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On a family of Euler-type numbers and polynomials

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Abstract. *We consider a family of Euler-type polynomials, depending on a real parameter $\alpha \neq 0, 1$. The case $\alpha = 2$ corresponds to standard Euler polynomials. We show some properties of these polynomials, and show also two generalized recurrences. As consequences of these results, we obtain several explicit formulas for Euler numbers and polynomials.*

1. Introduction

Euler polynomials $E_p(x)$ can be defined by the generating function

$$\frac{2e^{xt}}{e^t+1} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \tag{1.1}$$

or by the explicit formula [2, p. 48]

$$E_p(x) = \sum_{k=0}^{p} \binom{p}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2}\right)^{p-k}, \tag{1.2}$$

where E_k is the k-th Euler number: $E_k = 2^k E_k\left(\frac{1}{2}\right)$. Together with Bernoulli polynomials, Euler polynomials have been some of the predilect objects for mathemathicians during the last three centuries; the interest continues nowadays, obtaining new explicit formulas for them and new formulas where they are involved (see [3, 4, 5, 6, 8, 10, 12]).

In a previous work [9] we studied a generalization of Stirling numbers of the second kind, namely

$$S_{a,x}(p,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^j \binom{k}{j} \left(a(k-j)+x\right)^p, \tag{1.3}$$

where a, x are arbitrary complex numbers, $a \neq 0$. Two examples are

$$\begin{aligned} S_{1,0}(p,k) &= S(p,k), \\ S_{1,1}(p,k) &= S(p+1,k+1). \end{aligned} \tag{1.4}$$

If $k < 0$ or $k > p$ we have $S_{a,x}(p,k) = 0$.

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In this work we use results for the generalized Stirling numbers (GSN, for short) of the type $S_{1,x}(p,k)$, contained in [9], to study a generalization of Euler polynomials and numbers. (In [7] we used the GSN $S_{1,x}(p,k)$ to obtain some results involving Bernoulli polynomials.)

We summarize next some facts about the GSN $S_{1,x}(p,k)$.

- Some values:

$$\begin{aligned} S_{1,x}(p,0) =& x^p, \\ S_{1,x}(p,1) =& (x+1)^p - x^p, \\ S_{1,x}(p,2) =& \frac{1}{2}(x+2)^p - (x+1)^p + \frac{1}{2}x^p, \\ & \vdots \\ S_{1,x}(p,p) =& 1. \end{aligned}$$

- The GSN $S_{1,x}(p,k)$ can be written in terms of standard Stirling numbers as follows:

$$S_{1,x}(p,k) = \frac{1}{k!}\sum_{j=0}^{p}\binom{p}{j}(x-m)^{p-j}\sum_{t=0}^{m}\binom{m}{t}(k+t)!S(j,k+t), \tag{1.5}$$

where m is an arbitrary non-negative integer, and also as:

$$S_{1,x}(p,k) = \sum_{j=0}^{p}\binom{p}{j}(x-n)^{p-j}\sum_{t=0}^{n-1}(-1)^t s(n,n-t)S(j+n-t,k+n), \tag{1.6}$$

where n is an arbitrary positive integer, and $s(\cdot,\cdot)$ are the Stirling numbers of the first kind (with recurrence $s(q+1,k) = s(q,k-1) + qs(q,k)$). In particular, from (1.5) with $m=0$, and from (1.6) with $n=1$, we have

$$\begin{aligned} S_{1,x}(p,k) &= \sum_{j=k}^{p}\binom{p}{j}x^{p-j}S(j,k), & (1.7)\\ &= \sum_{j=k}^{p}\binom{p}{j}(x-1)^{p-j}S(j+1,k+1), & (1.8) \end{aligned}$$

respectively. (From (1.7) we see that $S_{1,x}(p,k)$ is a $(p-k)$-th degree polynomial in x.)

- The GSN $S_{1,x}(p,k)$ satisfy the identity:

$$S_{1,x+1}(p,k) = S_{1,x}(p,k) + (k+1)S_{1,x}(p,k+1). \tag{1.9}$$

- We have the identity:

$$S_{1,x}(p_1+p_2,l) = \sum_{m=0}^{p_2} S_{1,x}(p_2,m)S_{1,x+m}(p_1,l-m). \tag{1.10}$$

- The GSN $S_{1,x}(p,k)$ satisfy the recurrence:

$$S_{1,x}(p,k) = S_{1,x}(p-1,k-1) + (k+x)S_{1,x}(p-1,k). \tag{1.11}$$

- The GSN $S_{1,x}(p,k)$ can be written in terms of the GSN $S_{1,y}(p,k)$ as follows:

$$S_{1,x}(p,k) = \sum_{j=0}^{p} \binom{p}{j} (x-y)^{p-j} S_{1,y}(j,k). \tag{1.12}$$

- The derivative of the of GSN $S_{1,x}(p,k)$ with respect to x is

$$\frac{d}{dx} S_{1,x}(p,k) = pS_{1,x}(p-1,k). \tag{1.13}$$

(This formula is not included in [9], but its proof is straightforward.)

2. Definitions and preliminary results

In Theorem 3.1 of [6], the authors show the following explicit formula for Euler polynomials in terms of Stirling numbers of the second kind:

$$E_p(x) = \sum_{k=0}^{p} (-1)^{p-k} \binom{p}{k} \sum_{l=1}^{p-k+1} \frac{(-1)^{l-1}(l-1)!}{2^{l-1}} S(p-k+1,l)x^k. \tag{2.1}$$

We can write (2.1) as

$$E_p(x) = \sum_{k=0}^{p} \sum_{j=0}^{p} \binom{p}{j} x^{p-j} S(j,k) \frac{(-1)^k k!}{2^k}. \tag{2.2}$$

According to (1.7), formula (2.2) can be written in terms of GSN as

$$E_p(x) = \sum_{k=0}^{p} S_{1,x}(p,k) \frac{(-1)^k k!}{2^k}. \tag{2.3}$$

Three well-known properties of Euler polynomials are:
(*) Addition formula

$$E_p(x) = \sum_{j=0}^{p} \binom{p}{j} (x-y)^{p-j} E_j(y). \tag{2.4}$$

(*) Difference equation

$$E_p(x+1) + E_p(x) = 2x^p, \tag{2.5}$$

(*) Sum of powers

$$E_p(x+r)-(-1)^r E_p(x)=2\sum_{l=0}^{r-1}(-1)^{r-1-l}(x+l)^p. \tag{2.6}$$

For $\alpha\in\mathbb{R}$, $\alpha\neq 0,1$, we define the α *-Euler polynomial*, denoted as $E_p^{(\alpha)}(x)$, by

$$E_p^{(\alpha)}(x)=\sum_{k=0}^{p}S_{1,x}(p,k)\frac{(-1)^k k!}{\alpha^k}, \tag{2.7}$$

or, explicitly

$$E_p^{(\alpha)}(x)=\sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}x^{p-j}S(j,k)\frac{(-1)^k k!}{\alpha^k}. \tag{2.8}$$

It is clear that $E_p^{(\alpha)}(x)$ is a p-th degree monic polynomial.

