t \leq T} \|u(t)\|_{V^*} \leq R \right\},$$
$$\mathcal{W}_R(u_0) := \left\{ \tilde{u} \in \mathcal{V}(u_0)\,;\ \|u'\|_{L^2(0,T;V^*)} \leq R \right\}.$$

Then, we have Lemma 6.21.

Lemma 6.21. *For any $R > R_*$ the set $\mathcal{W}_R(u_0)$ is nonempty, convex and compact in $C([0,T]; V^*)$.*

Proof. Since we easily see that $\mathcal{W}_R(u_0)$ is nonempty and convex, we only show that $\mathcal{W}_R(u_0)$ is compact in $C([0,T]; V^*)$. Because of the compact imbedding $L^2(\Omega) \subset V^*$, we see that $\mathcal{W}_R(u_0)$ is relatively compact in $C([0,T]; V^*)$ by using the Ascoli-Arzelà theorem. In the following argumentation, we show that $\mathcal{W}_R(u_0)$ is closed in $C([0,T]; V^*)$. For this, we consider a sequence $\{z_m^*\}_{m\in\mathbb{N}} \subset \mathcal{W}_R(u_0)$ and a function $z^* \in C([0,T]; V^*)$ satisfying the following convergence as $m \to \infty$:

$$z_m^* \longrightarrow z^* \quad \begin{cases} \text{in} \quad C([0,T]; V^*) \\ \text{weakly in} \quad W^{1,2}([0,T]; V^*) \\ * - \text{weakly in} \quad L^\infty(0,T; L^\infty(\Omega)). \end{cases} \quad (6.64)$$

hence, from (6.27) in the proof of Lemma 6.5 we have

$$P_{c_0}^* z_m^* \longrightarrow P_{c_0}^* z^* \quad \text{in} \quad C([0,T]; V_0^*) \quad \text{as} \quad m \to \infty. \quad (6.65)$$

Because φ_{c_0} is nonnegative and l.s.c. on V_0^*, from (6.65) we get

$$\varphi_{c_0}\left(P_{c_0}^* z^*(t)\right) = 0, \quad \forall t \in [0,T]. \quad (6.66)$$

Hence, from (6.64) and (6.66) we have $z^* \in \mathcal{W}_R(u_0)$, so, $\mathcal{W}_R(u_0)$ is closed in $C([0,T]; V^*)$. □

Using the argumentation similar to that in Section 4, we show Theorem 6.22, which guarantees the existence of strong local-in-time solutions to (E).

Theorem 6.22. *There exists a time $T_0 \in (0,T]$ such that the Cauchy problem (E) has at least one strong solution u on $[0,T_0]$ satisfying (6.60)–(6.62) as $T = T_0$ and (6.63). Moreover, there exists a constant $K_9 > 0$ such that*

$$\|u'\|_{L^2(0,T_0;V^*)} + \sup_{0\le t\le T_0} \varphi_{c_0}\left(t,u,v_0,w_0\,;P^*_{c_0}u(t)\right) \le K_9,$$

which implies that the double obstacle conditions are satisfied:

$$0 \le u(x,t) \le \alpha, \quad \forall t \in [0,T_0], \quad a.a.\ x \in \Omega.$$

Proof. For each $\tilde{u} \in \mathcal{V}(u_0)$ we consider the following auxiliary problem (AE):

$$\text{(AE)}\begin{cases} z'(t) + \partial_{V_0^*(\tilde{v}(t))}\varphi(t,\tilde{u},v_0,w_0\,;z(t)) \ni -g\left(P^*_{c_0}\tilde{u}(t),\tilde{v}(t)\right) \\ \qquad\qquad \text{in} \quad V_0^*(\tilde{v}(t)), \quad \text{a.a. } t \in (0,T), \\ (\tilde{v}(t),\tilde{w}(t)) = S(\tilde{u}\,;t,0)(v_0,w_0) \quad \text{in} \quad A_v \times A_w, \\ z(0) = P^*_{c_0}u_0 \quad \text{in} \quad V_0^*. \end{cases}$$

Owing to Lemmas 6.3–6.21, we can apply [7, Theorem 3.1] and see that (AE) has one and only one strong solution $z(\tilde{u}) \in W^{1,2}(0,T;V_0^*)$. Fixing a constant $R > R_*$ and using the solution $z(\tilde{u})$ to (AE), we define an operator $\mathcal{S}\colon \mathcal{W}_R(u_0) \longmapsto W^{1,2}(0,T;V^*)$ by

$$(\mathcal{S}\tilde{u})(t) := (P^*_{c_0})^{-1}z(\tilde{u}\,;t) = z(\tilde{u}\,;t) + \frac{c_0}{|\Omega|}, \quad 0 \le \forall t \le T, \quad \forall \tilde{u} \in \mathcal{W}_R(u_0). \tag{6.67}$$

Repeating the argumentation similar to Lemma 3.6 and using Lemma 6.5, we see that the operator $\mathcal{S}$ is continuous with respect to the strong topology of $C([0,T];V^*)$.

Next, we use the energy inequalities which are obtained in Lemmas 3.2 and 3.4. At first, from the first energy type inequality in Lemma 3.2 and (b) in Lemma 6.7 we get the following inequality:

$$\begin{aligned} &\frac{d}{dt}\|z(\tilde{u}\,;t)\|^2_{V_0^*(\tilde{v}(t))} - 2(z'(\tilde{u}\,;t),z(\tilde{u}\,;t))_{V_0^*(\tilde{v}(t))} \\ &\qquad \le \tilde{c}_3\|\tilde{v}'(t)\|_{C(\overline{\Omega})}\|z(\tilde{u}\,;t)\|^2_{V_0^*(\tilde{v}(t))}, \quad \text{a.a. } t \in (0,T), \end{aligned}$$

hence, from Lemmas 6.10 and 6.19

$$\begin{aligned} &\frac{d}{dt}\|z(\tilde{u}\,;t)\|^2_{V_0^*(\tilde{v}(t))} + 2\varphi_{c_0}(t,\tilde{u},v_0,w_0\,;z(\tilde{u}\,;t)) \\ \le\ & -2\left(g\left(P^*_{c_0}\tilde{u}(t),\tilde{v}(t)\right),z(\tilde{u}\,;t)\right)_{V_0^*(\tilde{v}(t))} + \tilde{c}_3\|\tilde{v}'(t)\|_{C(\overline{\Omega})}\|z(\tilde{u}\,;t)\|^2_{V_0^*(\tilde{v}(t))} \\ \le\ & \phi(\tilde{v}(t))\|z(\tilde{u}\,;t)\|_{V_0^*(\tilde{v}(t))} + \tilde{c}_3\|\tilde{v}'(t)\|_{C(\overline{\Omega})}\|z(\tilde{u}\,;t)\|^2_{V_0^*(\tilde{v}(t))} \\ \le\ & \left(\tilde{c}_3 a\alpha K_1 + \frac{1}{2}\right)\|z(\tilde{u}\,;t)\|^2_{V_0^*(\tilde{v}(t))} + \frac{K_8^2}{2}. \end{aligned} \tag{6.68}$$

