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Strong convergence theorems by Martinez-Yanes–Xu projection method for mean-demiclosed mappings in Hilbert spaces

Atsumasa Kondo

Abstract. *Strong convergence theorems that approximate common fixed points of two nonlinear mappings are presented. Our method is based on the Martinez-Yanes–Xu iteration, which extends Nakajo and Takahashi's CQ method. In this paper, by exploiting the mean-valued iteration procedure, we further develop Nakajo and Takahashi's CQ method and Takahashi, Takeuchi, and Kubota's shrinking projection method. The approach of this paper does not require that the two mappings be continuous or commutative. The types of mappings considered in this paper include nonexpansive mappings and other well-known classes of mappings as special cases.*

1. Introduction

Let H be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\|\cdot\|$. A set that collects all fixed points of a mapping $T\colon C \to H$ is denoted by

$$F(T) = \{x \in C : Tx = x\},$$

where C is a nonempty subset of H. A mapping $T\colon C \to H$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. Many researchers have studied approximation methods for finding common fixed points of nonexpansive mappings. For two nonexpansive mappings $S, T\colon C \to C$, Atsushiba and Takahashi in [5] introduced the following iteration:

$$x_{n+1} = a_n x_n + (1 - a_n) \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n \tag{1.1}$$

for all $n \in \mathbb{N}$, where $a_n \in [0, 1]$ with certain conditions. Assuming $ST = TS$, they proved a weak convergence to a common fixed point of S and T. The idea of a mean-valued iteration procedure such as (1.1) has its roots in Baillon [6] and Shimizu and Takahashi [35]. More on mean-valued iteration can be found in [1, 11, 27]; see also papers cited in Kondo [21].

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In 2003, Nakajo and Takahashi [31] established a strong convergence theorem to find a fixed point of a nonexpansive mapping:

Theorem 1.1 ([31]). *Let C be a nonempty, closed, and convex subset of H and let $T: C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $a \in [0,1)$ and let $\{a_n\}$ be a sequence of real numbers such that $0 \leq a_n \leq a < 1$ for all $n \in \mathbb{N}$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given}, \\ y_n &= a_n x_n + (1 - a_n) T x_n \in C, \\ C_n &= \{h \in C : \|y_n - h\| \leq \|x_n - h\|\}, \\ Q_n &= \{h \in C : \langle x - x_n,\ x_n - h\rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{1.2}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(T)$, where $\widehat{x} = P_{F(T)} x$.

In Theorem 1.1, $P_{C_n \cap Q_n}$ and $P_{F(T)}$ are the metric projections from H onto $C_n \cap Q_n$ and $F(T)$, respectively. This iteration procedure is often called the "CQ method." In 2006, employing the idea of the Ishikawa iteration [14], Martinez-Yanes and Xu [29] proved the following theorem.

Theorem 1.2 ([29]). *Let C be a nonempty, closed, and convex subset of H and let $T: C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{a_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n \to 1$ and $0 \leq a_n \leq \delta < 1$ for some $\delta \in [0,1)$. Define a sequence $\{x_n\}$ in C as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given}, \\ z_n &= \lambda_n x_n + (1 - \lambda_n) T x_n, \\ y_n &= a_n x_n + (1 - a_n) T z_n, \\ C_n &= \{h \in C : \|y_n - h\|^2 \leq \|x_n - h\|^2 \\ &\qquad - (1 - a_n)\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ h\rangle\right)\}, \\ Q_n &= \{h \in C : \langle x - x_n,\ x_n - h\rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned} \tag{1.3}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(T)$, where $\widehat{x} = P_{F(T)} x$.

If $\lambda_n = 1$ for all $n \in \mathbb{N}$, then $z_n = x_n$. In this case, Theorem 1.2 coincides with Theorem 1.1. Thus, Theorem 1.2 is an extension of Theorem 1.1. The Martinez-Yanes–Xu iteration procedure has been studied by many researchers; see [1, 32, 36] for examples. For various convergence results using the Ishikawa iteration, see [1, 7, 21, 42, 43, 44]. In 2008, Takahashi, Takeuchi, and Kubota [38] introduced the following iteration:

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ C_1 &= C, \\ y_n &= a_n x_n + (1 - a_n) T x_n \in C, \\ C_{n+1} &= \{h \in C_n : \|y_n - h\| \le \|x_n - h\|\}, \text{ and} \\ x_{n+1} &= P_{C_{n+1}} x. \end{aligned} \tag{1.4}$$

This method is called the "shrinking projection method," most likely because the sequence of sets $\{C_n\}$ are shrinking, that is, $C_n \subset C_{n-1} \subset \cdots$. They showed the strong convergence of the sequence $\{x_n\}$ to a fixed point $\widehat{x} = P_{F(T)} x$ of T.

In 2019, Kondo and Takahashi [24] considered the following iteration: given $x_1 \in C$,

$$x_{n+1} = a_n x_n + b_n S x_n + c_n S^2 x_n + d_n T x_n + e_n T^2 x_n, \tag{1.5}$$

where $a_n, b_n, c_n, d_n, e_n \in [0,1]$ are such that $a_n + b_n + c_n + d_n + e_n = 1$. In (1.5), $S, T\colon C \to C$ are a more general type of nonlinear mappings than nonexpansive mappings, and $ST = TS$ was not assumed. Kondo and Takahashi proved a weak convergence theorem that approximates a common fixed point of S and T. Following (1.5) and the mean-valued iteration (1.1), Kondo and Takahashi [26] introduced iteration of the form

$$x_{n+1} = a_n x_n + b_n \frac{1}{n} \sum_{k=0}^{n-1} S^k x_n + c_n \frac{1}{n} \sum_{k=0}^{n-1} T^k x_n, \tag{1.6}$$

and proved a weak convergence theorem that approximates a common fixed point, where again S and T are not necessarily commutative. They also proved Halpern type strong convergence theorems in another paper [25]. Very recently, Kondo [19] combined iteration (1.6) with Nakajo and Takahashi's CQ method (1.2) and Takahashi, Takeuchi, and Kubota's shrinking projection method (1.4) to obtain strong convergence results.

In this paper, we develop the Martinez-Yanes–Xu's iteration method (1.3) by using the mean-valued iteration (1.6). Required conditions imposed on the mappings are relaxed, in other words, the types of mappings considered in this paper are more general than nonexpansive mappings. Common fixed points of two nonlinear mappings are approximated. Our approach does not require that the mappings to be continuous or commutative. After introducing basic information in Section 2, Nakajo and Takahashi's CQ method is developed in Section 3. In Section 4, a theorem of Takahashi, Takeuchi, and Kubota's type is presented. In Section 5, we briefly conclude the paper.

2. Preliminaries

In this section, we collect known definitions and results. Let H be a real Hilbert space. Maruyama et al. [30] proved that

$$\begin{aligned} &\|ax + by + cz\|^2 \\ &= a\|x\|^2 + b\|y\|^2 + c\|z\|^2 - ab\|x-y\|^2 - bc\|y-z\|^2 - ca\|z-x\|^2, \end{aligned} \tag{2.1}$$

for all $x, y, z \in H$ and $a, b, c \in \mathbb{R}$ such that $a + b + c = 1$. Let $x, y, z \in H$, let $d \in \mathbb{R}$, and let C be a nonempty, closed, and convex subset of H. According to Martinez-Yanes and Xu [29], a set defined by

$$D = \left\{ h \in C : \|y - h\|^2 \le \|x - h\|^2 + \langle z,\ h\rangle + d \right\} \tag{2.2}$$

is closed and convex.

Let Q be a nonempty, closed, and convex subset of H. In this paper, we use P_Q to denote a *metric projection* from H onto Q, which means that $\|x - P_Q x\| = \inf_{h\in Q}\|x - h\|$ for any $x \in H$. Metric projections are well-known to be nonexpansive and to satisfy the following inequalities:

$$\langle x - P_Q x,\ P_Q x - h\rangle \ge 0 \ \text{ and} \tag{2.3}$$

$$\|x - P_Q x\|^2 + \|P_Q x - h\|^2 \le \|x - h\|^2 \tag{2.4}$$

for all $x \in H$ and $h \in Q$.