Remark 2.1. The case $\alpha=1$ of (2.7) is not interesting, since $\sum_{k=0}^{p}(-1)^k k! S_{1,x}(p,k)$ $=(x-1)^p$, which can be proved easily.

Some examples are

$$\begin{aligned}
E_0^{(\alpha)}(x)&=1,\\
E_1^{(\alpha)}(x)&=x-\frac{1}{\alpha},\\
E_2^{(\alpha)}(x)&=x^2-\frac{2}{\alpha}x+\frac{2-\alpha}{\alpha^2},\\
E_3^{(\alpha)}(x)&=x^3-\frac{3}{\alpha}x^2+\frac{3(2-\alpha)}{\alpha^2}x-\frac{\alpha^2-6\alpha+6}{\alpha^3}.
\end{aligned}$$

Define the *α-Euler numbers* $E_p^{(\alpha)}$ as

$$E_p^{(\alpha)}=\alpha^p E_p^{(\alpha)}\left(\frac{1}{\alpha}\right). \tag{2.9}$$

That is, we have

$$E_p^{(\alpha)}=\sum_{k=0}^{p}\sum_{j=k}^{p}\binom{p}{j}\alpha^{j-k}S(j,k)(-1)^k k! \tag{2.10}$$

For example, we have $E_0^{(\alpha)}=1$, $E_1^{(\alpha)}=0$, $E_2^{(\alpha)}=1-\alpha$, $E_3^{(\alpha)}=(1-\alpha)(\alpha-2)$, $E_4^{(\alpha)}=(1-\alpha)(\alpha^2-9\alpha+9)$, and so on. Indeed, for $p>0$ we have $E_{2p+1}^{(\alpha)}=(1-\alpha)(\alpha-2)Q_{2p-2}(\alpha)$, where $Q_{2p-2}(\alpha)$ is a $(2p-2)$-th degree polynomial in α (we leave the proof of this fact to the reader).

Plainly $E_p^{(2)}(x)$ are the standard Euler polynomials $E_p(x)$, and $E_p^{(2)}$ are the standard Euler numbers E_p.

From (1.4), we see that the values of $E_p^{(\alpha)}(x)$ at $x = 0$ and $x = 1$ can be calculated as

$$E_p^{(\alpha)}(0) = \sum_{k=0}^{p} S(p,k)\frac{(-1)^k k!}{\alpha^k}, \tag{2.11}$$

$$E_p^{(\alpha)}(1) = \sum_{k=0}^{p} S(p+1,k+1)\frac{(-1)^k k!}{\alpha^k}. \tag{2.12}$$

By using the recurrence for Stirling numbers of the second kind in (2.12) we can see that

$$E_p^{(\alpha)}(1) = (1-\alpha)E_p^{(\alpha)}(0)\,,$$

where $p > 0$. Indeed, we have

$$\begin{aligned} E_p^{(\alpha)}(1) &= \sum_{k=0}^{p} S(p+1,k+1)\frac{(-1)^k k!}{\alpha^k} \\ &= \sum_{k=0}^{p} \left(S(p,k) + (k+1)S(p,k+1)\right)\frac{(-1)^k k!}{\alpha^k} \\ &= E_p^{(\alpha)}(0) - \alpha \sum_{k=0}^{p} S(p,k)\frac{(-1)^k k!}{\alpha^k} \\ &= (1-\alpha)E_p^{(\alpha)}(0), \end{aligned}$$

as claimed.

By using (1.5) and (1.6) we can write the following families of formulas for α-Euler polynomials

$$E_p^{(\alpha)}(x) = \sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}(x-m_1)^{p-j}\sum_{t=0}^{m_1}\binom{m_1}{t}S(j,k+t)\frac{(-1)^k(k+t)!}{\alpha^k}, \tag{2.13}$$

where m_1 is an arbitrary non-negative integer, and

$$E_p^{(\alpha)}(x) = \tag{2.14}$$
$$\sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}(x-m_2)^{p-j}\sum_{t=0}^{m_2-1}(-1)^t s(m_2,m_2-t)S(j+m_2-t,k+m_2)\frac{(-1)^k k!}{\alpha^k},$$

where m_2 is an arbitrary positive integer. The case $m_1 = 0$ of (2.13) is (2.7). The case $m_2 = 1$ of (2.14) is

$$E_p^{(\alpha)}(x) = \sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}(x-1)^{p-j}S(j+1,k+1)\frac{(-1)^k k!}{\alpha^k}. \tag{2.15}$$

In particular, from (2.13) we get the following formula for the value of $E_p^{(\alpha)}(x)$ at $x = m$ (a non-negative integer)

$$E_p^{(\alpha)}(m) = \sum_{k=0}^{p}\sum_{t=0}^{m}\binom{m}{t}S(p,k+t)\frac{(-1)^k(k+t)!}{\alpha^k}. \tag{2.16}$$

Similarly, from (2.14) we get the following formula for the value of $E_p^{(\alpha)}(x)$ at $x = n$ (a positive integer)

$$E_p^{(\alpha)}(n) = \sum_{k=0}^{p}\sum_{t=0}^{n-1}(-1)^t s(n,n-t)S(p+n-t,k+n)\frac{(-1)^k k!}{\alpha^k}. \tag{2.17}$$

We also obtain from (2.13) and (2.14) the following families of formulas for α-Euler numbers

$$E_p^{(\alpha)} = \sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}(1-\alpha m_1)^{p-j}\sum_{t=0}^{m_1}\binom{m_1}{t}S(j,k+t)\frac{(-1)^k(k+t)!}{\alpha^{k-j}} = \tag{2.18}$$

$$\sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}(1-\alpha m_2)^{p-j}\sum_{t=0}^{m_2-1}(-1)^t s(m_2,m_2-t)S(j+m_2-t,k+m_2)\frac{(-1)^k k!}{\alpha^{k-j}}.$$

where m_1 and m_2 are arbitrary integers, $m_1 \geq 0$, $m_2 > 0$.

By using (1.11) we obtain the following recurrence for α-Euler polynomials

$$E_{p+1}^{(\alpha)}(x) = \left(x - \frac{1}{\alpha}\right)E_p^{(\alpha)}(x) + \frac{\alpha-1}{\alpha}\sum_{k=0}^{p}S_{1,x}(p,k)\frac{(-1)^k k!k}{\alpha^k}. \tag{2.19}$$