Applying the Gronwall lemma to (6.68), we see that there exists a constant $K_{10} > 0$ such that

$$\sup_{0\le t\le T} \|z(\tilde{u}\,;t)\|^2_{V_0^*(\tilde{v}(t))} + \int_0^T \varphi_{c_0}(t,\tilde{u},v_0,w_0\,;z(\tilde{u}\,;t))dt \le K_{10}. \tag{6.69}$$

Secondly, we use the second type energy inequality in Lemma 3.4. Then, we see from Lemma 6.17 and (6.69) that there exists a constant $K_{11} > 0$ such that the following inequality holds for a.a. $t \in (0,T)$:

$$\begin{aligned}\frac{d}{dt}\varphi_{c_0}(t,\tilde{u},v_0,w_0\,;z(\tilde{u}\,;t)) &+ \left(z'(\tilde{u}\,;t), z(\tilde{u}\,;t) + g\left(P^*_{c_0}\tilde{u}(t),\tilde{v}(t)\right)\right)_{V_0^*(\tilde{v}(t))}\\ &\le K_{11}\left\{\varphi_{c_0}(t,\tilde{u},v_0,w_0\,;z(\tilde{u}\,;t)) + 1\right\}\left(\|\tilde{v}'(t)\|_{C(\overline{\Omega})} + 1\right),\end{aligned}$$

hence, from Lemmas 6.10 and 6.19

$$\begin{aligned}&\frac{1}{2}\|z'(\tilde{u}\,;t)\|^2_{V_0^*(\tilde{v}(t))} + \frac{d}{dt}\varphi_{c_0}(t,\tilde{u},v_0,w_0\,;z(\tilde{u}\,;t)) \\ &\le \frac{3}{2}\cdot\left\|g\left(P^*_{c_0}\tilde{u}(t),\tilde{v}(t)\right)\right\|^2_{V_0^*(\tilde{v}(t))}\\ &\quad + K_{11}\left\{\varphi_{c_0}(t,\tilde{u},v_0,w_0\,;z(\tilde{u}\,;t)) + 1\right\}\left(\|\tilde{v}'(t)\|_{C(\overline{\Omega})} + 1\right)\\ &\le K_{11}(a\alpha K_1 + 1)\varphi_{c_0}(t,\tilde{u},v_0,w_0\,;z(\tilde{u}\,;t)) + \frac{3K_8^2}{2} + K_{11}(a\alpha K_1 + 1).\end{aligned} \tag{6.70}$$

Applying the Gronwall lemma to (6.70), we see that there exists a constant $K_{12} > 0$ such that

$$\|z'(\tilde{u})\|_{L^2(0,T;V_0^*)} + \sup_{0\le t\le T}\varphi_{c_0}(t,\tilde{u},v_0,w_0\,;z(\tilde{u}\,;t)) \le K_{12}, \tag{6.71}$$

which implies

$$\varphi_{c_0}(z(\tilde{u}\,;t)) = 0, \quad \forall t \in [0,T].$$

Using (6.70) and (6.71), we see that the following inequality holds for all $t \in [0,T]$:

$$\int_0^t \|z'(\tilde{u}\,;s)\|^2_{V_0^*}ds \le \frac{2\varphi_{c_0}\left(0,u_0,v_0,w_0\,;P^*_{c_0}u_0\right)}{\tilde{c}_1^2} + \frac{K_{13}t}{\tilde{c}_1^2} = R_* + \frac{K_{13}t}{\tilde{c}_1^2},$$

where $\tilde{c}_1$ is the same constant that is obtained in Lemma 6.7 and $K_{13} > 0$ is a constant given by

$$K_{13} := K_{11}(K_{12}+1)(a\alpha K_1 + 1) + \frac{3K_8^2}{2}.$$

Choosing a time $T_0 > 0$ so that

$$\frac{K_{13}T_0}{\tilde{c}_1^2} < R^2 - R_*,$$

we have

$$\int_0^{T_0} \|z'(\tilde{u}\,;t)\|_{V_0^*}^2 dt \le R^2. \tag{6.72}$$

Now, for any $z_0^* \in C([0,T];V_0^*)$ we define a function Λz_0^* by

$$(\Lambda z_0^*)(t) := \begin{cases} z_0^*(t) & \text{if} \quad t \in [0,T_0], \\ z_0^*(T_0) & \text{if} \quad t \in (T_0,T]. \end{cases}$$

From (6.67) we get the following equalities:

$$P_{c_0}^*((\Lambda \circ \mathcal{S}\,)\tilde{u})(t) = \Lambda\left(P_{c_0}^*(\mathcal{S}\tilde{u})(t)\right) = \Lambda z(\tilde{u}\,;t), \quad \forall t \in [0,T], \tag{6.73}$$

$$\|(\Lambda\circ\mathcal{S}\tilde{u})'(t)\|_{V^*} = \|(P_{c_0}^*((\Lambda\circ\mathcal{S}\,)\tilde{u})'(t)\|_{V_0^*} = \|(\Lambda z(\tilde{u}))'(t)\|_{V_0^*}, \quad \forall t \in [0,T]. \tag{6.74}$$

Hence, we see from (6.72)–(6.74) that the following equality and boundedness hold:

$$\varphi_{c_0}(P_{c_0}^*((\Lambda \circ \mathcal{S}\,)\tilde{u})(t)) = 0, \quad \forall t \in [0,T],$$

$$\int_0^T \|((\Lambda \circ \mathcal{S}\,)\tilde{u})'(t)\|_{V^*}^2 dt \le R^2,$$

which imply $(\Lambda\circ\mathcal{S}\,)\tilde{u} \in \mathcal{W}_R(u_0)$. Since the operator $\Lambda\circ\mathcal{S}$ is continuous with respect to the strong topology of $C([0,T];V^*)$ and from Lemma 6.21 the set $\mathcal{W}_R(u_0)$ is nonempty, convex and compact in $C([0,T];V^*)$, we see that the operator $\Lambda \circ \mathcal{S}$ has at least one fixed point u, that is,

$$(\Lambda \circ \mathcal{S}\,)u = u, \quad \forall t \in [0,T], \qquad \text{hence,} \qquad z(u\,;t) = P_{c_0}^* u(t), \quad \forall t \in [0,T_0].$$

by applying the Schauder fixed-point theorem. Then, we see that the restriction on $[0,T_0]$ of the function u, which is the fixed point of $\Lambda \circ \mathcal{S}$, is a strong solution to (E) on $[0,T_0]$. □

In the rest of this section, we show the existence of strong solutions of (E) on $[0,T]$.