Next, we introduce various types of nonlinear mappings addressed in this paper. A mapping $S\colon C \to H$ with $F(S) \ne \emptyset$ is called *quasi-nonexpansive* if

$$\|Sx - q\| \le \|x - q\|$$

for all $x \in C$ and $q \in F(S)$, where C is a nonempty subset of H. According to Itoh and Takahashi [15], the set of all fixed points of a quasi-nonexpansive mapping is closed and convex. In main theorems of this paper, we require mappings to be quasi-nonexpansive.

Although a nonexpansive mapping with a fixed point is quasi-nonexpansive, the class of quasi-nonexpansive mappings includes various types of nonlinear mappings beyond nonexpansive mappings under the condition that a mapping has a fixed point. A mapping $S\colon C \to H$ is called *generalized hybrid* [16], if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Sx - Sy\|^2 + (1-\alpha)\|x - Sy\|^2 \le \beta\|Sx - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.5}$$

for all $x, y \in C$. A mapping $S\colon C \to H$ is also called an (α, β)-*generalized hybrid mapping*. A $(1, 0)$-generalized hybrid mapping is nonexpansive. The class of generalized hybrid mappings contains various types of mappings other than nonexpansive mappings. A $(2, 1)$-generalized hybrid mapping is a *nonspreading mapping*, while a $\left(\frac{3}{2}, \frac{1}{2}\right)$-generalized hybrid mapping is a *hybrid mapping* in the senses of Kohsaka and Takahashi [17] and Takahashi [37], respectively. For these points, see also Takahashi and Yao [41]. The nonspreading mappings are deduced from optimization problems. It is known that a nonspreading mapping is not necessarily continuous; see Igarashi et al. [13] and Example 2.3 in this section. It is also known that λ-*hybrid mappings* [2] are also generalized hybrid. Because generalized hybrid mappings with fixed points are quasi-nonexpansive, nonspreading mappings, hybrid mappings, λ-hybrid mappings are all quasi-nonexpansive if they have fixed points. Kocourek et al. [16] proved that a generalized hybrid mapping that has a fixed point is quasi-nonexpansive and established a fixed point theorem and weak convergence theorems for finding its fixed points.

The class of quasi-nonexpansive mappings includes more general types of nonlinear mappings than generalized hybrid mappings. A mapping $S\colon C \to C$ is called 2-*generalized hybrid* [30] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1 \left\|S^2x - Sy\right\|^2 + \alpha_2 \left\|Sx - Sy\right\|^2 + (1-\alpha_1-\alpha_2)\left\|x - Sy\right\|^2 \\ &\leq \beta_1 \left\|S^2x - y\right\|^2 + \beta_2 \left\|Sx - y\right\|^2 + (1-\beta_1-\beta_2)\left\|x - y\right\|^2 \end{aligned} \tag{2.6}$$

for all $x, y \in C$. The class of 2-generalized hybrid mappings contains generalized hybrid mappings as the special case $\alpha_1 = \beta_1 = 0$. A mapping $S\colon C \to C$ is called *normally* 2-*generalized hybrid* [22] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$, $\alpha_2 + \alpha_1 + \alpha_0 > 0$, and

$$\begin{aligned} &\alpha_2 \left\|S^2x - Sy\right\|^2 + \alpha_1 \left\|Sx - Sy\right\|^2 + \alpha_0 \left\|x - Sy\right\|^2 \\ &\qquad + \beta_2 \left\|S^2x - y\right\|^2 + \beta_1 \left\|Sx - y\right\|^2 + \beta_0 \left\|x - y\right\|^2 \leq 0 \end{aligned} \tag{2.7}$$

for all $x, y \in C$. The class of normally 2-generalized hybrid mappings contains 2-generalized hybrid mappings as the case with $\alpha_2 + \alpha_1 + \alpha_0 = 1$ and $\beta_2 + \beta_1 + \beta_0 = -1$. Therefore, this class of mappings also includes nonexpansive mappings, nonspreading mappings, hybrid mappings, and generalized hybrid mappings as special cases. It also includes *normally generalized hybrid mappings* [40] as a special case. It is easy to show that if $\sum_{n=0}^{2}(\alpha_n + \beta_n) > 0$, this class of mapping has at most one fixed point. Examples of these classes of mappings are provided in Hojo et al. [12] and Kondo [18]; see also Example 2.4 in this section. For results concerning 2-generalized hybrid mappings and normally 2-generalized hybrid mappings, see recent works by [1, 3, 4, 33, 34]. According to Kondo and Takahashi [22], a normally 2-generalized hybrid mapping with a fixed point is quasi-nonexpansive. A proof is also given in Kondo [21].

Lemma 2.1 ([22])**.** *Let $S\colon C \to C$ be a normally* 2*-generalized hybrid mapping with $F(S) \neq \emptyset$, where C is a nonempty subset of H. Then, S is quasi-nonexpansive.*

Furthermore, a normally 2-generalized hybrid mapping has the following property, which was shown by Kondo and Takahashi [23]. Alternative proofs were provided by Kondo [19, 21].

Lemma 2.2 ([23]). *Let $S\colon C \to C$ be a normally 2-generalized hybrid mapping with $F(S) \neq \emptyset$, where C is a nonempty, closed, and convex subset of H. For a bounded sequence $\{z_n\}$ in C, define $Z_n \equiv \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n \, (\in C)$ for each $n \in \mathbb{N}$. Suppose that $Z_{n_j} \rightharpoonup v \in H$, where $\{Z_{n_j}\}$ is a subsequence of $\{Z_n\}$. Then, $v \in F(S)$.*

Kondo [19] called a mapping $S\colon C \to C$ *mean-demiclosed* when Lemma 2.2 holds for the mapping, namely,

$$Z_{n_j} \rightharpoonup v \Longrightarrow v \in F(S) \tag{2.8}$$

under the assumptions of Lemma 2.2.

From Lemmas 2.1 and 2.2, a normally 2-generalized hybrid mapping that has a fixed point is quasi-nonexpansive and mean-demiclosed. The mappings on which we shed light in this paper are of this type. Such mappings include nonexpansive mappings, generalized hybrid mappings, and 2-generalized hybrid mappings as special cases because those classes of mappings are special cases of normally 2-generalized hybrid mappings.

The class of quasi-nonexpansive and mean-demiclosed mappings are not necessarily continuous. We present two examples here. Substituting $\alpha = 2$ and $\beta = 1$ in (2.5), we have

$$2\left\|Sx - Sy\right\|^2 \leq \left\|x - Sy\right\|^2 + \left\|Sx - y\right\|^2, \tag{2.9}$$

where $x, y \in C$. A mapping that satisfies (2.9) is a $(2,1)$-generalized hybrid mapping and hence, it is nonspreading.

Example 2.3. Let $H = C = \mathbb{R}$ and define a mapping $S\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$Sx = \begin{cases} 1 & \text{if } x > A, \\ 0 & \text{if } x \leq A, \end{cases} \tag{2.10}$$

where $A \geq 1$. Then, S is nonspreading if and only if $A \geq \sqrt{2}$. Indeed, let $A \geq \sqrt{2}$. If $x, y \leq A$ or $x, y > A$, the condition (2.9) holds true. Without loss of generality, assume that $x \leq A < y$. Then, $Sx = 0$, $Sy = 1$, and $S^2x = S^2y = 0$. Therefore, LHS of (2.9)$= 2$. On the other hand,

$$\begin{aligned} RHS &= \left\|x - Sy\right\|^2 + \left\|Sx - y\right\|^2 \\ &= \left\|x - 1\right\|^2 + \left\|y\right\|^2 \geq y^2 > A^2 \geq 2, \end{aligned}$$

which implies that the condition (2.9) holds. Conversely, if $(1 \leq)\, A < \sqrt{2}$, letting $x = 1$ and $A < y < \sqrt{2}$ breaks the condition (2.9). From the above, only in the case of $A \geq \sqrt{2}$, the mapping S defined by (2.10) is nonspreading and it is generalized hybrid. As generalized hybrid mappings belong to the class of normally 2-generalized hybrid mappings, from Lemmas 2.1 and 2.2, S is quasi-nonexpansive and mean-demiclosed because it has a fixed point $0 \in \mathbb{R}$.