In fact, we have

$$\begin{aligned}
E_{p+1}^{(\alpha)}(x) &= \sum_{k=0}^{p+1}S_{1,x}(p+1,k)\frac{(-1)^k k!}{\alpha^k} \\
&= \sum_{k=0}^{p+1}\left(S_{1,x}(p,k-1)+(k+x)S_{1,x}(p,k)\right)\frac{(-1)^k k!}{\alpha^k} \\
&= -\frac{1}{\alpha}\sum_{k=0}^{p}S_{1,x}(p,k)\frac{(-1)^k k!(k+1)}{\alpha^k} + \sum_{k=0}^{p}S_{1,x}(p,k)\frac{(-1)^k k!k}{\alpha^k} \\
&\quad + x\sum_{k=0}^{p}S_{1,x}(p,k)\frac{(-1)^k k!}{\alpha^k} \\
&= \left(x-\frac{1}{\alpha}\right)E_p^{(\alpha)}(x) + \frac{\alpha-1}{\alpha}\sum_{k=0}^{p}S_{1,x}(p,k)\frac{(-1)^k k!k}{\alpha^k},
\end{aligned}$$

as claimed. (In section 5 we will obtain general recurrences that includes (2.19) as particular case. See (5.2) and (5.3).) By setting $x = \frac{1}{\alpha}$ in (2.19) we obtain the following formula for the α-Euler numbers $E_{p+1}^{(\alpha)}$

$$E_{p+1}^{(\alpha)} = (\alpha-1)\sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}\alpha^{j-k}S(j,k)(-1)^k k!k. \tag{2.20}$$

In particular, from (2.19) we have the recurrence for standard Euler polynomials

$$E_{p+1}(x) = \left(x-\frac{1}{2}\right)E_p(x) + \sum_{k=0}^{p}S_{1,x}(p,k)\frac{(-1)^k k!k}{2^{k+1}}, \tag{2.21}$$

and from (2.20) we have the following formula for standard Euler numbers

$$E_{p+1} = \sum_{k=0}^{p}\sum_{j=k}^{p}\binom{p}{j}2^{j-k}S(j,k)(-1)^k k!k. \tag{2.22}$$

3. Basic properties

From (1.13) we see at once that $\frac{d}{dx}E_p^{(\alpha)}(x) = pE_{p-1}^{(\alpha)}(x)$. That is, α-Euler polynomials form an Appel sequence. We show next the properties for α-Euler polynomials, that generalize (2.4), (2.5) and (2.6).

Proposition 3.1. *The α-Euler polynomials $E_p^{(\alpha)}(x)$ have the following properties:*

(a) Addition formula:

$$E_p^{(\alpha)}(x) = \sum_{j=0}^{p}\binom{p}{j}(x-y)^{p-j}E_j^{(\alpha)}(y). \tag{3.1}$$

(b) Difference equation:

$$E_p^{(\alpha)}(x+1) + (\alpha-1)E_p^{(\alpha)}(x) = \alpha x^p. \tag{3.2}$$

(c) Sum of powers: If r is a given positive integer, then

$$E_p^{(\alpha)}(x+r) - (1-\alpha)^r E_p^{(\alpha)}(x) = \alpha\sum_{l=0}^{r-1}(1-\alpha)^{r-1-l}(x+l)^p. \tag{3.3}$$

Proof. (a) Formula (3.1) is equivalent to the fact that $E_p^{(\alpha)}(x)$ is an Appel sequence.

A direct and easy proof by using (1.12) is as follows

$$\begin{aligned}
E_p^{(\alpha)}(x) &= \sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^k k!}{\alpha^k} \\
&= \sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}(x-y)^{p-j}S_{1,y}(j,k)\frac{(-1)^k k!}{\alpha^k} \\
&= \sum_{j=0}^{p}\binom{p}{j}(x-y)^{p-j}\sum_{k=0}^{j}S_{1,y}(j,k)\frac{(-1)^k k!}{\alpha^k} \\
&= \sum_{j=0}^{p}\binom{p}{j}(x-y)^{p-j}E_j^{(\alpha)}(y),
\end{aligned}$$

as desired.

(b) We use (1.9) to write

$$\begin{aligned}
E_p^{(\alpha)}(x+1) &= \sum_{k=0}^{p} S_{1,x+1}(p,k)\frac{(-1)^k k!}{\alpha^k} \\
&= \sum_{k=0}^{p}\left(S_{1,x}(p,k)+(k+1)S_{1,x}(p,k+1)\right)\frac{(-1)^k k!}{\alpha^k} \\
&= \sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^k k!}{\alpha^k} + \sum_{k=0}^{p} S_{1,x}(p,k+1)\frac{(-1)^k (k+1)!}{\alpha^k} \\
&= E_p^{(\alpha)}(x) - \alpha\sum_{k=1}^{p} S_{1,x}(p,k)\frac{(-1)^k k!}{\alpha^k} \\
&= E_p^{(\alpha)}(x) - \alpha\left(E_p^{(\alpha)}(x) - S_{1,x}(p,0)\right) \\
&= (1-\alpha)E_p^{(\alpha)}(x) + \alpha x^p,
\end{aligned}$$

as claimed.

(c) The case $r=1$ of (3.3) is (3.2). If we suppose that (3.3) is true for a positive integer $r > 1$, then, by using (3.2) we have

$$\begin{aligned}
&E_p^{(\alpha)}(x+r+1) - (1-\alpha)^{r+1}E_p^{(\alpha)}(x) \\
&= E_p^{(\alpha)}(x+r+1) - (1-\alpha)^r E_p^{(\alpha)}(x+1) \\
&\quad +(1-\alpha)^r E_p^{(\alpha)}(x+1) - (1-\alpha)^{r+1}E_p^{(\alpha)}(x) \\
&= \alpha\sum_{l=0}^{r-1}(1-\alpha)^{r-1-l}(x+l+1)^p + (1-\alpha)^r\left(E_p^{(\alpha)}(x+1) - (1-\alpha)E_p^{(\alpha)}(x)\right) \\
&= \alpha\sum_{l=0}^{r-1}(1-\alpha)^{r-1-l}(x+l+1)^p + (1-\alpha)^r\alpha x^p \\
&= \alpha\sum_{l=0}^{r}(1-\alpha)^{r-l}(x+l)^p,
\end{aligned}$$

as wanted. □

If we set $y = \frac{1}{\alpha}$ in (3.1), we can write the α-Euler polynomial $E_p^{(\alpha)}(x)$ in terms of α-Euler numbers as

$$\begin{aligned}
E_p^{(\alpha)}(x) &= \sum_{j=0}^{p}\binom{p}{j}\left(x-\frac{1}{\alpha}\right)^{p-j}E_j^{(\alpha)}\left(\frac{1}{\alpha}\right) \\
&= \sum_{j=0}^{p}\binom{p}{j}\frac{E_j^{(\alpha)}}{\alpha^j}\left(x-\frac{1}{\alpha}\right)^{p-j}.
\end{aligned} \tag{3.4}$$

4. Different parameters

In this section we show some relations connecting α-Euler polynomials $E_p^{(\alpha)}(x)$ with β-Euler polynomials $E_p^{(\beta)}(x)$, where $\alpha, \beta \in \mathbb{R} - \{0, 1\}$ are given parameters.