Theorem 6.23. *The Cauchy problem (E) has at least one strong solution u on $[0,T]$.*

Proof. In order to show this theorem, we use the same notation and repeat the argumentation similar to that in Section 5. We define a set $\mathcal{Z}$, which is nonempty because of Theorem 6.22, by

$$\mathcal{Z} := \left\{ (\bar{u},\bar{v},\bar{w},\bar{T})\,; \begin{array}{l} \bar{u} \text{ is a strong solution of (E) on } [0,\bar{T}] \text{ and} \\ (\bar{v}(t),\bar{w}(t)) = S(\bar{u}\,;t,0)(v_0,w_0) \quad \text{in} \quad A_v \times A_w \\ \text{for all } t \in [0,\bar{T}] \end{array} \right\}$$

$$\subset \bigcup_{0\le\bar{T}\le T} W^{1,2}(0,\bar{T};V^*) \times W^{1,1}(0,\bar{T};C(\overline{\Omega})) \times C([0,\bar{T}];L^2(\Omega)) \times \{\bar{T}\},$$

and induce an order relation $\preceq$ on $\mathcal{Z}$ by

$$(\bar{u}_1, \bar{v}_1, \bar{w}_1, \bar{T}_1) \preceq (\bar{u}_2, \bar{v}_2, \bar{w}_2, \bar{T}_2)$$
$$\text{if and only if} \quad \bar{T}_1 \le \bar{T}_2 \quad \text{and} \quad \bar{u}_1 = \bar{u}_2 \quad \text{in} \quad W^{1,2}(0, \bar{T}_1; V^*).$$

Then, we see that the ordered set $(\mathcal{Z}, \preceq)$ is inductively ordered. Actually, we let $\mathcal{Y}$ be any linearly ordered subset of $\mathcal{Z}$ and define a quadruple $(\hat{u}, \hat{v}, \hat{w}, \hat{T})$ by

$$\hat{T} := \sup \left\{ \bar{T} \in (0, T] \,;\, (\bar{u}, \bar{v}, \bar{w}, \bar{T}) \in \mathcal{Y} \right\},$$

$$(\hat{u}(t), \hat{v}(t), \hat{w}(t)) = (\bar{u}(t), \bar{v}(t), \bar{w}(t)) \quad \text{in} \quad V^* \times A_v \times A_w, \quad \forall t \in [0, \bar{T}]$$
$$\text{whenever we have } (\bar{u}, \bar{v}, \bar{w}, \bar{T}) \in \mathcal{Y}.$$

We easily see that the triplet $(\hat{u}, \hat{v}, \hat{w}, \hat{T})$ is uniquely determined. Now, we can take out a sequence $\{(\bar{u}_m, \bar{v}_m, \bar{w}_m, \bar{T}_m)\}_{m \in \mathbb{N}} \subset \mathcal{Y}$ satisfying

$$\bar{T}_m \nearrow \hat{T} \quad \text{as} \quad m \to \infty,$$

$$\left(P^*_{c_0} \bar{u}_m\right)'(t) + \partial_{V_0^*(\bar{v}_m(t))} \varphi_{c_0}\left(t, \bar{u}_m, v_0, w_0 \,;\, P^*_{c_0} \bar{u}_m(t)\right) + g\left(P^*_{c_0} \bar{u}_m(t), \bar{v}_m(t)\right) \ni 0$$
$$\text{in} \quad V_0^*(\bar{v}_m(t)), \quad \text{a.a. } t \in (0, \bar{T}_m),$$

$$(\bar{v}_m(t), \bar{w}_m(t)) = S(\bar{u}_m \,; t, 0)(v_0, w_0) \quad \text{in} \quad A_v \times A_w, \quad \forall t \in [0, T],$$

$$\bar{u}_m(0) = u_0 \quad \text{in} \quad V^*.$$

Using Theorem 6.22, we see that there exists a sequence $\{K^*_m\} \subset (0, \infty)$ such that

$$\|\bar{u}'_m\|_{L^2(0, \bar{T}_m; V^*)} + \sup_{0 \le t \le \bar{T}_m} \varphi_{c_0}(t, \bar{u}_m, v_0, w_0 \,;\, P^*_{c_0} \bar{u}_m(t)) \le K^*_m.$$

Hence, we have

$$\varphi_{c_0}(P^*_{c_0} \bar{u}_m(t)) = 0, \quad \forall t \in [0, \bar{T}_m], \quad \forall m \in \mathbb{N}, \tag{6.75}$$

which implies that there exists a constant $K_{14} > 0$, which is independent of $m \in \mathbb{N}$, such that

$$\sup_{m \in \mathbb{N}} \left(\sup_{0 \le t \le \bar{T}_m} \|\bar{u}_m(t)\|_{V^*} \right) \le K_{14} \tag{6.76}$$

because of the double constraints condition (cf. (6.75)):

$$0 \le \bar{u}_m(x, t) \le \alpha, \quad \forall t \in [0, \bar{T}_m], \quad \text{a.a. } x \in \Omega, \quad \forall m \in \mathbb{N}.$$

For any $m \in \mathbb{N}$ we consider a sequence $\{\tilde{u}_m\}_{m \in \mathbb{N}} \subset \mathcal{V}(u_0)$ defined by

$$\tilde{u}_m(t) := (\bar{u}_m)_{\bar{T}_m}(t) = \begin{cases} \bar{u}_m(t) & \text{if} \quad t \in [0, \bar{T}_m], \\ \bar{u}_m(\bar{T}_m) & \text{if} \quad t \in (\bar{T}_m, T], \end{cases}$$

and consider the Cauchy problem $(\mathrm{E})_m := \{(6.77) - (6.79)\}$:

$$\tilde{z}_m'(t) + \partial_{V_0^*(\tilde{v}_m(t))}\varphi_{c_0}\left(t, \tilde{u}_m, v_0, w_0\,; \tilde{z}_m(t)\right) \ni -g\left(P_{c_0}^*\tilde{u}_m(t), \tilde{v}_m(t)\right) \quad \text{in} \quad V_0^*(\tilde{v}(t)), \text{ a.a. } t \in (0,T), \tag{6.77}$$

$$(\tilde{v}_m(t), \tilde{w}_m(t)) = S(\tilde{u}_m\,; t, 0)(v_0, w_0) \quad \text{in} \quad A_v \times A_w, \quad \forall t \in [0,T], \tag{6.78}$$

$$\tilde{z}_m(0) = P_{c_0}^* u_0 \quad \text{in} \quad V_0^*. \tag{6.79}$$

We see from Lemma 6.5 that there exists a constant $K_{15} > 0$, which depends on $\|v_0\|_{W^{1,\infty}(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$, such that the following uniform estimate holds:

$$\sup_{m\in\mathbb{N}}\left\{\sup_{0\le t\le T}\left(\|\tilde{w}_m(t)\|_{W^{1,\infty}(\Omega)} + \|\nabla\tilde{v}_m(t)\|_{(L^\infty(\Omega))^N} + \|\tilde{v}_m'(t)\|_{C(\overline{\Omega})}\right)\right\} \le K_{15}. \tag{6.80}$$