Next, letting $\alpha_1 = \beta_1 = 1$ and $\alpha_2 = \beta_2 = 0$ in (2.6), we have

$$\left\|S^2x - Sy\right\| \leq \left\|S^2x - y\right\|, \tag{2.11}$$

where $x, y \in C$. The mapping that satisfies the condition (2.11) is 2-generalized hybrid and therefore, it is normally 2-generalized hybrid.

Example 2.4. Consider the mapping $S\colon \mathbb{R} \to \mathbb{R}$ defined by (2.10) with $A = 1$. Note that S is not nonspreading because $A < \sqrt{2}$. We show that the mapping S satisfies the condition (2.11). As $S^2x = 0$ for all $x \in \mathbb{R}$, our aim is to show $\|Sy\| \leq \|y\|$. If $y \leq A = 1$, then $LHS = \|Sy\| = 0$, and therefore, (2.11) holds true. If $y > A = 1$, then $LHS = \|Sy\| = 1 < y = RHS$. Thus, the condition (2.11) is met. This demonstrates that S is a 2-generalized hybrid mapping and therefore, it is normally 2-generalized hybrid. As the mapping S with $A = 1$ has a fixed point $0 \in \mathbb{R}$, it is quasi-nonexpansive and mean-demiclosed from Lemmas 2.1 and 2.2. This example, together with Example 2.3, illustrates that the class of mappings that are quasi-nonexpansive and mean-demiclosed contains mappings that are not continuous even in one dimensional real space $\mathbb{R}$.

In the following two sections, we assume that two given quasi-nonexpansive and mean-demiclosed mappings have a common fixed point. A set of sufficient conditions for this assumption is provided by the next theorem.

Theorem 2.5 ([8]; see also [10]). *Let C be a nonempty, closed, and convex subset of H and let $S, T\colon C \to C$ be normally 2-generalized hybrid mappings such that $ST = TS$. Assume that there exists an element $x \in C$ such that $\{S^kT^lx : k, l \in \mathbb{N} \cup \{0\}\}$ is bounded. Then, $F(S) \cap F(T)$ is nonempty.*

3. CQ method

In this section, we present a Martinez-Yanes–Xu type convergence theorem, which is based on the Nakajo–Takahashi's CQ method and the Ishikawa iteration. Its purpose is to find a common fixed point of two quasi-nonexpansive and mean-demiclosed mappings. That class of mappings contains nonexpansive mappings and other more general classes of nonlinear mappings as special cases; see the previous section and a remark after Corollary 3.4. One of the highlights is that the two mappings are not necessarily commutative. The basic elements of the proof were developed in many works [1, 9, 12, 19, 20, 29, 31, 39]. Before proving the main theorem of this section, we prepare the following lemma for convenience.

Lemma 3.1. *Let C be a nonempty and convex subset of H and let $S\colon C \to C$ be a quasi-nonexpansive mapping with $F(S) \neq \emptyset$. Then,*

$$\left\|\frac{1}{n}\sum_{l=m}^{n+m-1} S^l x - q\right\| \leq \|x - q\|$$

for all $x \in C$ and $q \in F(S)$, where $m \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N}$.

Proof. Let $x \in C$ and $q \in F(S)$. As S is quasi-nonexpansive, we have

$$\begin{aligned}\left\|\frac{1}{n}\sum_{l=m}^{n+m-1} S^l x - q\right\| &= \frac{1}{n}\left\|\sum_{l=m}^{n+m-1} S^l x - nq\right\| = \frac{1}{n}\left\|\sum_{l=m}^{n+m-1}\left(S^l x - q\right)\right\| \\ &\leq \frac{1}{n}\sum_{l=m}^{n+m-1}\left\|S^l x - q\right\| \leq \frac{1}{n}\sum_{l=m}^{n+m-1}\|x-q\| = \|x-q\|.\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and T be quasi-nonexpansive and mean-demiclosed mappings from C into itself such that $F(S)\cap F(T)\neq\emptyset$. Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, and $\{\theta_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n+\mu_n+\nu_n+\xi_n+\theta_n=1$ for all $n\in\mathbb{N}$ and $\lambda_n\to 1$. Let $\{\lambda_n'\}$, $\{\mu_n'\}$, $\{\nu_n'\}$, $\{\xi_n'\}$, and $\{\theta_n'\}$ be sequences of real numbers in $[0,1]$ such that $\lambda_n'+\mu_n'+\nu_n'+\xi_n'+\theta_n'=1$ for all $n\in\mathbb{N}$ and $\lambda_n'\to 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0,1]$ such that $a_n+b_n+c_n=1$ for all $n\in\mathbb{N}$,*

$$\underline{\lim}_{n\to\infty} a_n b_n > 0, \quad \text{and} \quad \underline{\lim}_{n\to\infty} a_n c_n > 0. \tag{3.1}$$

Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n}\sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n}\sum_{l=0}^{n-1} T^l x_n, \qquad (3.2) \\ w_n &= \lambda_n' x_n + \mu_n' S x_n + \nu_n' T x_n + \xi_n' \frac{1}{n}\sum_{l=0}^{n-1} S^l x_n + \theta_n' \frac{1}{n}\sum_{l=0}^{n-1} T^l x_n, \\ y_n &= a_n x_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n, \\ C_n &= \{h \in C : \|y_n - h\|^2 \leq \|x_n - h\|^2 \\ &\quad -b_n\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ h\rangle\right) \\ &\quad -c_n\left(\|x_n\|^2 - \|w_n\|^2 - 2\langle x_n - w_n,\ h\rangle\right)\}, \\ Q_n &= \{h \in C : \langle x - x_n,\ x_n - h\rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned}$$

for all $n\in\mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{x}$ of $F(S)\cap F(T)$, where $\widehat{x}=P_{F(S)\cap F(T)}x$.

Proof. Assume that $x_n \in C$ is given momentarily. As S and T are quasi-nonexpansive mappings, from Lemma 3.1, the following holds:

$$\left\| \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n - q \right\| \leq \| x_n - q \| \quad \text{and} \quad \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| \leq \| x_n - q \| \tag{3.3}$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$. Using these inequalities, we can prove that

$$\| z_n - q \| \leq \| x_n - q \| \quad \text{and} \quad \| w_n - q \| \leq \| x_n - q \| \tag{3.4}$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$. Indeed, using S and T are quasi-nonexpansive yields

$$\begin{aligned}
&\| z_n - q \| \\
&= \left\| \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| \\
&= || \lambda_n (x_n - q) + \mu_n (S x_n - q) + \nu_n (T x_n - q) \\
&\quad + \xi_n \left(\frac{1}{n} \sum_{l=0}^{n-1} S^l x_n - q \right) + \theta_n \left(\frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right) || \\
&\leq \lambda_n \| x_n - q \| + \mu_n \| S x_n - q \| + \nu_n \| T x_n - q \| \\
&\quad + \xi_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n - q \right\| + \theta_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| \\
&\leq \lambda_n \| x_n - q \| + \mu_n \| x_n - q \| + \nu_n \| x_n - q \| + \xi_n \| x_n - q \| + \theta_n \| x_n - q \| \\
&= \| x_n - q \|
\end{aligned}$$

as claimed. The second part $\| w_n - q \| \leq \| x_n - q \|$ of (3.4) can be demonstrated in a similar way. Define

$$Z_n = \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n \quad \text{and} \quad W_n = \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n.$$

From the convexity of C, $\{Z_n\}$ and $\{W_n\}$ are sequences in C. Using these notations, we can simply write $y_n = a_n x_n + b_n Z_n + c_n W_n$. From Lemma 3.1, it holds that

$$\| Z_n - q \| \leq \| z_n - q \| \quad \text{and} \quad \| W_n - q \| \leq \| w_n - q \| \tag{3.5}$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$. We must check that the sequence $\{x_n\}$ is properly defined. It is obvious that Q_n is closed and convex in C for all $n \in \mathbb{N}$. Also, C_n is closed and convex in C for all $n \in \mathbb{N}$ once x_n, y_n, z_n, and $w_n \in C$ are given. Indeed, an easy calculation reveals that

$$\begin{aligned}
\| y_n - h \|^2 \leq \| x_n - h \|^2 &- b_n \left(\| x_n \|^2 - \| z_n \|^2 - 2 \langle x_n - z_n ,\ h \rangle \right) \\
&- c_n \left(\| x_n \|^2 - \| w_n \|^2 - 2 \langle x_n - w_n ,\ h \rangle \right)
\end{aligned}$$

$$\iff \|y_n - h\|^2 \leq \|x_n - h\|^2 + 2\langle b_n(x_n - z_n) + c_n(x_n - w_n), \ h\rangle \\ - b_n\left(\|x_n\|^2 - \|z_n\|^2\right) - c_n\left(\|x_n\|^2 - \|w_n\|^2\right).$$

From (2.2), this implies that C_n is closed and convex in C.

