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Existence and uniqueness results for nonlinear hybrid Ψ-Caputo-type fractional differential equations with nonlocal periodic boundary conditions

Fatima Ezzahra Bourhim, Ali El Mfadel, M'hamed Elomari and Naoufel Hatime

Abstract. *In this paper, we consider a nonlinear fractional hybrid differential equation involving the Ψ-Caputo fractional operator with nonlocal periodic boundary conditions. Based on Lipschitz and Carathéodory conditions and via the Krasnoselskii fixed point theorem and some basic fractional analysis techniques, we discuss the existence and uniqueness of solutions to the proposed problem. Moreover, for a specific class of continuous functions, we prove some fundamental fractional differential inequalities. We finish this work with a non-trivial example.*

1. Introduction

Fractional calculus has emerged as an important area of investigation given its extensive applications in the mathematical modeling of many complex systems with long-term memory effects. An important characteristic of fractional order operators is their nonlocal nature, which accounts for the hereditary properties of the underlying phenomena. The interactions among macromolecules in the damping phenomenon give rise to a macroscopic stress-strain relationship in terms of fractional differential operators. It is known that fractional order models are used for a better description of phenomena having both discrete and continuous behaviors and are applied in different sciences and engineering fields such as material theory, transport processes, earthquakes, electrochemical processes, wave propagation, signal theory, biology, electromagnetic theory, fluid flow phenomena, thermodynamics, mechanics, geology, astrophysics, economics, and control theory (see [3, 9, 13, 29]). Fractional differential equations have been of great interest recently, such as boundary value problems for nonlinear fractional differential equations, which can be employed in modeling and describing non-homogeneous physical phenomena that take place in their form. Almeida et al. [5] investigated the existence and uniqueness results of nonlinear fractional differential equations involving a Caputo type fractional derivative with respect to another function by using fixed point theorems and the Picard iteration method. Zhang in [33] proved the existence and uniqueness results for nonlinear fractional boundary value problems involving Caputo type fractional derivatives by using some fixed point the-

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orems. Many researchers have obtained some interesting results on the existence and uniqueness of solutions to boundary value problems for fractional differential equations involving different fractional operators, such as Riemann–Liouville [25], Caputo [2], Hilfer [24], Erdelyi–Kober [27] and Hadamard [1]. For more details, we refer the reader to [6, 7, 10, 12, 15, 16, 18, 20, 28, 32, 34] and the references therein.

Based on the works that we mentioned, we look into expanding the results found in [23] in the frame of Ψ-Caputo type fractional derivatives of order $2 < \alpha \leq 3$. More specifically, we aim to study the existence and uniqueness of solutions for the following fractional boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{\alpha;\Psi}\left(\dfrac{x(t)}{f(t,x(t))}\right) = g(t,x(t)), \quad t \in J = [0,T], \\ \\ x'(0) = x'(T), \quad a\dfrac{x(0)}{f(0,x(0))} + b\dfrac{x(T)}{f(T,x(T))} = c. \end{cases} \tag{1.1}$$

Where ${}^C D^{\alpha;\Psi}$ is the Ψ-Caputo fractional derivative of order $2 < \alpha \leq 3$, $f \in \mathcal{C}^2(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ and a, b, c are real constants with $a + b \neq 0$.

Our paper is structured as follows:

- In Section 2, we give an overview of the Ψ-Caputo fractional calculus that will be used in subsequent parts of the paper.
- In Section 3, based on mixed Lipschitz and Carathéodory conditions, we establish some existence and uniqueness results for the main problem (1.1).
- In Section 4, we prove some fundamental fractional differential inequalities.
- As an application, an illustrative example is presented in Section 5.
- To summarize, we finish the paper by a conclusion.

2. Preliminaries

We start this section by introducing some necessary definitions and the basic results required for further development.

Let $J = [0,T], T > 0$ and $X = \mathcal{C}(J, \mathbb{R})$ the Banach space of continuous functions from J into $\mathbb{R}$ endowed with supremum norm

$$||y|| = \sup\{|y(t)|, t \in J\}.$$

We denote by $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ the class of function $g\colon J \times \mathbb{R} \to \mathbb{R}$ such that

1. The map $t \mapsto g(t,x)$ is Lebesgue-measurable for all $x \in \mathbb{R}$.
2. The map $x \mapsto g(t,x)$ is continuous for almost every $t \in J$.

The class $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$. Let $L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable real-valued functions on J with norm $||.||_{L^1}$ defined by

$$||x||_{L^1} = \int_0^T |x(s)|ds.$$

We define the multiplication in X by $(xy)(t) = x(t)y(t), \forall x, y \in X$.