Let us write (2.8) as

$$E_p^{(\alpha)}(x) = \sum_{j=0}^{p}\sum_{k=0}^{p-j}\binom{p}{j}S(p-j,k)\frac{(-1)^k k!}{\alpha^k}x^j. \tag{4.1}$$

We denote by $c_j^{(p;\alpha)}$, the coefficient of x^j in $E_p^{(\alpha)}(x)$, $j = 0, 1, \ldots, p$. That is, we have

$$E_p^{(\alpha)}(x) = \sum_{j=0}^{p} c_j^{(p;\alpha)}x^j, \tag{4.2}$$

where

$$c_j^{(p;\alpha)} = \sum_{k=0}^{p-j}\binom{p}{j}S(p-j,k)\frac{(-1)^k k!}{\alpha^k}. \tag{4.3}$$

Proposition 4.1. *We have*

$$E_p^{(\alpha)}(x) - E_p^{(\beta)}(x) = \frac{\beta-\alpha}{\beta}\sum_{j=0}^{p-1} c_j^{(p;\alpha)} E_j^{(\beta)}(x). \tag{4.4}$$

Proof. Both sides of (4.4) are $(p-1)$-th degree polynomials in x. We will show that the values of these polinomials are equal when x is a non-negative integer, and then we conclude that the polinomials are equal for all $x \in \mathbb{R}$. We proceed by induction on x. First observe that if (4.4) is true for a positive integer x, then, by using (3.2) we have

$$\begin{aligned}
&\frac{\beta-\alpha}{\beta}\sum_{j=0}^{p-1} c_j^{(p;\alpha)} E_j^{(\beta)}(x+1)\\
&= \frac{\beta-\alpha}{\beta}\sum_{j=0}^{p-1} c_j^{(p;\alpha)} \left(\beta x^j - (\beta-1)E_j^{(\beta)}(x)\right)\\
&= (\beta-\alpha)\sum_{j=0}^{p-1} c_j^{(p;\alpha)} x^j - (\beta-1)\frac{\beta-\alpha}{\beta}\sum_{j=0}^{p-1} c_j^{(p;\alpha)} E_j^{(\beta)}(x)\\
&= (\beta-\alpha)\left(E_p^{(\alpha)}(x) - x^p\right) - (\beta-1)\left(E_p^{(\alpha)}(x) - E_p^{(\beta)}(x)\right)\\
&= (\alpha-\beta)x^p - (\alpha-1)E_p^{(\alpha)}(x) + (\beta-1)E_p^{(\beta)}(x)\\
&= \alpha x^p - (\alpha-1)E_p^{(\alpha)}(x) - \left(\beta x^p - (\beta-1)E_p^{(\beta)}(x)\right)\\
&= E_p^{(\alpha)}(x+1) - E_p^{(\beta)}(x+1),
\end{aligned} \tag{4.5}$$

which shows that (4.4) is also true for $x+1$.

Now we prove that (4.4) is valid for $x = 0$. Observe that

$$\begin{aligned}
E_p^{(\alpha)}(x) - E_p^{(\beta)}(x) &= \sum_{k=0}^{p} S_{1,x}(p,k)(-1)^k k!\left(\frac{1}{\alpha^k} - \frac{1}{\beta^k}\right)\\
&= \sum_{k=0}^{p} S_{1,x}(p,k)(-1)^k k!\frac{\beta^k-\alpha^k}{\alpha^k\beta^k}\\
&= \frac{\beta-\alpha}{\beta}\sum_{k=1}^{p} S_{1,x}(p,k)(-1)^k k!\frac{\sum_{l=1}^{k}\beta^{k-l}\alpha^{l-1}}{\alpha^k\beta^{k-1}}.
\end{aligned} \tag{4.6}$$

That is, we have to prove that

$$\sum_{j=0}^{p-1} c_j^{(p;\alpha)} E_j^{(\beta)}(0) = \sum_{k=1}^{p} S(p,k)(-1)^k k!\frac{\sum_{l=1}^{k}\beta^{k-l}\alpha^{l-1}}{\alpha^k\beta^{k-1}},$$

or, according to (2.11) and (4.3), we have to prove that

$$\sum_{j=0}^{p-1}\sum_{m=1}^{p-j}\binom{p}{j}S(p-j,m)\frac{(-1)^m m!}{\alpha^m}\sum_{l=0}^{j}S(j,l)\frac{(-1)^l l!}{\beta^l} = \sum_{k=1}^{p}S(p,k)(-1)^k k!\frac{\sum_{l=1}^{k}\beta^{k-l}\alpha^{l-1}}{\alpha^k\beta^{k-1}}. \tag{4.7}$$

Let us begin with the left-hand side of (4.7), that we denote simply as $\mathrm{LHS}_{(4.7)}$. We have

$$\mathrm{LHS}_{(4.7)} = \sum_{m=1}^{p}\frac{(-1)^m m!}{\alpha^m}\sum_{j=0}^{p-1}\sum_{l=0}^{j}\binom{p}{j}S(p-j,m)S(j,l)\frac{(-1)^l l!}{\beta^l}. \tag{4.8}$$

Introduce the new index $n=m+l$ in (4.8), to write

$$\mathrm{LHS}_{(4.7)} = \sum_{m=1}^{p}\frac{(-1)^m m!}{\alpha^m}\sum_{n=m}^{p-1}\sum_{j=0}^{p-1}\binom{p}{j}S(p-j,m)S(j,n-m)\frac{(-1)^{n-m}(n-m)!}{\beta^{n-m}}. \tag{4.9}$$

Now we use the convolution formula for Stirling numbers of the second kind

$$\sum_{j=0}^{p-1}\binom{p}{j}S(p-j,m)S(j,N-m) = \binom{N}{m}S(p,N), \tag{4.10}$$

(see [1, p. 825]) to obtain from (4.9) that

$$\mathrm{LHS}_{(4.7)} = \sum_{m=1}^{p}\frac{(-1)^m m!}{\alpha^m}\sum_{N=m}^{p}\binom{N}{m}S(p,N)\frac{(-1)^{N-m}(N-m)!}{\beta^{N-m}}. \tag{4.11}$$

Some additional elementary simplifications give us

$$\begin{aligned}\mathrm{LHS}_{(4.7)} &= \sum_{m=1}^{p}\sum_{N=m}^{p}S(p,N)\frac{(-1)^N N!}{\alpha^m\beta^{N-m}}\\ &= \sum_{N=1}^{p}S(p,N)(-1)^N N!\sum_{m=1}^{N}\alpha^{-m}\beta^{m-N}\end{aligned} \tag{4.12}$$

Introduce the new index $l=N+1-m$ in (4.12) to obtain

$$\mathrm{LHS}_{(4.7)} = \sum_{N=1}^{p}S(p,N)(-1)^N N!\sum_{l=1}^{N}\alpha^{l-N-1}\beta^{1-l},$$

which is the right-hand side of (4.7). □

Formula (4.4) says that if

$$E_p^{(\alpha)}(x) = x^p + \sum_{j=0}^{p-1} c_j^{(p;\alpha)} x^j, \tag{4.13}$$

then we can use the polynomials $E_j^{(\beta)}(x)$, $0 \leq j \leq p$, to obtain $E_p^{(\alpha)}(x)$, "by means of an Umbral Substitution" (of x^j by $E_j^{(\beta)}(x)$) in the right-hand side of (4.13), with the additional factor $\frac{\beta-\alpha}{\beta}$ in the terms of degree $< p$.