Using mathematical induction, we show that

$$F(S) \cap F(T) \subset C_n \cap Q_n \quad \text{for all } n \in \mathbb{N}.$$

(i) As $Q_1 = C$, clearly $F(S) \cap F(T) \subset Q_1$. Let $q \in F(S) \cap F(T)$. Using (3.5) and the hypothesis $a_1 + b_1 + c_1 = 1$, we have the following:

$$\begin{aligned} \|y_1 - q\|^2 &= \|a_1 x_1 + b_1 Z_1 + c_1 W_1 - q\|^2 \\ &= \|a_1(x_1 - q) + b_1(Z_1 - q) + c_1(W_1 - q)\|^2 \\ &\leq a_1\|x_1 - q\|^2 + b_1\|Z_1 - q\|^2 + c_1\|W_1 - q\|^2 \\ &\leq a_1\|x_1 - q\|^2 + b_1\|z_1 - q\|^2 + c_1\|w_1 - q\|^2 \\ &= \|x_1 - q\|^2 + b_1\left(\|z_1 - q\|^2 - \|x_1 - q\|^2\right) \\ &\quad + c_1\left(\|w_1 - q\|^2 - \|x_1 - q\|^2\right) \\ &= \|x_1 - q\|^2 - b_1\left(\|x_1\|^2 - \|z_1\|^2 - 2\langle x_1 - z_1, \ q\rangle\right) \\ &\quad - c_1\left(\|x_1\|^2 - \|w_1\|^2 - 2\langle x_1 - w_1, \ q\rangle\right). \end{aligned} \tag{3.6}$$

This shows that $q \in C_1$, and thus, $F(S) \cap F(T) \subset C_1$. (ii) Assume that

$$F(S) \cap F(T) \subset C_k \cap Q_k,$$

where $k \in \mathbb{N}$. From the assumption $F(S) \cap F(T) \neq \emptyset$, $C_k \cap Q_k$ is also nonempty. As $C_k \cap Q_k$ is a nonempty, closed, and convex subset of $C\,(\subset H)$, the metric projection $P_{C_k \cap Q_k}$ from H onto $C_k \cap Q_k$ exists. Consequently, x_{k+1} is defined as $x_{k+1} = P_{C_k \cap Q_k} x$. Furthermore, z_{k+1}, w_{k+1}, Z_{k+1}, W_{k+1}, $y_{k+1}\,(\in C)$, C_{k+1}, and $Q_{k+1}\,(\subset C)$ are properly defined. We now demonstrate that

$$F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}.$$

Let $q \in F(S) \cap F(T)$. We can prove $q \in C_{k+1}$ in a similar way as (3.6), omitting it here. As $x_{k+1} = P_{C_k \cap Q_k} x$ and $q \in F(S) \cap F(T) \subset C_k \cap Q_k$, using (2.3) yields $\langle x - x_{k+1}, \ x_{k+1} - q\rangle \geq 0$. This shows that $q \in Q_{k+1}$. Therefore, it holds true that $F(S) \cap F(T) \subset C_{k+1} \cap Q_{k+1}$ as claimed. We have demonstrated that $F(S) \cap F(T) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. As $F(S) \cap F(T) \neq \emptyset$ is assumed, $C_n \cap Q_n$ is nonempty for all $n \in \mathbb{N}$, and thus, the sequence $\{x_n\}$ is properly defined inductively.

From the definition of Q_n, it holds that $x_n = P_{Q_n} x$ for all $n \in \mathbb{N}$. Consequently, we have the following:

$$\|x - x_n\| \leq \|x - q\| \tag{3.7}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. Indeed, since $x_n = P_{Q_n} x$ and $q \in F(S) \cap F(T) \subset C_n \cap Q_n \subset Q_n$, the inequality (3.7) follows. From (3.7), $\{x_n\}$ is bounded, so from (3.4), $\{z_n\}$ and $\{w_n\}$ are also bounded.

Observe that

$$\|x - x_n\| \leq \|x - x_{n+1}\| \tag{3.8}$$

for all $n \in \mathbb{N}$. Indeed, from $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \in Q_n$, we obtain (3.8), which implies that the sequence $\{\|x - x_n\|\}$ of real numbers is monotone increasing. As $\{x_n\}$ is bounded, $\{\|x - x_n\|\}$ is therefore convergent in $\mathbb{R}$.

We demonstrate that

$$x_n - x_{n+1} \to 0. \tag{3.9}$$

As $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \in Q_n$, using (2.4), we have

$$\|x - x_n\|^2 + \|x_n - x_{n+1}\|^2 \leq \|x - x_{n+1}\|^2 .$$

As $\{\|x - x_n\|\}$ is convergent, we obtain (3.9) as claimed.

Note that $\{Sx_n\}$ is bounded. Indeed, as S is quasi-nonexpansive, the following holds for $q \in F(S)$:

$$\begin{aligned} \|Sx_n\| &\leq \|Sx_n - q\| + \|q\| \\ &\leq \|x_n - q\| + \|q\| . \end{aligned}$$

As $\{x_n\}$ is bounded, $\{Sx_n\}$ is also bounded. Similarly, $\{Tx_n\}$ is bounded because T is quasi-nonexpansive. From (3.3), $\left\{\frac{1}{n}\sum_{l=0}^{n-1} S^l x_n\right\}$ and $\left\{\frac{1}{n}\sum_{l=0}^{n-1} T^l x_n\right\}$ are also bounded because $\{x_n\}$ is bounded. Using these facts, we can prove that

$$z_n - x_n \to 0 \quad \text{and} \quad w_n - x_n \to 0. \tag{3.10}$$

Indeed, as $\lambda_n \to 1$ is assumed, it follows that $\mu_n, \nu_n, \xi_n, \theta_n \to 0$. Therefore,

$$\begin{aligned} &\|z_n - x_n\| \\ &= \left\| \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - x_n \right\| \\ &\leq (1 - \lambda_n) \|x_n\| + \mu_n \|Sx_n\| + \nu_n \|Tx_n\| \\ &\quad + \xi_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n \right\| + \theta_n \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n \right\|. \\ &\to 0 \end{aligned}$$

Similarly, we can obtain $w_n - x_n \to 0$ because $\lambda_n' \to 1$ is assumed. As $\{x_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded, it follows from (3.10) that

$$\|x_n\|^2 - \|z_n\|^2 \to 0 \quad \text{and} \quad \|x_n\|^2 - \|w_n\|^2 \to 0. \tag{3.11}$$

Indeed,

$$\left| \|x_n\|^2 - \|z_n\|^2 \right| = (\|x_n\| + \|z_n\|) \cdot |\|x_n\| - \|z_n\|| \\ \leq (\|x_n\| + \|z_n\|) \|x_n - z_n\| \to 0.$$

Similarly, we can obtain the second part $\|x_n\|^2 - \|w_n\|^2 \to 0$.