Using Lemma 3.4, we see from Lemmas 6.10 and 6.17 that there exists a constant $K_{16} > 0$, which depend on $\|u_0\|_{V_0^*}$, $\|v_0\|_{W^{1,\infty}(\Omega)}$ and $\|w_0\|_{W^{1,\infty}(\Omega)}$, such that the following inequality holds for all $m \in \mathbb{N}$ and a.a. $t \in (0,T)$:

$$\frac{d}{dt}\varphi_{c_0}(t, \tilde{u}_m, v_0, w_0\,; \tilde{z}_m(t)) + \left(\tilde{z}_m'(t), \tilde{z}_m'(t) + g\left(P_{c_0}^*\tilde{u}_m(t), \tilde{v}_m(t)\right)\right)_{V_0^*(\tilde{v}_m(t))} \le K_{16}\left\{\varphi_{c_0}(t, \tilde{u}_m, v_0, w_0\,; \tilde{z}_m(t)) + 1\right\},$$

hence, from Lemma 6.14

$$\begin{aligned}
&\frac{1}{2}\|\tilde{z}_m'(t)\|^2_{V_0^*(\tilde{v}_m(t))} + \frac{d}{dt}\varphi_{c_0}(t, \tilde{u}_m, v_0, w_0\,; \tilde{z}_m(t)) \\
&\le \frac{3\tilde{c}_2^2}{2}\cdot\|g(\tilde{u}_m(t), \tilde{v}_m(t))\|^2_{V_0^*} + K_{16}\left\{\varphi_{c_0}(t, \tilde{u}_m, v_0, w_0\,; \tilde{z}_m(t)) + 1\right\} \\
&\le K_{16}\left\{\varphi_{c_0}(t, \tilde{u}_m, v_0, w_0\,; \tilde{z}_m(t)) + 1\right\} \\
&\quad + \frac{3c_2^2}{2}\left(\sup_{0\le t\le T}\phi(\tilde{v}_m(t))\right)^2\left\{\varphi_{c_0}\left(P_{c_0}^*\tilde{u}_m(t)\right) + K_7\right\}.
\end{aligned}$$

Using (1) in Lemma 6.14 and (6.75), we get the following inequality for a.a. $t \in (0,T)$:

$$\frac{d}{dt}\varphi_{c_0}(t, \tilde{u}_m, v_0, w_0\,; \tilde{z}_m(t)) + \frac{1}{2}\|\tilde{z}_m'(t)\|^2_{V_0^*(\tilde{v}_m(t))} \le K_{16}\varphi_{c_0}(t, \tilde{u}_m, v_0, w_0\,; \tilde{z}_m(t)) + K_{17}, \tag{6.81}$$

where the constant $K_{17} > 0$ is given by

$$K_{17} := \frac{3\tilde{c}_2^2 K_7 K_8^2}{2}.$$

Applying the Gronwall lemma to (6.81), we get the following uniform estimates:

$$\sup_{m\in\mathbb{N}}\left(\sup_{0\le t\le T}\varphi_{c_0}(t,\tilde{u}_m,v_0,w_0\,;\tilde{z}_m(t))\right) \le \left\{\varphi_{c_0}\left(0,u_0,v_0,w_0\,;P^*_{c_0}u_0\right)+\frac{K_{17}}{K_{16}}\right\}e^{K_{16}T}, \tag{6.82}$$

$$\sup_{m\in\mathbb{N}}\int_0^T\|\tilde{z}'_m(t)\|^2_{V_0^*}dt \le \frac{2}{c_1^2}\left\{\varphi_{c_0}\left(0,u_0,v_0,w_0\,;P^*_{c_0}u_0\right)+\frac{K_{17}}{K_{16}}\right\}e^{K_{16}T}. \tag{6.83}$$

From (6.75), (6.76), (6.82), (6.83) and the following equality

$$\forall m\in\mathbb{N},\quad \tilde{z}_m(t)=P^*_{c_0}\bar{u}_m(t)\quad\text{in}\quad V_0^*(\bar{v}_m(t)),\quad \forall t\in[0,\bar{T}_m], \tag{6.84}$$

we get the following uniform estimate:

$$\sup_{m\in\mathbb{N}}\left\{\|\tilde{u}'_m\|_{L^2(0,T;V^*)}+\sup_{0\le t\le T}\varphi_{c_0}\left(P^*_{c_0}\tilde{u}_m(t)\right)\right\} \le \sqrt{\frac{2}{c_1^2}\left\{\varphi_{c_0}\left(0,u_0,v_0,w_0\,;P^*_{c_0}u_0\right)+\frac{K_{17}}{K_{16}}\right\}e^{K_{16}T}},$$

which implies that the sequence $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ is relatively compact in $C([0,T];V^*)$ and bounded in $W^{1,2}(0,T;V^*)$. Hence, we see that there exist a subsequence $\{\tilde{u}_{m_k}\}_{k\in\mathbb{N}}$ of $\{\tilde{u}_m\}_{m\in\mathbb{N}}$ and an element $\tilde{u}\in C([0,T];V^*)\cap W^{1,2}(0,T;V^*)$ such that

$$\begin{aligned}\tilde{u}_{m_k}\longrightarrow\tilde{u}\quad&\text{in}\quad C([0,T];V^*)\quad\text{and}\\ &\text{weakly in}\quad W^{1,2}(0,T;V^*)\quad\text{as}\quad k\to\infty,\end{aligned} \tag{6.85}$$

hence,

$$\mathcal{S}\tilde{u}_{m_k}\longrightarrow\mathcal{S}\tilde{u}\quad\text{in}\quad C([0,T];V_0^*)\quad\text{as}\quad k\to\infty. \tag{6.86}$$

From (6.84), (6.85) and (6.86) we get the following equality:

$$(\tilde{u}(t),\tilde{v}(t),\tilde{w}(t))=(\hat{u}(t),\hat{v}(t),\hat{w}(t))\quad\text{on}\quad V_0^*\times A_v\times A_w,\quad\forall t\in[0,\hat{T}), \tag{6.87}$$

$$\tilde{u}=\tilde{u}_{\hat{T}}\quad\text{in}\quad C([0,T];V_0^*). \tag{6.88}$$

We see from (6.87) and (6.88) that the function $\tilde{u}$ is a strong solution of (E) below on $[0,\hat{T}]$:

$$\begin{cases}(P^*_{c_0}\tilde{u})'(t)+\partial_{V_0^*(\tilde{v}(t))}\varphi\left(t,\tilde{u},v_0,w_0\,;P^*_{c_0}\tilde{u}(t)\right)+g\left(P^*_{c_0}\tilde{u}(t),\tilde{v}(t)\right)\ni 0\\ \qquad\qquad\text{in}\quad V_0^*(\tilde{v}(t)),\ \text{a.a. } t\in(0,\hat{T}),\\ (\tilde{v}(t),\tilde{w}(t))=S(\tilde{u}\,;t,0)(v_0,w_0)\quad\text{in}\quad A_v\times A_w,\quad\forall t\in[0,\hat{T}],\\ \tilde{u}(0)=u_0\quad\text{in}\quad V_0^*.\end{cases}$$

Hence we see that the triplet $(\tilde{u},\tilde{v},\tilde{w},\hat{T})$ is an upper bound of $\mathcal{Y}$.