Corollary 4.2. *(a) The α-Euler polynomial $E_p^{(\alpha)}(x)$ can be written in terms of the standard Euler polynomials $E_j(x)$, $j = 0, 1, \ldots, p$ as*

$$E_p^{(\alpha)}(x) = E_p(x) + \frac{2-\alpha}{2} \sum_{j=0}^{p-1} \sum_{k=0}^{p-j} \binom{p}{j} S(p-j,k) \frac{(-1)^k k!}{\alpha^k} E_j(x). \tag{4.14}$$

(b) The standard Euler polynomial $E_p(x)$ can be written in terms of the α-Euler polynomials $E_j^{(\alpha)}(x)$, $j = 0, 1, \ldots, p$ as

$$E_p(x) = E_p^{(\alpha)}(x) + \frac{\alpha-2}{\alpha} \sum_{j=0}^{p-1} \sum_{k=0}^{p-j} \binom{p}{j} S(p-j,k) \frac{(-1)^k k!}{2^k} E_j^{(\alpha)}(x). \tag{4.15}$$

Proof. It is a direct consequence of (4.4). □

We can use (4.14) to write the sum of powers (3.3) in terms of standard Euler polynomials as

$$\begin{aligned} E_p(x+r) &- (1-\alpha)^r E_p(x) \\ &+ \frac{2-\alpha}{2} \sum_{j=0}^{p-1} \sum_{k=0}^{p-j} \binom{p}{j} S(p-j,k) \frac{(-1)^k k!}{\alpha^k} \left(E_j(x+r) - (1-\alpha)^r E_j(x)\right) \\ &= \alpha \sum_{l=0}^{r-1} (1-\alpha)^{r-1-l} (x+l)^p. \end{aligned} \tag{4.16}$$

In the case $\alpha = 1$, formula (4.16) gives us the identiy (see Remark 2.1)

$$\sum_{j=0}^{p} \sum_{k=0}^{p-j} \binom{p}{j} (-1)^{k+1} k! S(p-j,k) E_j(x+r) = E_p(x+r) - 2(x+r-1)^p,$$

and in the case $\alpha = -1$, we have a formula for the weighted sum of powers $\sum_{l=0}^{r-1} 2^{r-l}(x+l)^p$ in terms of Euler polynomials and Fubini numbers $F_m = \sum_{k=0}^{m} k! S(m,k)$, namely

$$\sum_{l=0}^{r-1} 2^{r-l}(x+l)^p = E_p(x+r) - 2^r E_p(x) - 3 \sum_{j=0}^{p} \binom{p}{j} \left(E_j(x+r) - 2^r E_j(x)\right) F_{p-j}.$$

5. Generalized recurrences

We begin this section with a formula for $E^{(\alpha)}_{p_1+p_2}(x)$, where p_1, p_2 are arbitrary non-negative integers.

Proposition 5.1. *We have*

$$E^{(\alpha)}_{p_1+p_2}(x) = \sum_{k_1=0}^{p_1} \sum_{k_2=0}^{p_2} S_{1,x}(p_2, k_2) S_{1,x+k_2}(p_1, k_1) \frac{(-1)^{k_1+k_2}(k_1+k_2)!}{\alpha^{k_1+k_2}}. \tag{5.1}$$

Proof. By using (2.3) and (1.10) we have

$$\begin{aligned} E^{(\alpha)}_{p_1+p_2}(x) &= \sum_{k=0}^{p_1+p_2} S_{1,x}(p_1+p_2, k) \frac{(-1)^k k!}{\alpha^k} \\ &= \sum_{k_2=0}^{p_2} \sum_{k_1=k_2}^{p_1+p_2} S_{1,x}(p_2, k_2) S_{1,x+k_2}(p_1, k_1-k_2) \frac{(-1)^{k_1} k_1!}{\alpha^{k_1}} \\ &= \sum_{k_1=0}^{p_1} \sum_{k_2=0}^{p_2} S_{1,x}(p_2, k_2) S_{1,x+k_2}(p_1, k_1) \frac{(-1)^{k_1+k_2}(k_1+k_2)!}{\alpha^{k_1+k_2}}, \end{aligned}$$

as wanted. □

In terms of standard Stirling numbers, formula (5.1) looks as

$$E^{(\alpha)}_{p_1+p_2}(x) =$$

$$\sum_{k_1=0}^{p_1} \sum_{k_2=0}^{p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (x+k_2)^{p_1-j_2} x^{p_2-j_2} S(j_1, k_1) \times$$

$$\times S(j_2, k_2) \frac{(-1)^{k_1+k_2}(k_1+k_2)!}{\alpha^{k_1+k_2}}.$$

The case $p_1 = 1$ of (5.1) is

$$E^{(\alpha)}_{p+1}(x) = (x-1)E^{(\alpha)}_p(x) + (\alpha-1) \sum_{k=0}^{p} \sum_{j=0}^{p} \binom{p}{j} x^{p-j} S(j,k) \frac{(-1)^k (k+1)!}{\alpha^{k+1}}. \tag{5.2}$$

(The case $\alpha = 2$ of (5.2) is essentially the recurrence (2.21).) Formula (5.2) is included in the following general result.

Proposition 5.2. *For arbitrary non-negative integers p, q, we have*

$$\sum_{k=0}^{q} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) = (\alpha-1)^q\sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^{k+q}(k+q)!}{\alpha^{k+q}}. \quad (5.3)$$

Proof. We proceed by induction on q. The case $q = 1$ of (5.3) is (5.2). If we suppose that (5.3) is true for a given $q \in \mathbb{N}$, then

$$\begin{aligned}
&\sum_{k=0}^{q+1} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q+1}(x-j) \\
&= \sum_{k=0}^{q+1} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\left((x-q-1)\prod_{j=1}^{q}(x-j)\right) \\
&= \sum_{k=0}^{q+1} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\left((x-q-1)\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) + k\frac{d^{k-1}}{dx^{k-1}}\prod_{j=1}^{q}(x-j)\right) \\
&= (x-q-1)\sum_{k=0}^{q} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) \\
&\quad -\sum_{k=0}^{q} E_{p+1+k;\alpha}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) \\
&= (x-q-1)(\alpha-1)^q\sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^{k+q}(k+q)!}{\alpha^{k+q}} \\
&\quad -(\alpha-1)^q\sum_{k=0}^{p+1} S_{1,x}(p+1,k)\frac{(-1)^{k+q}(k+q)!}{\alpha^{k+q}},
\end{aligned} \quad (5.4)$$

where we used the induction hypothesis in the last step. According to (1.11) we have from (5.4) that