Next, observe that

$$y_n - x_{n+1} \to 0. \tag{3.12}$$

Indeed, as $x_{n+1} = P_{C_n \cap Q_n} x \in C_n$, the following holds:

$$\begin{aligned} &\|y_n - x_{n+1}\|^2 \\ &\leq \|x_n - x_{n+1}\|^2 - b_n \left(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n, \ x_{n+1} \rangle \right) \\ &\quad -c_n \left(\|x_n\|^2 - \|w_n\|^2 - 2 \langle x_n - w_n, \ x_{n+1} \rangle \right). \end{aligned}$$

From (3.9)–(3.11), we obtain (3.12). From (3.9) and (3.12), it holds true that

$$x_n - y_n \to 0. \tag{3.13}$$

Our next aim is to show that

$$x_n - Z_n \to 0 \quad \text{and} \quad x_n - W_n \to 0. \tag{3.14}$$

To demonstrate this, choose $q \in F(S) \cap F(T)$ arbitrarily. It follows from (2.1), (3.5), and (3.4) that

$$\begin{aligned} &\|y_n - q\|^2 \\ &= \|a_n (x_n - q) + b_n (Z_n - q) + c_n (W_n - q)\|^2 \\ &= a_n \|x_n - q\|^2 + b_n \|Z_n - q\|^2 + c_n \|W_n - q\|^2 \\ &\quad -a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\ &\leq a_n \|x_n - q\|^2 + b_n \|z_n - q\|^2 + c_n \|w_n - q\|^2 \\ &\quad -a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\ &\leq a_n \|x_n - q\|^2 + b_n \|x_n - q\|^2 + c_n \|x_n - q\|^2 \\ &\quad -a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2 \\ &= \|x_n - q\|^2 \\ &\quad -a_n b_n \|x_n - Z_n\|^2 - b_n c_n \|Z_n - W_n\|^2 - c_n a_n \|W_n - x_n\|^2. \end{aligned}$$

Using $b_n c_n \|Z_n - W_n\|^2 \geq 0$, we have

$$\begin{aligned} &a_n b_n \|x_n - Z_n\|^2 + a_n c_n \|x_n - W_n\|^2 \\ &\leq \|x_n - q\|^2 - \|y_n - q\|^2 \\ &\leq (\|x_n - q\| + \|y_n - q\|) |\|x_n - q\| - \|y_n - q\|| \\ &\leq (\|x_n - q\| + \|y_n - q\|) \|x_n - y_n\|. \end{aligned}$$

As $\{x_n\}$ is bounded, it follows from (3.13) that $\{y_n\}$ is also bounded, so we obtain from (3.13) and the assumption (3.1) on the parameters a_n, b_n, c_n that $x_n - Z_n \to 0$ and $x_n - W_n \to 0$ as claimed.

Our goal is to prove that $x_n \to \widehat{x}\left(= P_{F(S)\cap F(T)}x\right)$. It suffices to show that, for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to \widehat{x}$. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. As $\{x_{n_i}\}$ is bounded, there exist a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ and $v \in H$ such that $x_{n_j} \rightharpoonup v$. From (3.14), it follows that $Z_{n_j} \rightharpoonup v$ and $W_{n_j} \rightharpoonup v$. As S and T are mean-demiclosed (2.8), we obtain $v \in F(S) \cap F(T)$.

We show that $\{x_{n_j}\}$ converges strongly to v. As $v \in F(S)\cap F(T)$, using (3.7), we obtain

$$\begin{aligned}\left\|x_{n_j} - v\right\|^2 &= \left\|x_{n_j} - x\right\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle + \|x - v\|^2 \\ &\le \|x - v\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle + \|x - v\|^2 \\ &= 2\|x - v\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle.\end{aligned}$$

As $x_{n_j} \rightharpoonup v$, it follows that

$$\begin{aligned}\left\|x_{n_j} - v\right\|^2 &\le 2\|x - v\|^2 + 2\left\langle x_{n_j} - x,\ x - v\right\rangle \\ &\to 2\|x - v\|^2 + 2\langle v - x,\ x - v\rangle = 0.\end{aligned}$$

Hence, $x_{n_j} \to v$, as claimed.

Finally, we show that

$$v\left(= \lim_{j\to\infty} x_{n_j}\right) = \widehat{x}\left(= P_{F(S)\cap F(T)}x\right).$$

Because $\widehat{x} = P_{F(S)\cap F(T)}x$ and $v \in F(S)\cap F(T)$, it suffices to demonstrate that $\|x - v\| \le \|x - \widehat{x}\|$. As $\widehat{x} \in F(S)\cap F(T)$, it holds from (3.7) that

$$\left\|x - x_{n_j}\right\| \le \|x - \widehat{x}\|$$

for all $j \in \mathbb{N}$. As $x_{n_j} \to v$, we obtain $\|x - v\| \le \|x - \widehat{x}\|$. Consequently, $v = \widehat{x}$. We have proved that for any subsequence $\{x_{n_i}\}$ of $\{x_n\}$, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_i}\}$ such that $x_{n_j} \to \widehat{x}\,(= v)$. Therefore, $x_n \to \widehat{x}$. This concludes the proof. □

As an illustration, we describe the following corollary, which is directly derived from Theorem 3.2, because Theorem 3.2 seems to be a bit complicated.

Corollary 3.3. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and T be quasi-nonexpansive and mean-demiclosed mappings from C into itself such that $F(S)\cap F(T) \neq \emptyset$. Let $\{\lambda_n\}$, $\{\mu_n\}$, and $\{\theta_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n + \mu_n + \theta_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$,*

$$\varliminf_{n\to\infty} a_n b_n > 0, \quad \textit{and} \quad \varliminf_{n\to\infty} a_n c_n > 0.$$

Define a sequence $\{x_n\}$ *in* C *as follows:*

$$
\begin{aligned}
x_1 &= x \in C \text{ is given,}\\
z_n &= \lambda_n x_n + \mu_n S x_n + \theta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n,\\
y_n &= a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n,\\
C_n &= \{h \in C : \|y_n - h\|^2 \le \|x_n - h\|^2\\
&\qquad - (b_n + c_n)\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ h\rangle\right)\},\\
Q_n &= \{w \in C : \langle x - x_n,\ x_n - w\rangle \ge 0\},\ \text{and}\\
x_{n+1} &= P_{C_n \cap Q_n} x
\end{aligned}
$$

for all $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{x}$ *of* $F(S) \cap F(T)$*, where* $\widehat{x} = P_{F(S)\cap F(T)} x$*.*

Proof. Let $\lambda_n = \lambda'_n$, $\mu_n = \mu'_n$, $\nu_n = \nu'_n$, $\xi_n = \xi'_n$, and $\theta_n = \theta'_n$ in Theorem 3.2. Then, $z_n = w_n$, and hence, the set C_n becomes

$$
\begin{aligned}
C_n = \{h \in C : \|y_n - h\|^2 &\le \|x_n - h\|^2\\
&- (b_n + c_n)\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ h\rangle\right)\}.
\end{aligned}
$$

Furthermore, substituting $\nu_n = \nu'_n = 0$ and $\xi_n = \xi'_n = 0$, we obtain the desired result. □

Some additional remarks related to Theorem 3.2 are given below. First, the required conditions on the parameters are only $\lambda_n \to 1$ and $\lambda'_n \to 1$ other than (3.1) as Martinez-Yanes and Xu [29]. Second, the constructions of z_n and w_n shown in (3.2) can be generalized, for example

$$
\begin{aligned}
z_n = \lambda_n x_n &+ \mu_n S^L x_n + \nu_n T^M x_n\\
&+ \xi_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n^2} \sum_{k=0}^{n-1} \sum_{l=0}^{n-1} S^k T^l x_n,
\end{aligned}
\tag{3.15}
$$

where $L, M \in \mathbb{N} \cup \{0\}$. Third, letting $\lambda_n = \lambda'_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.2 yields $z_n = w_n = x_n$. This special case corresponds to Theorem 3.1 in Kondo [19]:

Corollary 3.4 ([19])**.** *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H*. Let* S *and* T *be quasi-nonexpansive and mean-demiclosed mappings from* C *into itself such that* $F(S) \cap F(T) \neq \emptyset$*. Let* $\{a_n\}$*,* $\{b_n\}$*, and* $\{c_n\}$ *be sequences of real numbers in the interval* $[0, 1]$ *such that* $a_n + b_n + c_n = 1$ *for all* $n \in \mathbb{N}$*,*

$$
\varliminf_{n\to\infty} a_n b_n > 0, \quad \text{and} \quad \varliminf_{n\to\infty} a_n c_n > 0.
$$

Define a sequence $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ y_n &= a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \\ C_n &= \{h \in C : \|y_n - h\| \leq \|x_n - h\|\}, \\ Q_n &= \{h \in C : \langle x - x_n,\ x_n - h \rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned}$$

for all $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{x}$ *of* $F(S) \cap F(T)$*, where* $\widehat{x} = P_{F(S) \cap F(T)} x$*.*

Our fourth remark on Theorem 3.2 is as follows: Theorem 3.2 holds true for normally 2-generalized hybrid mappings (2.7) because such mappings, when they have fixed points, are quasi-nonexpansive and mean-demiclosed from Lemmas 2.1 and 2.2. Because nonexpansive mappings, generalized hybrid mappings, and 2-generalized hybrid mappings are all special cases of normally 2-generalized hybrid mappings, the theorem is effective for those classes of mappings.