Finally, we can easily show Theorem 6.22 by repeating the argumentation similar to the proof of Theorem 1.3 and omit it in this proof. □

6.7. Mass-conservative tumor invasion model

At the beginning of this subsection, we give the definition of strong solutions to (T) on $[0,T]$.

Definition 6.24. A triplet (u,v,w) is called a strong solution to (T) on $[0,T]$ if and only if the following conditions are satisfied:

(1) $u \in W^{1,2}(0,T;V^*) \cap L^\infty(0,T;L^2(\Omega))$ with $u(0)=u_0$ in V^*, and there exists a function $\eta \in L^2(0,T;V)$ such that for any $z \in V$ and a.a. $t \in (0,T)$ the following variational equality holds with the quasi-variational double obstacle condition (6.90):

$$\langle u'(t), z\rangle_{V^*,V} + \langle F(v(t))\eta(t), z\rangle_{V^*V} - \left(\int_\Omega \eta(t)\,dx\right)\left(\int_\Omega z\,dx\right) = \int_\Omega u(t)\nabla\lambda(v(t))\cdot\nabla\xi\,dx, \tag{6.89}$$

$$\eta \in \beta(v(t)\,;u(t)) \quad \text{a.a. } x\in\Omega, \quad \forall t\in[0,T]. \tag{6.90}$$

(2) $v \in C([0,T];C(\overline{\Omega})\cap V) \cap W^{1,\infty}(0,T;L^\infty(\Omega))$, and it is expressed by

$$\begin{aligned} v(x,t) &= (S(u,w_0\,;t,0)v_0)(x,t) \\ &= v_0(x)\exp\left(\int_0^t w(x,s)ds\right), \quad \forall (x,t)\in\Omega\times[0,T]. \end{aligned}$$

(3) $w \in W^{1,2}(0,T;L^2(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega))$, and it is expressed by

$$w(t) = e^{t(d_w\Delta - b)}w_0 + c\int_0^t e^{(t-s)(d_w\Delta-b)}u(s)ds, \quad \forall t\in[0,T].$$

Next, we give Theorem 6.23, which is given in [9, Theorem 1.3].

Theorem 6.25. *A triplet (u,v,w) is a strong solution to (T) on $[0,T]$ if and only if it satisfies (1)' instead of (1), which is stated below, and (2), (3) in Definition 6.24:*

(1)' $u(t) \in W^*(c_0)$ for all $t\in[0,T]$, which is called a mass conservative property of (T) in this paper and enables us to consider $P^*_{c_0}u$ instead of u. Then, $P^*_{c_0}u \in W^{1,2}(0,T;V_0^*)\cap L^\infty(0,T;(L^2(\Omega))_0)$ with $P^*_{c_0}u(0) = \pi_{c_0}u_0$ in V_0^* and there exists a function $\eta\in L^2(0,T;V)$ such that for any $z_0\in V_0$ and a.a. $t\in(0,T)$ the following quasi-variational equality holds with (6.90):

$$\left\langle \left(P^*_{c_0}u\right)'(t), z_0\right\rangle_{V_0^*,V_0} + \langle (F_0(v(t))\circ P)\eta(t), z_0\rangle_{V_0^*,V_0} = \left\langle g\left(P^*_{c_0}u(t), v(t)\right), z_0\right\rangle_{V_0^*,V_0}. \tag{6.91}$$

Proof. Let a triplet (u, v, w) be a strong solution to (T) on $[0, T]$. Substituting $z = 1 \in V$ in (6.89), we get

$$\langle u'(t), 1\rangle_{V^*,V} = \frac{d}{dt}\int_\Omega u(t)\,dt = 0, \quad \text{a.a. } t \in (0, T), \tag{6.92}$$

hence,

$$\langle u(t), 1\rangle_{V^*,V} = \int_\Omega u(t)\,dt = \int_\Omega u_0\,dx = c_0, \quad \forall t \in [0, T],$$

which implies $u(t) \in W^*(c_0)$ for all $t \in [0, T]$. Moreover, we see from Proposition 6.6 and (6.92) that for any $z_0 \in V_0$ and all $t \in [0, T]$ we have

$$\langle u'(t), z_0\rangle_{V^*,V} = \langle u'(t), z_0\rangle_{V_0^*,V_0} = \left\langle \left(P_{c_0}^* u\right)'(t), z_0\right\rangle_{V_0^*,V_0}.$$

Hence, we see from Lemma 6.8 that (6.91) holds.

Conversely, we assume that (1)' holds. Then, we see from Lemma 6.8 and (6.91) that the following equality holds for all $z \in V$ and a.a. $t \in (0, T)$:

$$\begin{aligned}\left\langle (P_{c_0}u^*)'(t), Pz\right\rangle_{V_0^*,V_0} + \langle F(v(t))\eta(t), z\rangle_{V^*\,V} - \left(\int_\Omega \eta(t)\,dx\right)\left(\int_\Omega z\,dx\right) \\ = \langle g(u(t), v(t)), Pz\rangle_{V_0^*\,V_0} = \int_\Omega u(t)\nabla\lambda(v(t))\cdot\nabla z\,dx.\end{aligned} \tag{6.93}$$

Because of $u(t) \in W^*(c_0)$ for all $t \in [0, T]$, we have $u(s) - u(t) \in W^*(0)$ for all $s, t \in [0, T]$. Using Proposition 6.6 and (6.25), we see that the following equality holds for all $z \in V$:

$$\begin{aligned}\langle u(s) - u(t), z\rangle_{V^*,V} &= \langle u(s) - u(t), Pz\rangle_{V_0^*,V_0} \\ &= \left\langle P_{c_0}^* u(s) - P_{c_0}^* u(t), Pz\right\rangle_{V_0^*,V_0}.\end{aligned} \tag{6.94}$$

Dividing the both sides of (6.94) by $(t - s)$ and taking the limit $s \to t$, we get

$$\langle u'(t), z\rangle_{V^*,V} = \left\langle \left(P_{c_0}^* u\right)'(t), Pz\right\rangle_{V_0^*,V_0}, \quad \text{a.a. } t \in (0, T), \tag{6.95}$$

$$\begin{aligned}\|\langle u'(t), z\rangle_{V^*,V}| &\le \left\|\left(P_{c_0^*}u\right)'(t)\right\|_{V_0^*}\|Pz\|_{V_0} \\ &\le \left\|\left(P_{c_0}^* u\right)'(t)\right\|_{V_0^*}\|z\|_V, \quad \text{a.a. } t \in (0, T),\end{aligned}$$

which implies

$$\|u'(t)\|_{V^*} \le \left\|\left(P_{c_0}^* u\right)'(t)\right\|_{V_0^*}, \quad \text{a.a. } t \in (0, T),$$

hence, $u' \in W^{1,2}(0, T; V^*)$. Finally, we see from (6.93) and (6.95) that (6.89) holds. □

Concerning the relation between (E) and (T), we show Theorem 6.26 at first.