$$
\begin{aligned}
&\sum_{k=0}^{q+1} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q+1}(x-j)\\
&=(\alpha-1)^q(x-q-1)\sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^{k+q}(k+q)!}{\alpha^{k+q}}\\
&\quad-(\alpha-1)^q\sum_{k=0}^{p+1}\left(S_{1,x}(p,k-1)+(k+x)S_{1,x}(p,k)\right)\frac{(-1)^{k+q}(k+q)!}{\alpha^{k+q}}\\
&=(\alpha-1)^q\,(x-q-1)\sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^{k+q}(k+q)!}{\alpha^{k+q}}\\
&\quad+(\alpha-1)^q\sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^{k+q}(k+q+1)!}{\alpha^{k+q+1}}\\
&\quad-(\alpha-1)^q\sum_{k=0}^{p}(k+x)S_{1,x}(p,k)\frac{(-1)^{k+q}(k+q)!}{\alpha^{k+q}}\\
&=\alpha(\alpha-1)^q\sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^{k+q+1}(k+q+1)!}{\alpha^{k+q+1}}\\
&\quad+(\alpha-1)^q\sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^{k+q}(k+q+1)!}{\alpha^{k+q+1}}\\
&=(\alpha-1)^{q+1}\sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^{k+q+1}(k+q+1)!}{\alpha^{k+q+1}},
\end{aligned}
$$

as wanted. □

If we set $p_2 = 1$ in (5.1) we get

$$
E_{p+1}^{(\alpha)}(x) = xE_p^{(\alpha)}(x) + \sum_{k=0}^{p}\sum_{j=k}^{p}\binom{p}{j}(x+1)^{p-j}S(j,k)\frac{(-1)^{k+1}(k+1)!}{\alpha^{k+1}}. \tag{5.5}
$$

Formula (5.5) is included in the following general result.

Proposition 5.3. *For arbitrary non-negative integers p, q, we have*

$$
\sum_{k=0}^{q} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=0}^{q-1}(x+j) = \sum_{k=0}^{p} S_{1,x+q}(p,k)\frac{(-1)^k(k+q)!}{\alpha^{k+q}}. \tag{5.6}
$$

Proof. We proceed by induction on q. The case $q = 1$ of (5.6) is (5.5). If we suppose formula (5.6) is true for a given $q \in \mathbb{N}$, then

$$\begin{aligned}
&\sum_{k=0}^{q+1} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=0}^{q}(x+j) \\
&= \sum_{k=0}^{q+1} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\left((x+q)\prod_{j=0}^{q-1}(x+j)\right) \\
&= \sum_{k=0}^{q+1} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\left((x+q)\frac{d^k}{dx^k}\prod_{j=0}^{q-1}(x+j)+k\frac{d^{k-1}}{dx^{k-1}}\prod_{j=0}^{q-1}(x+j)\right) \\
&= (x+q)\sum_{k=0}^{q} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=0}^{q-1}(x+j) \\
&\quad -\sum_{k=0}^{q} E_{p+1+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=0}^{q-1}(x+j) \\
&= (x+q)\sum_{k=0}^{p} S_{1,x+q}(p,k)\frac{(-1)^k(k+q)!}{\alpha^{k+q}}-\sum_{k=0}^{p+1} S_{1,x+q}(p+1,k)\frac{(-1)^k(k+q)!}{\alpha^{k+q}},
\end{aligned}$$

where we used induction hypothesis in the last step. By using first (1.11) and then (1.9) we have

$$\begin{aligned}
&\sum_{k=0}^{q+1} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=0}^{q}(x+j) \\
&= (x+q)\sum_{k=0}^{p} S_{1,x+q}(p,k)\frac{(-1)^k(k+q)!}{\alpha^{k+q}} \\
&\quad -\sum_{k=0}^{p+1}\left(S_{1,x+q}(p,k-1)+(k+x+q)S_{1,x+q}(p,k)\right)\frac{(-1)^k(k+q)!}{\alpha^{k+q}} \\
&= -\sum_{k=1}^{p+1}\left(S_{1,x+q}(p,k-1)+kS_{1,x+q}(p,k)\right)\frac{(-1)^k(k+q)!}{\alpha^{k+q}} \\
&= \sum_{k=0}^{p}\left(S_{1,x+q}(p,k)+(k+1)S_{1,x+q}(p,k+1)\right)\frac{(-1)^k(k+q+1)!}{\alpha^{k+q+1}} \\
&= \sum_{k=0}^{p} S_{1,x+q+1}(p,k)\frac{(-1)^k(k+q+1)!}{\alpha^{k+q+1}},
\end{aligned}$$

as desired. □

We show next some particular cases of (5.3) and (5.6). We will use that

$$\left[\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j)\right]_{x=0} = (-1)^q s(q+1,k+1),$$
$$\left[\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j)\right]_{x=1} = (-1)^q s(q,k).$$

First observe that we can write formula (5.6) as

$$\sum_{k=0}^{q} E_{p+k}^{(\alpha)}(x-q)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) = \sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^k(k+q)!}{\alpha^{k+q}}. \tag{5.7}$$

Thus, formulas (5.3) and (5.7) can be written together as

$$\begin{aligned}&\frac{1}{(\alpha-1)^q}\sum_{k=0}^{q} E_{p+k}^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j)\\ &= \sum_{k=0}^{q} E_{p+k}^{(\alpha)}(x-q)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j)\\ &= \sum_{k=0}^{p} S_{1,x}(p,k)\frac{(-1)^k(k+q)!}{\alpha^{k+q}}. \end{aligned}\tag{5.8}$$

The cases $x = 0$ and $x = 1$ of (5.8) are

$$\begin{aligned}&\frac{1}{(\alpha-1)^q}\sum_{k=0}^{q} s(q+1,k+1)E_{p+k}^{(\alpha)}(0)\\ &= (-1)^q\sum_{k=0}^{q} s(q+1,k+1)E_{p+k}^{(\alpha)}(-q)\\ &= \sum_{k=0}^{p} S(p,k)\frac{(-1)^k(k+q)!}{\alpha^{k+q}}, \end{aligned}\tag{5.9}$$

and

$$\begin{aligned}&\frac{1}{(\alpha-1)^q}\sum_{k=0}^{q} s(q,k)E_{p+k}^{(\alpha)}(1)\\ &= (-1)^q\sum_{k=0}^{q} s(q,k)E_{p+k}^{(\alpha)}(1-q)\\ &= \sum_{k=0}^{p} S(p+1,k+1)\frac{(-1)^k(k+q)!}{\alpha^{k+q}}, \end{aligned}\tag{5.10}$$

respectively.

In the case $p = 0$, formulas (5.8), (5.9) and (5.10) produce the same result: $q!/\alpha^q$. We can write together the corresponding identities as

$$\begin{aligned}
&\frac{1}{(1-\alpha)^q}\sum_{k=0}^{q}E_k^{(\alpha)}(x)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) \\
&= \sum_{k=0}^{q}E_k^{(\alpha)}(x-q)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) \\
&= \frac{1}{(\alpha-1)^q}\sum_{k=0}^{q}s(q+1,k+1)E_k^{(\alpha)}(0) \\
&= (-1)^q\sum_{k=0}^{q}s(q+1,k+1)E_k^{(\alpha)}(-q) \\
&= \frac{1}{(\alpha-1)^q}\sum_{k=0}^{q}s(q,k)E_k^{(\alpha)}(1) \\
&= (-1)^q\sum_{k=0}^{q}s(q,k)E_k^{(\alpha)}(1-q) \\
&= \frac{q!}{\alpha^q}. \qquad (5.11)
\end{aligned}$$

In particular, we have the following identities involving standard Euler polynomials:

$$\begin{aligned}
&\sum_{k=0}^{q}E_k(x)\frac{(-1)^{k+q}}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) \\
&= \sum_{k=0}^{q}E_k(x-q)\frac{(-1)^k}{k!}\frac{d^k}{dx^k}\prod_{j=1}^{q}(x-j) \\
&= \sum_{k=0}^{q}s(q+1,k+1)E_k(0) \\
&= (-1)^q\sum_{k=0}^{q}s(q+1,k+1)E_k(-q) \\
&= \sum_{k=0}^{q}s(q,k)E_k(1) \\
&= \sum_{k=0}^{q}(-1)^{k+q}s(q,k)E_k(q) \\
&= \frac{q!}{2^q}. \qquad (5.12)
\end{aligned}$$

6. Some formulas for Euler numbers

From the results of previous sections, we obtain several explicit formulas for even Euler numbers, and sum and difference of two even Euler numbers as well. (We will be considering the case $\alpha = 2$ of the previous general results.)