As a final remark, Theorem 3.2 is close to the following theorem proved by Alizadeh and Moradlou [1].

Theorem 3.5 ([1])**.** *Let* C *be a nonempty, closed, and convex subset of a real Hilbert space* H*. Let* T *be a 2-generalized hybrid mapping from* C *into itself such that* $F(T) \neq \emptyset$*. Let* $\{\lambda_n\}$ *and* $\{a_n\}$ *be sequences of real numbers in the interval* $[0, 1]$ *such that* $\lambda_n \to 1$ *and* $0 \leq a_n \leq \delta < 1$ *for some* $\delta \in [0, 1)$*. Define a sequence* $\{x_n\}$ *in* C *as follows:*

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ z_n &= \lambda_n x_n + (1 - \lambda_n) T x_n, \\ y_n &= a_n x_n + (1 - a_n) \frac{1}{n} \sum_{l=0}^{n-1} T^l z_n, \\ C_n &= \{h \in C : \|y_n - h\|^2 \leq \|x_n - h\|^2 \\ &\qquad - (1 - a_n) \left(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n,\ h \rangle \right) \}, \\ Q_n &= \{h \in C : \langle x - x_n,\ x_n - h \rangle \geq 0\}, \text{ and} \\ x_{n+1} &= P_{C_n \cap Q_n} x \end{aligned}$$

for all $n \in \mathbb{N}$*. Then,* $\{x_n\}$ *converges strongly to a point* $\widehat{x}$ *of* $F(T)$*, where* $\widehat{x} = P_{F(T)} x$*.*

Note that Alizadeh and Moradlou [1] dealt with a m-generalized hybrid mapping. When $S = I$, $\lambda_n = \lambda'_n$, $\mu_n = \mu'_n = 0$, $\xi_n = \xi'_n = 0$, $\theta_n = \theta'_n = 0$, Theorem 3.2 almost implies Theorem 3.5. There is only one difference that is found in the conditions on the parameters $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$.

4. Shrinking projection method

In this section, we prove a theorem that shows how to construct a sequence that strongly approximates a common fixed point of two nonlinear mappings by employing the Martinez-Yanes–Xu iteration procedure together with the shrinking projection method developed by Takahashi, Takeuchi, and Kubota [38]. The proof has been improved by many researchers; see, for instance, [9, 12, 19, 20, 28, 39].

In proving the main theorem in this section, we can relax a condition required on the mappings as compared to Theorem 3.2. Recall the setting of Lemma 2.2: Let C be a nonempty, closed, and convex subset of H, let $S\colon C \to C$ with $F(S) \neq \emptyset$, and let $\{z_n\}$ be a bounded sequence in C. Define $Z_n = \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n \,(\in C)$. Following [19], consider the following condition:

$$Z_{n_j} \to v \Longrightarrow v \in F(S), \tag{4.1}$$

where $\{Z_{n_j}\}$ is a subsequence of $\{Z_n\}$. Mean-demiclosed mappings satisfy the condition (4.1), and thus, broad classes of mappings, including nonexpansive mappings, generalized hybrid mappings, and normally 2-generalized hybrid mappings, satisfy this condition (4.1). In the main theorem of this section, quasi-nonexpansive mappings with the condition (4.1) are considered.

Theorem 4.1. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and T be quasi-nonexpansive mappings from C into itself that satisfy $F(S) \cap F(T) \neq \emptyset$ and the condition (4.1). Let $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\xi_n\}$, and $\{\theta_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n + \mu_n + \nu_n + \xi_n + \theta_n = 1$ for all $n \in \mathbb{N}$ and $\lambda_n \to 1$. Let $\{\lambda'_n\}$, $\{\mu'_n\}$, $\{\nu'_n\}$, $\{\xi'_n\}$, and $\{\theta'_n\}$ be sequences of real numbers in $[0,1]$ such that $\lambda'_n + \mu'_n + \nu'_n + \xi'_n + \theta'_n = 1$ for all $n \in \mathbb{N}$ and $\lambda'_n \to 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$,*

$$\varliminf_{n\to\infty} a_n b_n > 0, \quad \textit{and} \quad \varliminf_{n\to\infty} a_n c_n > 0. \tag{4.2}$$

Let $\{u_n\}$ be a sequence in H such that $u_n \to u \,(\in H)$. Define a sequence $\{x_n\}$

in C as follows:

$$
\begin{aligned}
x_1 &= x \in C \text{ is given,} \\
C_1 &= C, \\
z_n &= \lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \qquad (4.3) \\
w_n &= \lambda'_n x_n + \mu'_n S x_n + \nu'_n T x_n + \xi'_n \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n + \theta'_n \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n, \\
y_n &= a_n x_n + b_n \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n, \\
C_{n+1} &= \{ h \in C_n : \|y_n - h\|^2 \leq \|x_n - h\|^2 \\
&\quad - b_n \left(\|x_n\|^2 - \|z_n\|^2 - 2 \langle x_n - z_n, \ h \rangle \right) \\
&\quad - c_n \left(\|x_n\|^2 - \|w_n\|^2 - 2 \langle x_n - w_n, \ h \rangle \right) \}, \text{ and} \\
x_{n+1} &= P_{C_{n+1}} u_{n+1}
\end{aligned}
$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S) \cap F(T)} u$.

Proof. For convenience, we use again the notation

$$
Z_n = \frac{1}{n} \sum_{l=0}^{n-1} S^l z_n \quad \text{and} \quad W_n = \frac{1}{n} \sum_{l=0}^{n-1} T^l w_n.
$$

The averaged sequences $\{Z_n\}$ and $\{W_n\}$ are in C because C is convex. We can now simply write $y_n = a_n x_n + b_n Z_n + c_n W_n \in C$. Note that the following hold:

$$
\left\| \frac{1}{n} \sum_{l=0}^{n-1} S^l x_n - q \right\| \leq \|x_n - q\|, \quad \left\| \frac{1}{n} \sum_{l=0}^{n-1} T^l x_n - q \right\| \leq \|x_n - q\|, \qquad (4.4)
$$

$$
\|z_n - q\| \leq \|x_n - q\|, \quad \|w_n - q\| \leq \|x_n - q\|, \qquad (4.5)
$$

$$
\|Z_n - q\| \leq \|z_n - q\|, \quad \text{and} \quad \|W_n - q\| \leq \|w_n - q\| \qquad (4.6)
$$

for all $n \in \mathbb{N}$ and $q \in F(S) \cap F(T)$. The inequalities (4.4) and (4.6) follow from Lemma 3.1, and (4.5) can be demonstrated in a similar way as (3.4).

Next, we use mathematical induction to verify that C_n is a closed convex subset of C and

$$
F(S) \cap F(T) \subset C_n
$$

for all $n \in \mathbb{N}$.

(i) For $n = 1$, the results follow because $C_1 = C$.