Theorem 6.26. *Let a triplet (u, v, w) be a strong solution to (T) on $[0, T]$. Then, the function u is a strong solution to (E) on $[0, T]$.*

Before giving the proof of Theorem 6.26, we prepare Lemma 6.27.

Lemma 6.27. *Assume that $z_0^* \in D(\varphi_{c_0}(\tilde{v}))$ and there exists a function $\eta \in V$ such that*

$$\eta(x) \in \beta\left(\tilde{v}(x)\,; z_0^*(x) + \frac{c_0}{|\Omega|}\right), \quad a.a\ x \in \Omega. \tag{6.96}$$

Then, we have $(F_0(\tilde{v}) \circ P)\,\eta \in \partial_{V_0^*(\tilde{v})}\varphi_{c_0}(\tilde{v}\,; z_0^*)$.

Proof. From (6.96) we have

$$\eta(x)\left\{r - \left(z_0^* + \frac{c_0}{|\Omega|}\right)\right\} \le \hat{\beta}(\tilde{v}\,; r) - \hat{\beta}\left(\tilde{v}\,; z_0^* + \frac{c_0}{|\Omega|}\right), \tag{6.97}$$
$$\forall r \in \mathbb{R}, \quad \text{a.a. } x \in \Omega.$$

Hence, we see from (6.97) that the following inequality holds for all $y_0^* \in D(\varphi_{c_0}(\tilde{v}))$:

$$\int_\Omega \eta(y_0^* - z_0^*)dx \le \int_\Omega \hat{\beta}\left(\tilde{v}\,; y_0^* + \frac{c_0}{|\Omega|}\right)dx - \int_\Omega \hat{\beta}\left(\tilde{v}\,; z_0^* + \frac{c_0}{|\Omega|}\right)dx. \tag{6.98}$$

Moreover, we have

$$\begin{aligned}\int_\Omega \eta(y_0^* - z_0^*)dx &= \int_\Omega (y_0^* - z_0^*)P\eta\,dx = (y_0^* - z_0^*, P\eta)_{(L^2(\Omega))_0} \\ &= \langle y_0^* - z_0^*, P\eta\rangle_{V_0^*, V_0} = ((F_0(\tilde{v}) \circ P)\,\eta, y_0^* - z_0^*)_{V_0^*}.\end{aligned} \tag{6.99}$$

Hence, from (6.98) and (6.99) we get

$$((F_0(\tilde{v}) \circ P)\,\eta, y_0^* - z_0^*)_{V_0^*} \le \varphi_{c_0}(y_0^*) - \varphi_{c_0}(z_0^*), \quad \forall y_0^* \in D(\varphi_{c_0}(\tilde{v})),$$

which implies $(F_0(\tilde{v}) \circ P)\,\eta \in \partial_{V_0^*(\tilde{v})}\varphi_{c_0}(\tilde{v}\,; z_0^*)$. □

Using Lemma 6.27, we show Theorem 6.26.

Proof of Theorem 6.26. We let a triplet (u, v, w) a strong solution to (T) on $[0, T]$. From Theorem 6.25 we have $P_{c_0}^* u(t) \in D(\varphi_{c_0}(t, u, v_0, w_0))$ for all $t \in (0, T)$ and the following equality holds for all $z_0^* \in V_0^*$ and a.a. $t \in (0, T)$:

$$\begin{aligned}&\left\langle \left(P_{c_0}^* u\right)'(t), F_0^{-1}(v(t))z_0^*\right\rangle_{V_0^*, V_0} + \left\langle (F_0(v(t)) \circ P)\eta(t), F_0^{-1}(v(t))z_0^*\right\rangle_{V_0^*, V_0} \\ &\qquad = \left\langle g\left(P_{c_0}^* u(t), v(t)\right), F_0^{-1}(v(t))z_0^*\right\rangle_{V_0^*, V_0},\end{aligned}$$

hence,

$$\begin{aligned}&\left(\left(P_{c_0}^* u\right)'(t), z_0^*\right)_{V_0^*(v(t))} + ((F_0(v(t)) \circ P)\eta(t), z_0^*)_{V_0^*(v(t))} \\ &\qquad = \left(g\left(P_{c_0}^* u(t), v(t)\right), z_0^*\right)_{V_0^*(v(t))},\end{aligned}$$

with

$$\eta \in \beta\left(v(t)\,;P^*_{c_0}u(t)+\frac{c_0}{|\Omega|}\right), \quad \text{a.a. } x\in\Omega, \quad \forall t\in[0,T].$$

Using Lemma 6.27, we get the following inclusion for a.a. $t\in(0,T)$:

$$\begin{aligned}(F_0(v(t))\circ P)\eta(t) &= -\left(P^*_{c_0}u\right)'(t)+g\left(P^*_{c_0}u(t),v(t)\right)\\ &\in \partial_{V_0^*(v(t))}\varphi_{c_0}\left(t,u,v_0,w_0\,;P^*_{c_0}u(t)\right).\end{aligned}$$

Hence, we see that the function u is a strong solution to (E) on $[0,T]$. □

In the rest of this subsection, we find the condition under which strong solutions to (E) on $[0,T]$ also become strong solutions to (T) on $[0,T]$. In order to do this, for $v\in A_v$ we define a proper, nonnegative, l.s.c. and convex function $\varphi(v)\colon V^*\longmapsto\mathbb{R}\cup\{\infty\}$ by

$$\varphi(\tilde{v}\,;z^*):=\begin{cases}\displaystyle\int_\Omega \hat{\beta}(\tilde{v}\,;z^*)dx & \\ \qquad\qquad \text{if} \quad z^*\in D(\varphi(\tilde{v})):=\left\{\tilde{z}^*\in L^2(\Omega)\,;\,\hat{\beta}(\tilde{v}\,;\tilde{z}^*)\in L^1(\Omega)\right\}, \\ \infty, \quad \text{if} \quad z^*\in V^*\setminus D(\varphi(\tilde{v})).\end{cases}$$

Since we can show that the function $\varphi(\tilde{v})$ is l.s.c. on V^* by using the argumentation similar to $\varphi_{c_0}(\tilde{v})$ in Lemma 6.10, we omit its proof here.