Proposition 6.1. *We have the following formulas for the even Euler numbers E_{2p}, where $p > 0$,*

$$E_{2p} = \sum_{k=0}^{2p} \sum_{j=k}^{2p} \binom{2p}{j} 2^{j-k} S(j,k)(-1)^k (k+1)! \tag{6.1}$$

$$= \sum_{k=0}^{2p-1} \sum_{j=0}^{2p-1} \binom{2p-1}{j} 2^{j-k} S(j,k)(-1)^k (k+1)! \tag{6.2}$$

$$= \sum_{k=0}^{2p-2} \sum_{j=0}^{2p-2} \binom{2p-2}{j} 2^{j-k} S(j,k)(-1)^k (k+2)!(k-1) \tag{6.3}$$

$$= \sum_{k=0}^{2p-1} \sum_{j=0}^{2p-1} \binom{2p-1}{j} 2^{j-k-2} S(j,k)(-1)^k (k+2)! \tag{6.4}$$

$$= \sum_{k=0}^{2p} \sum_{j=0}^{2p} \binom{2p}{j} 3^{2p-j} 2^{j-k} S(j,k)(-1)^k (k+1)! \tag{6.5}$$

$$= \sum_{k=0}^{2p-1} \sum_{j=0}^{2p-1} \binom{2p-1}{j} 3^{2p-1-j} 2^{j-k} S(j,k)(-1)^{k+1} (k+1)! \tag{6.6}$$

$$= \sum_{k=0}^{2p-1} \sum_{j=0}^{2p-1} \binom{2p-1}{j} 5^{2p-1-j} 2^{j-k-2} S(j,k)(-1)^{k+1} (k+2)! \tag{6.7}$$

Proof. From (5.2) we can see that

$$E_{p+1} + E_p = \sum_{k=0}^{p} \sum_{j=k}^{p} \binom{p}{j} 2^{j-k} S(j,k)(-1)^k (k+1)! \tag{6.8}$$

Replace p by $2p$ in (6.8) to obtain (6.1). Replace p by $2p-1$ in (6.8) to obtain (6.2).

From the case $q = 2$ of (5.3) we see that

$$3E_p + 4E_{p+1} + E_{p+2} = \sum_{k=0}^{p} \sum_{j=0}^{p} \binom{p}{j} 2^{j-k} S(j,k)(-1)^k (k+2)! \tag{6.9}$$

Replace p by $2p-1$ in (6.9) to obtain (6.4). Replace p by $2p-2$ in (6.9) to obtain

$$3E_{2p-2} + E_{2p} = \sum_{k=0}^{2p-2} \sum_{j=0}^{2p-2} \binom{2p-2}{j} 2^{j-k} S(j,k)(-1)^k (k+2)! \tag{6.10}$$

From (6.1) we see (replacing p by $p-1$) that

$$E_{2p-2}=\sum_{k=0}^{2p-2}\sum_{j=k}^{2p-2}\binom{2p-2}{j}2^{j-k}S(j,k)(-1)^{k}(k+1)! \tag{6.11}$$

From (6.10) and (6.11) we obtain (6.3).
From (5.5) we see that

$$E_{p+1}-E_{p}=\sum_{k=0}^{p}\sum_{j=k}^{p}\binom{p}{j}3^{p-j}2^{j-k}S(j,k)(-1)^{k+1}(k+1)! \tag{6.12}$$

Replace p by $2p$ in (6.12) to obtain (6.5). Replace p by $2p-1$ in (6.12) to obtain (6.6).
From (5.6) with $q=2$ we obtain

$$3E_{p}-4E_{p+1}+E_{p+2}=\sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}5^{p-j}2^{j-k}S(j,k)(-1)^{k}(k+2)! \tag{6.13}$$

Replace p by $2p-1$ in (6.13) to obtain (6.7). □

Proposition 6.2. *We have the following formulas for the sum of two consecutive even Euler numbers*

$$E_{2p+2}+E_{2p}=\sum_{k=0}^{2p}\sum_{j=0}^{2p}\binom{2p}{j}2^{j-k}S(j,k)(-1)^{k}(k+1)!k \tag{6.14}$$

$$=\frac{1}{6}\sum_{k=0}^{2p}\sum_{j=0}^{2p}\binom{2p}{j}2^{j-k}S(j,k)(-1)^{k}(k+2)!k \tag{6.15}$$

$$=\sum_{k=0}^{2p}\sum_{j=0}^{2p}\binom{2p}{j}2^{j-k-2}S(j,k)(-1)^{k}(k+1)!k^{2}. \tag{6.16}$$

Proof. Replace p by $2p$ in (6.9) to obtain

$$3E_{2p}+E_{2p+2}=\sum_{k=0}^{2p}\sum_{j=0}^{2p}\binom{2p}{j}2^{j-k}S(j,k)(-1)^{k}(k+1)!(k+2). \tag{6.17}$$

From (6.17) and (6.1) we obtain (6.14). From the case $q=3$ of (5.3) we see that

$$15E_{p}+23E_{p+1}+9E_{p+2}+E_{p+3}=\sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}2^{j-k}S(j,k)(-1)^{k}(k+3)! \tag{6.18}$$

Replace p by $2p$ in (6.18), and use (6.17), to get

$$\begin{aligned}15E_{2p}+9E_{2p+2} &= \sum_{k=0}^{2p}\sum_{j=0}^{2p}\binom{2p}{j}2^{j-k}S(j,k)(-1)^k(k+3)! \\ &= \sum_{k=0}^{2p}\sum_{j=0}^{2p}\binom{2p}{j}2^{j-k}S(j,k)(-1)^k(k+2)!(k+3) \\ &= \sum_{k=0}^{2p}\sum_{j=0}^{2p}\binom{2p}{j}2^{j-k}S(j,k)(-1)^k(k+2)!k+3(3E_{2p}+E_{2p+2}),\end{aligned}$$

from where we obtain (6.15). From (6.14) and (6.15) we obtain (6.16). □

Proposition 6.3. *We have the following formulas for the difference of two consecutive even Euler numbers*

$$E_{2p+2}-E_{2p} = \sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)! \tag{6.19}$$

$$= \frac{1}{5}\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!k \tag{6.20}$$

$$= \frac{1}{8}\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+3)! \tag{6.21}$$

Proof. From (6.17) and (6.4) we get

$$3E_{2p}+E_{2p+2} = \sum_{k=0}^{2p}\sum_{j=0}^{2p}\left(\binom{2p-1}{j}+\binom{2p-1}{j-1}\right)2^{j-k}S(j,k)(-1)^k(k+2)! \tag{6.22}$$

$$= 4E_{2p}+\sum_{k=0}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!,$$

from where we obtain (6.19).