(ii) Assume that C_k is closed and convex and

$$F(S) \cap F(T) \subset C_k,$$

where $k \in \mathbb{N}$. As $F(S) \cap F(T) \neq \emptyset$ is assumed, the induction assumption $F(S) \cap F(T) \subset C_k$ implies that $C_k \neq \emptyset$. Consequently, the metric projection P_{C_k} exists, and x_k, z_k, w_k, Z_k, W_k, y_k, and C_{k+1} are defined properly. It follows that C_{k+1} is closed and convex from the induction assumption that C_k is closed and convex and (2.2). We show that

$$F(S) \cap F(T) \subset C_{k+1}.$$

Choose $q \in F(S) \cap F(T)$ arbitrarily. From (4.6), the following holds:

$$\begin{aligned}
&\|y_k - q\|^2 \\
&= \|a_k x_k + b_k Z_k + c_k W_k - q\|^2 \\
&= \|a_k (x_k - q) + b_k (Z_k - q) + c_k (W_k - q)\|^2 \\
&\leq a_k \|x_k - q\|^2 + b_k \|Z_k - q\|^2 + c_k \|W_k - q\|^2 \\
&\leq a_k \|x_k - q\|^2 + b_k \|z_k - q\|^2 + c_k \|w_k - q\|^2 \\
&= \|x_k - q\|^2 + b_k \left(\|z_k - q\|^2 - \|x_k - q\|^2 \right) + c_k \left(\|w_k - q\|^2 - \|x_k - q\|^2 \right) \\
&= \|x_k - q\|^2 - b_k \left(\|x_k\|^2 - \|z_k\|^2 - 2 \langle x_k - z_k, q \rangle \right) \\
&\quad - c_k \left(\|x_k\|^2 - \|w_k\|^2 - 2 \langle x_k - w_k, q \rangle \right).
\end{aligned}$$

This implies that $q \in C_{k+1}$, and it follows that $F(S) \cap F(T) \subset C_{k+1}$ as claimed. We have shown that C_n is a closed and convex subset of C and $F(S) \cap F(T) \subset C_n$ for all $n \in \mathbb{N}$. From the hypothesis $F(S) \cap F(T) \neq \emptyset$, we have $C_n \neq \emptyset$ for all $n \in \mathbb{N}$. Hence, the sequence $\{x_n\}$ is properly defined inductively.

Define $\overline{u}_n = P_{C_n} u \in C_n$. As the sequence $\{C_n\}$ of sets is shrinking, that is, $C_n \subset C_{n-1} \subset \cdots \subset C_1 = C$, $\{\overline{u}_n\}$ is a sequence in C. Observe that

$$\|u - \overline{u}_n\| \leq \|u - q\| \tag{4.7}$$

for all $q \in F(S) \cap F(T)$ and $n \in \mathbb{N}$. This follows from the definition $\overline{u}_n = P_{C_n} u$ and the fact that $q \in F(S) \cap F(T) \subset C_n$, and implies that $\{\overline{u}_n\}$ is bounded. Next, we show that

$$\|u - \overline{u}_n\| \leq \|u - \overline{u}_{n+1}\| \tag{4.8}$$

for all $n \in \mathbb{N}$. Because $\overline{u}_n = P_{C_n} u$ and $\overline{u}_{n+1} = P_{C_{n+1}} u \in C_{n+1} \subset C_n$, the inequality (4.8) follows, which means that $\{\|u - \overline{u}_n\|\}$ is monotone increasing. As $\{\overline{u}_n\}$ is bounded, so is $\{\|u - \overline{u}_n\|\}$. Therefore, $\{\|u - \overline{u}_n\|\}$ is a convergent sequence in $\mathbb{R}$.

We claim that the sequence $\{\overline{u}_n\}$ is convergent in C, namely, there exists $\overline{u} \in C$ such that

$$\overline{u}_n \to \overline{u}. \tag{4.9}$$

To prove this claim, we show that $\{\overline{u}_n\}$ is a Cauchy sequence in C. Let $m, n \in \mathbb{N}$ with $m \geq n$. As $\overline{u}_n = P_{C_n} u$ and $\overline{u}_m = P_{C_m} u \in C_m \subset C_n$, using (2.4), we have

$$\|u - \overline{u}_n\|^2 + \|\overline{u}_n - \overline{u}_m\|^2 \leq \|u - \overline{u}_m\|^2 .$$

As $\{\|u - \overline{u}_n\|\}$ is convergent, it follows that $\overline{u}_n - \overline{u}_m \to 0$ as $m, n \to \infty$, and thus that $\{\overline{u}_n\}$ is indeed a Cauchy sequence in C. As C is closed in H, it is complete. Consequently, there exists $\overline{u} \in C$ such that $\overline{u}_n \to \overline{u}$ as claimed. Next, we prove that $\{x_n\}$ has the same limit point, that is,

$$x_n \to \overline{u}. \tag{4.10}$$

Because the metric projection is nonexpansive and $u_n \to u$ is assumed, (4.9) implies that

$$\begin{aligned} \|x_n - \overline{u}\| &\leq \|x_n - \overline{u}_n\| + \|\overline{u}_n - \overline{u}\| \\ &= \|P_{C_n} u_n - P_{C_n} u\| + \|\overline{u}_n - \overline{u}\| \\ &\leq \|u_n - u\| + \|\overline{u}_n - \overline{u}\| \to 0. \end{aligned}$$

Thus, (4.10) is true as claimed. This implies that $\{x_n\}$ is bounded. From (4.5), $\{z_n\}$ and $\{w_n\}$ are also bounded.

We verify that $\{Sx_n\}$ is bounded. Let $q \in F(S)$. As S is quasi-nonexpansive,

$$\begin{aligned} \|Sx_n\| &\leq \|Sx_n - q\| + \|q\| \\ &\leq \|x_n - q\| + \|q\| . \end{aligned}$$

This shows that $\{Sx_n\}$ is bounded. That $\{Tx_n\}$ is bounded follows in a similar manner. Furthermore, (4.4) implies that $\left\{\frac{1}{n}\sum_{l=0}^{n-1} S^l x_n\right\}$ and $\left\{\frac{1}{n}\sum_{l=0}^{n-1} T^l x_n\right\}$ are also bounded. Using these facts, we show that

$$z_n - x_n \to 0 \quad \text{and} \quad w_n - x_n \to 0. \tag{4.11}$$

As $\lambda_n \to 1$ is assumed, $\mu_n, \nu_n, \xi_n, \theta_n \to 0$. Thus, we can prove the first part of (4.11) as follows:

$$\begin{aligned} &\|z_n - x_n\| \\ &= \left\|\lambda_n x_n + \mu_n S x_n + \nu_n T x_n + \xi_n \frac{1}{n}\sum_{l=0}^{n-1} S^l x_n + \theta_n \frac{1}{n}\sum_{l=0}^{n-1} T^l x_n - x_n\right\| \\ &\leq (1 - \lambda_n)\|x_n\| + \mu_n \|Sx_n\| + \nu_n \|Tx_n\| \\ &\quad + \xi_n \left\|\frac{1}{n}\sum_{l=0}^{n-1} S^l x_n\right\| + \theta_n \left\|\frac{1}{n}\sum_{l=0}^{n-1} T^l x_n\right\| \\ &\to 0. \end{aligned}$$

The second part $w_n - x_n \to 0$ can be demonstrated in a similar way because $\lambda_n' \to 1$ is assumed. As the sequences $\{x_n\}$, $\{z_n\}$, and $\{w_n\}$ are bounded, from (4.11), the following hold:

$$\|x_n\|^2 - \|z_n\|^2 \to 0 \quad \text{and} \quad \|x_n\|^2 - \|w_n\|^2 \to 0. \tag{4.12}$$

This can be verified as follows:

$$\begin{aligned}\left|\|x_n\|^2 - \|z_n\|^2\right| &= (\|x_n\| + \|z_n\|) \cdot |\|x_n\| - \|z_n\|| \\ &\leq (\|x_n\| + \|z_n\|) \|x_n - z_n\| \to 0.\end{aligned}$$

In much the same way, it also follows that $\|x_n\|^2 - \|w_n\|^2 \to 0$.