Clearly, $X = \mathcal{C}(J, \mathbb{R})$ is a Banach algebra with respect to the above norm and the defined multiplication.

Now, we give some results and properties from the theory of fractional calculus.

Defintion 2.1 (Ψ-Riemann–Liouville fractional integral [4]). Let $\alpha > 0$, f a Lebesgue integrable function defined on $[a, b]$ and $\Psi\colon [a, b] \to \mathbb{R}$ that is an increasing differentiable function such that $\Psi'(t) \neq 0$, for all $t \in [a, b]$.

The Ψ-Riemann–Liouville fractional integral operator of order α of a function f is defined by

$$I_{a^+}^{\alpha;\Psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} f(s)ds.$$

Defintion 2.2 (Ψ-Riemann–Liouville fractional derivative [4]). Let $n \in \mathbb{N}$, f and Ψ two functions of $\mathcal{C}^n([a, b])$ such that Ψ is increasing with $\Psi'(t) \neq 0$, for all $t \in [a, b]$. The Ψ-Riemann–Liouville fractional derivative of order α of a function f is defined by

$$\begin{aligned} D_{a^+}^{\alpha;\Psi} f(t) &= \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n (I_{a^+}^{n-\alpha;\Psi} f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n \int_a^t \Psi'(s)(\Psi(t) - \Psi(s))^{n-\alpha-1} f(s)ds, \end{aligned}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Defintion 2.3 (Ψ-Caputo fractional derivative [4]). Let $n \in \mathbb{N}$, f and Ψ two functions of $\mathcal{C}^n([a, b])$ such that Ψ is increasing with $\Psi'(t) \neq 0$, for all $t \in [a, b]$. The Ψ-Caputo fractional derivative of order α of a function f is defined by

$$\begin{aligned} {}^C D_{a^+}^{\alpha;\Psi} f(t) &= (I_{a^+}^{n-\alpha;\Psi} f_\Psi^{[n]})(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \Psi'(s)(\Psi(t) - \Psi(s))^{n-\alpha-1} f_\Psi^{[n]}(s)ds, \end{aligned}$$

where $n = [\alpha] + 1$, for $\alpha \notin \mathbb{N}$ and $n = \alpha$ otherwise, and

$$f_\Psi^{[n]}(t) = \left(\frac{1}{\Psi'(t)} \frac{d}{dt}\right)^n f(t) \quad on \quad [a, b].$$

From this definition, it is clear that when $\alpha = n \in \mathbb{N}$, we have

$${}^C D_{a^+}^{\alpha;\Psi} f(t) = f_\Psi^{[n]}(t).$$

We note that if $f \in \mathcal{C}^n([a,b])$, then the Ψ-Caputo fractional derivative of order α of f is giving as

$$ {}^C D_{a^+}^{\alpha;\Psi} f(t) = D_{a^+}^{\alpha;\Psi}\left(f(t) - \sum_{k=0}^{n-1} \frac{f_\Psi^{[k]}(a)}{k!}(\Psi(t)-\Psi(a))^k \right). $$

Theorem 2.4 ([4]). *Let $f \in \mathcal{C}^n([a,b])$ and $\alpha > 0$, then we have*

$$ I_{a^+}^{\alpha;\Psi}\ {}^C D_{a^+}^{\alpha;\Psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\Psi^{[k]}(a)}{k!}(\Psi(t)-\Psi(a))^k. $$

In particular, given $\alpha \in (0,1)$ we have:

$$ I_{a^+}^{\alpha;\Psi}\ {}^C D_{a^+}^{\alpha;\Psi} f(t) = f(t) - f(a). $$

Remark 2.5. For the sake of simplicity, in the rest of the paper, we use the notation $D^{\alpha;\Psi}$ instead of $D_{0^+}^{\alpha;\Psi}$.

To prove the existence and uniqueness of the results of the problem (1.1), we assume the following assumptions throughout the rest of this paper:

(H_0) $x \mapsto \dfrac{x}{f(t,x)}$ is increasing on $\mathbb{R}$, for all $t \in J$.

(H_1) There exists a constant $L > 0$ such that:

$$ |f(t,x) - f(t,y)| \leq L|x-y|, \text{ for all } t \in J \text{ and } x, y \in \mathbb{R}. $$

(H_2) There exist some constants $P, Q > 0$ and $p \in (0,1)$ such that the function g satisfies the following growth condition :

$$ |g(t,x)| \leq P|x|^p + Q, \text{ for each } t \in J \text{ and each } x \in \mathbb{R}. $$

3. Main results

In this section, we prove some existence and uniqueness results for the problem (1.1) on the real interval J. First of all, we define the meaning of solution for the boundary fractional hybrid differential equations (1.1).