Next, we consider conjugate functions $\varphi^*_{c_0}:V_0\longmapsto\mathbb{R}\cup\{\infty\}$ and $\tilde{\varphi}:V\longmapsto\mathbb{R}\cup\{\infty\}$, which are defined by (6.100) and (6.101), respectively, and investigate the relation between $\varphi_{c_0}(\tilde{v})$ and $\varphi(\tilde{v})$:

$$\varphi^*_{c_0}(\tilde{v}\,;z_0):=\sup\left\{\langle y_0^*,z_0\rangle_{V_0^*,V_0}-\varphi_{c_0}(\tilde{v}\,;y_0^*)\,;\,y_0^*\in V_0^*\right\}, \quad \forall z_0\in V_0, \tag{6.100}$$

$$\varphi^*(\tilde{v}\,;z):=\sup\left\{\langle y^*,z\rangle_{V^*,V}-\varphi(\tilde{v}\,;y^*)\,;\,y^*\in V^*\right\}, \quad \forall z\in V. \tag{6.101}$$

Then, we have Lemma 6.28.

Lemma 6.28. *The following inequality holds for all $z\in V$:*

$$\varphi^*_{c_0}(\tilde{v}\,;Pz)\le\varphi^*(\tilde{v}\,;z)-\frac{c_0}{|\Omega|}\int_\Omega z\,dx.$$

Proof. From (6.101) we have

$$\langle y^*,z\rangle_{V^*,V}-\varphi(\tilde{v}\,;y^*)\le\varphi^*(\tilde{v}\,;z), \quad \forall y^*\in D(\varphi(\tilde{v})). \tag{6.102}$$

From Proposition 6.6 and (6.102) the following inequality holds for all $y^*\in D(\varphi(\tilde{v}))\cap W^*(c_0)$:

$$\langle P^*_{c_0}y^*,Pz\rangle_{V_0^*,V_0}+\frac{c_0}{|\Omega|}\int_\Omega z\,dx-\varphi\left(\tilde{v}\,;y_0^*+\frac{c_0}{|\Omega|}\right)\le\varphi^*(\tilde{v}\,;z). \tag{6.103}$$

Since we have $D(\varphi_{c_0}(\tilde{v})) = P^*_{c_0}(D(\varphi(\tilde{v})) \cap W^*(c_0)))$, we see from (6.103) that the following inequality holds for all $y^*_0 \in D(\varphi_{c_0}(\tilde{v}))$:

$$\langle y^*_0, Pz\rangle_{V^*_0,V_0} - \varphi_{c_0}(\tilde{v}\,;y^*_0) + \frac{c_0}{|\Omega|}\int_\Omega z\,dx \le \varphi^*(\tilde{v}\,;z). \tag{6.104}$$

We see from (6.101) and (6.104) that this lemma holds. □

Next, we show Proposition 6.29.

Proposition 6.29. *Assume that the following equality holds for all $\eta \in V$ and $\tilde{v} \in A_v$:*

$$\varphi^*_{c_0}(\tilde{v}\,;P\eta) = \varphi^*(\tilde{v}\,;\eta) - \frac{c_0}{|\Omega|}\int_\Omega \eta\,dx. \tag{6.105}$$

Then, the following conditions (a) and (b) are equivalent.

(a) $z^* \in D(\varphi(\tilde{v})) \cap W^*(c_0)$ and there exists a function $z \in V$ such that the following equality holds:

$$\varphi^*(\tilde{v}\,;z) + \varphi(\tilde{v}\,;z^*) = (z^*, z)_{L^2(\Omega)}. \tag{6.106}$$

(b) $z^*_0 \in D(\varphi_{c_0}(\tilde{v}))$ and there exists a function $z \in V$ such that the following equality holds:

$$\varphi^*_{c_0}(\tilde{v}\,;Pz) + \varphi_{c_0}(\tilde{v}\,;z^*_0) = (z^*_0, Pz)_{(L^2(\Omega))_0}. \tag{6.107}$$

Proof. We assume that (a) holds. Using (6.106) and substituting the following equality;

$$z^* = P^*_{c_0}z^* + \frac{c_0}{|\Omega|},$$

we have the following equality:

$$\varphi^*(v\,;z) + \varphi\left(\tilde{v}\,;P^*_{c_0}z^* + \frac{c_0}{|\Omega|}\right) = \left(P^*_{c_0}z^* + \frac{c_0}{|\Omega|}, z\right)_{L^2(\Omega)},$$

hence, from (6.105)

$$\begin{aligned}\varphi^*_{c_0}(\tilde{v}\,;Pz) + \varphi_{c_0}(v\,;P^*_{c_0}z^*) &= \left(P^*_{c_0}z^* + \frac{c_0}{|\Omega|}, z\right)_{L^2(\Omega)} - \frac{c_0}{|\Omega|}\int_\Omega z\,dx \\ &= (P^*_{c_0}z^*, z)_{L^2(\Omega)} = (P^*_{c_0}z^*, Pz)_{(L^2(\Omega))_0}.\end{aligned}$$

Hence, we see that $P^*_{c_0}z^* \in D(\varphi_{c_0}(\tilde{v}))$ is a required one as z^*_0 in (b).

Conversely, we assume that (b) holds. Using (6.107) and considering the following function

$$z^* = z^*_0 + \frac{c_0}{|\Omega|}, \tag{6.108}$$

we have $z^* \in D(\varphi(\tilde{v})) \cap W^*(c_0)$ and

$$\varphi^*_{c_0}(\tilde{v}\,;Pz) + \varphi_{c_0}\left(\tilde{v}\,;z^* - \frac{c_0}{|\Omega|}\right) = \left(z^* - \frac{c_0}{|\Omega|}, Pz\right)_{(L^2(\Omega))_0}.$$

We see from (6.105) that the following equality holds:

$$\begin{aligned}
&\varphi^*_{c_0}(\tilde{v}\,;Pz) + \varphi_{c_0}\left(\tilde{v}\,;z^* - \frac{c_0}{|\Omega|}\right) = \psi(\tilde{v}\,;z) + \varphi(\tilde{v}\,;z^*) \\
&= \int_\Omega \left(z^* - \frac{c_0}{|\Omega|}\right)(Pz)\,dx + \frac{c_0}{|\Omega|}\int_\Omega z\,dx \\
&= \int_\Omega \left(z^* - \frac{c_0}{|\Omega|}\right)\left(z - \frac{1}{|\Omega|}\int_\Omega z\,dx\right)dx + \frac{c_0}{|\Omega|}\int_\Omega z\,dx = (z^*, z)_{L^2(\Omega)}.
\end{aligned}$$

Hence, we see that the function z^* given by (6.108) is a required one in (a). □

Using [8, Proposition 3.5] and [9, Lemma 3.5], we have Corollary 6.30 to Proposition 6.29.