Next, observe that

$$y_n - x_{n+1} \to 0. \tag{4.13}$$

Indeed, as $x_{n+1} = P_{C_{n+1}} u_{n+1} \in C_{n+1}$, we obtain

$$\begin{aligned}\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 & \\ -b_n & \left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ x_{n+1}\rangle\right) \\ -c_n & \left(\|x_n\|^2 - \|w_n\|^2 - 2\langle x_n - w_n,\ x_{n+1}\rangle\right).\end{aligned}$$

From (4.10), $x_n - x_{n+1} \to 0$. Therefore, (4.11) and (4.12) implies that (4.13) holds true. As $x_n - x_{n+1} \to 0$ and $y_n - x_{n+1} \to 0$, it follows that

$$x_n - y_n \to 0. \tag{4.14}$$

Our next aim is to demonstrate that

$$x_n - Z_n \to 0 \quad \text{and} \quad x_n - W_n \to 0. \tag{4.15}$$

Let $q \in F(S) \cap F(T)$. From (2.1), (4.6), and (4.5), we have that

$$\begin{aligned}&\|y_n - q\|^2 \\ &= \|a_n(x_n - q) + b_n(Z_n - q) + c_n(W_n - q)\|^2 \\ &= a_n\|x_n - q\|^2 + b_n\|Z_n - q\|^2 + c_n\|W_n - q\|^2 \\ &\quad -a_nb_n\|x_n - Z_n\|^2 - b_nc_n\|Z_n - W_n\|^2 - c_na_n\|W_n - x_n\|^2 \\ &\leq a_n\|x_n - q\|^2 + b_n\|z_n - q\|^2 + c_n\|z_n - q\|^2 \\ &\quad -a_nb_n\|x_n - Z_n\|^2 - b_nc_n\|Z_n - W_n\|^2 - c_na_n\|W_n - x_n\|^2 \\ &\leq a_n\|x_n - q\|^2 + b_n\|x_n - q\|^2 + c_n\|x_n - q\|^2 \\ &\quad -a_nb_n\|x_n - Z_n\|^2 - b_nc_n\|Z_n - W_n\|^2 - c_na_n\|W_n - x_n\|^2 \\ &= \|x_n - q\|^2 \\ &\quad -a_nb_n\|x_n - Z_n\|^2 - b_nc_n\|Z_n - W_n\|^2 - c_na_n\|W_n - x_n\|^2.\end{aligned}$$

Using $b_nc_n\|Z_n - W_n\|^2 \geq 0$, we have

$$\begin{aligned}&a_nb_n\|x_n - Z_n\|^2 + a_nc_n\|x_n - W_n\|^2 \\ &\leq \|x_n - q\|^2 - \|y_n - q\|^2 \\ &\leq (\|x_n - q\| + \|y_n - q\|)\,|\|x_n - q\| - \|y_n - q\|| \\ &\leq (\|x_n - q\| + \|y_n - q\|)\,\|x_n - y_n\|.\end{aligned}$$

As $\{x_n\}$ and $\{y_n\}$ are bounded, we have from (4.14) and the assumption (4.2) on the parameters a_n, b_n, c_n that $x_n - Z_n \to 0$ and $x_n - W_n \to 0$ as claimed. From (4.10) and (4.15), it follows that $Z_n \to \overline{u}$ and $W_n \to \overline{u}$. As S and T satisfy the condition (4.1), we obtain $\overline{u} \in F(S) \cap F(T)$.

From (4.10), it suffices to show that

$$\overline{u}\left(= \lim_{n\to\infty} \overline{u}_n = \lim_{n\to\infty} x_n\right) = \widehat{u}\left(= P_{F(S)\cap F(T)}u\right).$$

As $\overline{u} \in F(S) \cap F(T)$ and $\widehat{u} = P_{F(S)\cap F(T)}u$, our aim becomes to show that

$$\|u - \overline{u}\| \le \|u - \widehat{u}\|.$$

Applying (4.7) for $q = \widehat{u} \in F(S) \cap F(T)$, we have $\|u - \overline{u}_n\| \le \|u - \widehat{u}\|$ for all $n \in \mathbb{N}$. From (4.9), we obtain $\|u - \overline{u}\| \le \|u - \widehat{u}\|$. This indicates that $\overline{u} = \widehat{u}$. From (4.10), we obtain $x_n \to \widehat{u}$. This completes the proof. □

From Theorem 4.1, the following corollary is obtained:

Corollary 4.2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and T be quasi-nonexpansive mappings from C into itself that satisfy $F(S) \cap F(T) \neq \emptyset$ and the condition (4.1). Let $\{\lambda_n\}$ and $\{\lambda'_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $\lambda_n \to 1$ and $\lambda'_n \to 1$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$,*

$$\underline{\lim}_{n\to\infty} a_n b_n > 0, \quad \text{and} \quad \underline{\lim}_{n\to\infty} a_n c_n > 0.$$

Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned}
x_1 &= x \in C \text{ is given},\\
C_1 &= C,\\
z_n &= \lambda_n x_n + (1-\lambda_n)\,T x_n,\\
w_n &= \lambda'_n x_n + (1-\lambda'_n)\,S x_n,\\
y_n &= a_n x_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l z_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l w_n,\\
C_{n+1} &= \{h \in C_n : \|y_n - h\|^2 \le \|x_n - h\|^2\\
&\quad -b_n\left(\|x_n\|^2 - \|z_n\|^2 - 2\langle x_n - z_n,\ h\rangle\right)\\
&\quad -c_n\left(\|x_n\|^2 - \|w_n\|^2 - 2\langle x_n - w_n,\ h\rangle\right)\}, \text{ and}\\
x_{n+1} &= P_{C_{n+1}} u_{n+1}
\end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)}u$.

The constructions of z_n and w_n in (4.3) can be replaced by a more general one such as (3.15). Also, the following corollary is obtained as a special case when $\lambda_n = \lambda'_n = 1$ in Theorem 4.1 (or Corollary 4.2).

Corollary 4.3 ([19]). *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let S and T be quasi-nonexpansive mappings from C into itself that satisfy $F(S) \cap F(T) \neq \emptyset$ and the condition (4.1). Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers in the interval $[0,1]$ such that $a_n + b_n + c_n = 1$ for all $n \in \mathbb{N}$,*

$$\varliminf_{n\to\infty} a_n b_n > 0, \quad \text{and} \quad \varliminf_{n\to\infty} a_n c_n > 0.$$

Let $\{u_n\}$ be a sequence in H such that $u_n \to u\,(\in H)$. Define a sequence $\{x_n\}$ in C as follows:

$$\begin{aligned} x_1 &= x \in C \text{ is given,} \\ C_1 &= C, \\ y_n &= a_n x_n + b_n \frac{1}{n}\sum_{l=0}^{n-1} S^l x_n + c_n \frac{1}{n}\sum_{l=0}^{n-1} T^l x_n, \\ C_{n+1} &= \{h \in C_n : \|y_n - h\| \leq \|x_n - h\|\}, \text{ and} \\ x_{n+1} &= P_{C_{n+1}} u_{n+1} \end{aligned}$$

for all $n \in \mathbb{N}$. Then, $\{x_n\}$ converges strongly to a point $\widehat{u}$ of $F(S) \cap F(T)$, where $\widehat{u} = P_{F(S)\cap F(T)} u$.

5. Concluding Remarks

This paper establishes strong convergence theorems for finding common fixed points of two nonlinear mappings. Our method draws on iterative methods due to Ishikawa, Martinez-Yanes and Xu, Nakajo and Takahashi, and Takahashi, Takeuchi, and Kubota, as well as the mean-valued iterative method. The two mappings are not necessarily continuous nor commutative (examples of mappings that are not continuous are given in Section 2). Because nonexpansive mappings, generalized hybrid mappings, 2-generalized hybrid mappings, and normally 2-generalized hybrid mappings are special cases of the class of mappings considered in this paper, the main theorems in this paper are applicable to those classes of mappings. Required conditions on the convex coefficients to prove the main theorems are at a minimum level as the Martinez-Yanes and Xu's work. Finally, all results in this paper can be extended to any finite number of mappings.

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Department of Economics, Shiga University, Banba 1-1-1, Hikone, Shiga 522-0069, Japan
a-kondo@biwako.shiga-u.ac.jp