Replace p by $2p-1$ in (6.18) and use (6.4) to get

$$23E_{2p}+E_{2p+2} = \sum_{k=0}^{2p-1}\sum_{j=0}^{2p-1}\binom{2p-1}{j}2^{j-k}S(j,k)(-1)^k(k+2)!k+12E_{2p}.$$

That is, we have

$$11E_{2p}+E_{2p+2} = \sum_{k=0}^{2p-1}\sum_{j=0}^{2p-1}\binom{2p-1}{j}2^{j-k}S(j,k)(-1)^k(k+2)!k. \tag{6.23}$$

From (6.15) and (6.23) we get

$$6(E_{2p+2}+E_{2p}) = \sum_{k=0}^{2p}\sum_{j=0}^{2p}\left(\binom{2p-1}{j}+\binom{2p-1}{j-1}\right)2^{j-k}S(j,k)(-1)^k(k+2)!k$$
$$= 11E_{2p}+E_{2p+2}+\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!k,$$

from where (6.20) follows.

From (6.19) and (6.20) we have

$$E_{2p+2}-E_{2p} = \frac{1}{5}\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!(k+3-3)$$
$$= \frac{1}{5}\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+3)!-\frac{3}{5}(E_{2p+2}-E_{2p}),$$

from where (6.21) follows. □

Proposition 6.4. *We have the following family of formulas for the difference of Euler numbers $E_{2p+4}-E_{2p}$*

$$E_{2p+4}-E_{2p} = \sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!(k^2+\beta k-5\beta-23), \quad (6.24)$$

where β is an arbitrary constant.

Proof. From the case $q=4$ of (5.3) we see that

$$105E_p+176E_{p+1}+86E_{p+2}+16E_{p+3}+E_{p+4} = \sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}2^{j-k}S(j,k)(-1)^k(k+4)! \quad (6.25)$$

Replace p by $2p-1$ in (6.25) to get

$$176E_{2p}+16E_{2p+2} = \sum_{j=0}^{2p-1}\sum_{k=0}^{2p-1}\binom{2p-1}{j}2^{j-k}S(j,k)(-1)^k(k+4)! \quad (6.26)$$

Replace p by $2p$ in (6.25) and use (6.26) to get

$$\begin{aligned}
&105E_{2p}+86E_{2p+2}+E_{2p+4}\\
&=\sum_{j=0}^{2p}\sum_{k=0}^{2p}\binom{2p}{j}2^{j-k}S(j,k)(-1)^k(k+4)!\\
&=\sum_{j=0}^{2p-1}\sum_{k=0}^{2p-1}\binom{2p-1}{j}2^{j-k}S(j,k)(-1)^k(k+4)!\\
&\quad+\sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+4)!\\
&=176E_{2p}+16E_{2p+2}+\sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+4)!
\end{aligned}$$

That is, we have

$$-71E_{2p}+70E_{2p+2}+E_{2p+4}=\sum_{j=1}^{2p-1}\sum_{k=1}^{2p-1}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+4)! \tag{6.27}$$

or

$$E_{2p+4}-E_{2p}=\sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+4)!-70(E_{2p+2}-E_{2p}). \tag{6.28}$$

By using (6.19), we obtain from (6.28) that

$$\begin{aligned}
&E_{2p+4}-E_{2p}\\
&=\sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+4)! \qquad (6.29)\\
&-70\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!\\
&=\sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!(k^2+7k-58).
\end{aligned}$$

Finally, observe that (6.19) and (6.20) imply that

$$\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!(k-5)=0.$$

Thus, for any constant α we have

$$\begin{aligned}E_{2p+4}-E_{2p} &= \sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!(k^2+7k-58)\\ &\quad+\alpha\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!(k-5)\\ &=\sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2)!(k^2+(7+\alpha)k-58-5\alpha).\end{aligned} \tag{6.30}$$

Set $\beta=7+\alpha$ in (6.30) to obtain the desired conclusion (6.24). □

Let us consider now the case $q=6$ of (5.3), namely

$$\begin{aligned}10395E_p+19524E_{p+1}+12139E_{p+2}+3480E_{p+3}+505E_{p+4}+36E_{p+5}+E_{p+6}\\ =\sum_{k=0}^{p}\sum_{j=0}^{p}\binom{p}{j}2^{j-k}S(j,k)(-1)^k(k+6)!\end{aligned} \tag{6.31}$$

and proceed as we did to obtain (6.27). Replace p by $2p-1$ in (6.31) to get

$$19524E_{2p}+3480E_{2p+2}+36E_{2p+4}=\sum_{k=0}^{2p-1}\sum_{j=0}^{2p-1}\binom{2p-1}{j}2^{j-k}S(j,k)(-1)^k(k+6)! \tag{6.32}$$

Replace p by $2p$ in (6.31) to get

$$10395E_{2p}+12139E_{2p+2}+505E_{2p+4}+E_{2p+6}=\sum_{k=0}^{2p}\sum_{j=0}^{2p}\binom{2p}{j}2^{j-k}S(j,k)(-1)^k(k+6)! \tag{6.33}$$

Beginning with (6.33), and then using (6.32) we have

$$\begin{aligned}&10395E_{2p}+12139E_{2p+2}+505E_{2p+4}+E_{2p+6}\\ &=\sum_{k=0}^{2p}\sum_{j=0}^{2p}\left(\binom{2p-1}{j}+\binom{2p-1}{j-1}\right)2^{j-k}S(j,k)(-1)^k(k+6)!\\ &=19524E_{2p}+3480E_{2p+2}+36E_{2p+4}\\ &\quad+\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+6)!\end{aligned}$$

That is, we have the formula

$$
\begin{aligned}
&-(8659+469+1)E_{2p}+8659E_{2p+2}+469E_{2p+4}+E_{2p+6} \\
&=\sum_{k=1}^{2p}\sum_{j=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+6)!
\end{aligned}
\tag{6.34}
$$

Formulas (6.19), (6.27) and (6.35), suggest that for any $m\in\mathbb{N}$, there exist $a_1,a_2,\ldots,a_m\in\mathbb{N}$, with $a_m=1$, such that

$$
\begin{aligned}
&-\left(\sum_{j=1}^{m}a_j\right)E_{2p}+a_1E_{2p+2}+a_2E_{2p+4}+\cdots+a_mE_{2p+2m} \\
&=\sum_{j=1}^{2p}\sum_{k=1}^{2p}\binom{2p-1}{j-1}2^{j-k}S(j,k)(-1)^k(k+2m)!
\end{aligned}
\tag{6.35}
$$

We leave this final comment as a conjecture.

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