Corollary 6.30. *Assume that (6.105) holds for all $z \in V$. Then, the following three conditions (c), (d) and (e) are equivalent.*

(c) $z^* \in D(\varphi(\tilde{v})) \cap W^*(c_0)$ *and there exists a function* $z \in V$ *such that*

$$z(x) \in \beta(\tilde{v}(x)\,;z^*(x)) \quad a.a.\ x \in \Omega.$$

(d) $z^* \in D(\varphi(\tilde{v})) \cap W^*(c_0)$ *and there exists a function* $z \in V$ *such that* $F(\tilde{v})z \in \partial_{V^*(\tilde{v})}\varphi(\tilde{v}\,;z^*)$.

(e) $z^*_0 \in D(\varphi_{c_0}(\tilde{v}))$ *and there exists a function* $z \in V$ *such that* $(F_0(\tilde{v}) \circ P)z \in \partial_{V^*_0(\tilde{v})}\varphi_{c_0}(\tilde{v}\,;z^*_0)$.

Proof. We entrust the proof of (c) ⇔ (d) to [8, Proposition 3.5] and [9, Lemma 3.5], and omit it in this proof. We only show (d) ⇔ (e).

We assume that (d) holds. Because of $F(\tilde{v})z \in \partial_{V^*(\tilde{v})}\varphi(\tilde{v}\,;z^*)$, we see that the following equality holds:

$$\varphi^*(\tilde{v}\,;z) + \varphi(\tilde{v}\,;z^*) = (z^*, F(\tilde{v})z)_{V^*(\tilde{v})} = \langle z^*, z\rangle_{V^*,V} = (z^*, z)_{L^2(\Omega)}. \tag{6.109}$$

As you see from the proof of (a) ⇒ (b) in Proposition 6.29, we see from (6.109) that the following equality holds:

$$\begin{aligned}
\varphi^*_{c_0}(\tilde{v}\,;Pz) + \varphi_{c_0}(\tilde{v}\,;P^*_{c_0}z^*) &= (P^*_{c_0}z^*, Pz)_{(L^2(\Omega))_0} \\
&= \langle P^*_{c_0}z^*, Pz\rangle_{V^*_0,V_0} = (P^*_{c_0}z^*, (F_0(\tilde{v}) \circ P)z)_{V^*_0(\tilde{v})}.
\end{aligned} \tag{6.110}$$

We see that (6.110) implies $(F_0(\tilde{v}) \circ P)z \in \partial_{V^*_0(\tilde{v})}\varphi_{c_0}(\tilde{v}\,;P^*_{c_0}z^*)$. That is, $P^*_{c_0}z^* \in D(\varphi_{c_0}(\tilde{v}))$ is a required one as z^*_0, hence, we see that (e) holds .

Conversely, we assume that (e) holds. Because of $(F_0(\tilde{v})\circ P)z \in \partial_{V_0^*(\tilde{v})}\varphi_{c_0}(\tilde{v}\,;z_0^*)$, we see that the following equality holds:

$$\begin{aligned}\varphi_{c_0}^*(\tilde{v}\,;Pz)+\varphi_{c_0}(\tilde{v}\,;z_0^*) &= (z_0^*,(F_0(\tilde{v})\circ P)z)_{V_0^*(\tilde{v})} \\ &= \langle z_0^*, Pz\rangle_{V_0^*,V_0} = (z_0^*,Pz)_{(L^2(\Omega))_0}.\end{aligned} \tag{6.111}$$

As you see from the proof of (b) $\Rightarrow$ (a) in Proposition 6.29 again, we see from (6.111) that the following equality holds:

$$\begin{aligned}\varphi^*(\tilde{v}\,;z)+\varphi\left(\tilde{v}\,;z_0^*+\frac{c_0}{|\Omega|}\right) &= \left(z_0^*+\frac{c_0}{|\Omega|},z\right)_{L^2(\Omega)} \\ = \left\langle z_0^*+\frac{c_0}{|\Omega|},z\right\rangle_{V^*,V} &= \left(z_0^*+\frac{c_0}{|\Omega|},F(\tilde{v})z\right)_{V^*(\tilde{v})},\end{aligned}$$

which implies

$$F(\tilde{v})z \in \partial_{V^*(\tilde{v})}\varphi\left(\tilde{v}\,;z_0^*+\frac{c_0}{|\Omega|}\right).$$

That is, the following function

$$z_0^*+\frac{c_0}{|\Omega|} \in D(\varphi(\tilde{v}))\cap W^*(c_0)$$

is a required one as z^*, hence, we see that (d) holds. □

Finally, we obtain Theorem 6.23 as a result of Theorem 6.22, Proposition 6.29 and Corollary 6.30.

Theorem 6.31. *Assume that (6.105) holds for all $\eta \in V^*$ and $\tilde{v} \in A_v$. Then, the triplet (u,v,w) is a strong solution to (T) on $[0,T]$ if a function u is a strong solution to (E) on $[0,T]$.*

Proof. We assume that a function u is a strong solution to (E) on $[0,T]$. From (6.61) we have

$$\begin{aligned}-\left(P_{c_0}^*u\right)'(t)-g\left(P_{c_0}^*u(t),v(t)\right) &\in \partial_{V_0^*(v(t))}\varphi_{c_0}\left(t,u,v_0,w_0\,;P_{c_0}^*u(t)\right), \\ \text{in}\quad V_0^*(v(t)),&\quad \text{a.a. } t\in(0,T).\end{aligned}$$

We have $P_{c_0}^*u(t) \in D(\varphi_{c_0}(t,u,v_0,w_0))$ for a.a. $t\in(0,T)$ and see from Corollary 6.30 that there exists a function $\eta \in L^2(0,T;V)$ such that the following equality holds for a.a. $t\in(0,T)$:

$$\begin{aligned}(F_0(v(t))\circ P)\eta(t) &= -\left(P_{c_0}^*u\right)'(t)-g\left(P_{c_0}^*u(t),v(t)\right) \\ &\in \partial_{V_0^*(v(t))}\varphi_{c_0}\left(t,u,v_0,w_0\right);P_{c_0}^*u(t)\big) \quad \text{in} \quad V_0^*(v(t)).\end{aligned}$$

Hence, we see that (6.91) in Theorem 6.25 is satisfied. Using Corollary 6.30, we get

$$u(t)\in D(\varphi(v(t))\cap W^*(c_0), \quad \eta\in\beta(v(t)\,;u(t)) \quad \text{a.a. } x\in\Omega, \quad \forall t\in[0,T].$$

Hence, we see that the triplet (u,v,w) is a strong solution to (T) on $[0,T]$. □

Remark 6.32. From Theorems 6.26 and 6.31 we see that the initial-boundary value problem (T) of mass-conservative tumor invasion model is equivalent to the Cauchy problem (E) of an evolution inclusion with quasi-variational structures under the condition (6.105). In order to show (6.105), from Lemma 6.28 it is enough to show that the following inequality holds for all $\eta \in V$ and $\tilde{v} \in A_v$:

$$\varphi^*_{c_0}(\tilde{v}\,; P\eta) \geq \varphi^*(\tilde{v}\,; \eta) - \frac{c}{|\Omega|} \int_\Omega \eta \, dx.$$

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