Defintion 3.1. Let $x \in \mathcal{C}^2(J,\mathbb{R})$. If the following conditions are fulfilled:

i) The function $t \mapsto \dfrac{x}{f(t,x)} \in \mathcal{C}^2(J,\mathbb{R})$ for each $x \in X$,

ii) x satisfies (1.1).

Then x called a solution to (1.1).

Theorem 3.2 ([11]). *Let S be a non-empty, closed, convex, and bounded subset of the Banach algebra X, and let $A\colon X \to X, B\colon X \to X$ be two operators such that*

1. *A is Lipschitzian with a Lipschitz constant L.*
2. *B is completely continuous.*
3. *$x = AxBy \Rightarrow x \in S$, for all $y \in S$.*
4. *$LM < 1$, where $M = ||B(S)||$.*

Then, the operator equation $AxBx = x$ has a solution in S.

In order to get our target, we need the following lemma:

Lemma 3.3. *Let a, b, c are real constants such that $a + b \neq 0$. Assume that hypothesis $(\boldsymbol{H_0})$ holds for a given $h \in L^1(J, \mathbb{R})$, then the unique solution of the hybrid fractional differential equation*

$$\begin{cases} {}^{C}D^{\alpha;\Psi}\left(\dfrac{x(t)}{f(t,x(t))}\right) = h(t),\ t \in J = [0,T], \\[2ex] a\dfrac{x(0)}{f(0,x(0))} + b\dfrac{x(T)}{f(T,x(T))} = c. \end{cases} \tag{3.1}$$

is given by

$$\begin{aligned} x(t) = [f(t,x(t))] \Bigg\{ & c_1[\Psi(t) - \frac{a}{a+b}\Psi(T) - \frac{a}{a+b}\Psi(0)] \\ & + c_2[(\Psi(t) - \Psi(0))^2 - \frac{b}{a+b}(\Psi(T) - \Psi(0))^2] \\ & + \frac{1}{\Gamma(\alpha)}\int_0^{\boldsymbol{t}} \Psi'(s)(\Psi(\boldsymbol{t}) - \Psi(s))^{\alpha-1}h(s)ds \\ & - \frac{b}{a+b}\frac{1}{\Gamma(\alpha)}\int_0^{\boldsymbol{T}} \Psi'(s)(\Psi(\boldsymbol{T}) - \Psi(s))^{\alpha-1}h(s)ds + \frac{c}{a+b} \Bigg\}. \end{aligned}$$

Where $c_1 = \dfrac{1}{\Psi'(0)}\left(\dfrac{x(0)}{f(0,x(0))}\right)'$ *and* $c_2 = \dfrac{1}{2\Psi'(0)}\left(\dfrac{x(0)}{f(0,x(0))}\right)''$.

Proof. Taking the Ψ-Riemann–Liouville fractional integral of order α to the first equation of (3.1), and using Theorem 2.4, we get

$$\begin{aligned} \frac{x(t)}{f(t,x(t))} = \frac{x(0)}{f(0,x(0))} & + \underbrace{\frac{1}{\Psi'(0)}\left(\frac{x(0)}{f(0,x(0))}\right)'}_{c_1}(\Psi(t) - \Psi(0)) \\ & + \underbrace{\frac{1}{2}\frac{1}{\Psi'(0)}\left(\frac{x(0)}{f(0,x(0))}\right)''}_{c_2}(\Psi(t) - \Psi(0))^2 + I^{\alpha;\Psi}h(t). \end{aligned}$$

If we set

$$c_1 = \frac{1}{\Psi'(0)} \left(\frac{x(0)}{f(0, x(0))} \right)' \quad and \quad c_2 = \frac{1}{2\Psi'(0)} \left(\frac{x(0)}{f(0, x(0))} \right)''.$$

Hence we get

$$b\frac{x(T)}{f(T, x(T))} = b\frac{x(0)}{f(0, x(0))} + bc_1(\Psi(T) - \Psi(0)) + bc_2(\Psi(T) - \Psi(0))^2 + bI^{\alpha;\Psi}h(T).$$

Thus

$$\frac{x(0)}{f(0, x(0))} = \frac{1}{a+b} \left\{ c - bc_1(\Psi(T) - \Psi(0)) - bc_2(\Psi(T) - \Psi(0))^2 - bI^{\alpha;\Psi}h(T) \right\}.$$

Consequently,

$$\begin{aligned} x(t) = [f(t, x(t)] \Big\{ & c_1[\Psi(t) - \frac{a}{a+b}\Psi(T) - \frac{a}{a+b}\Psi(0)] \\ & + c_2[(\Psi(t) - \Psi(0))^2 - \frac{b}{a+b}(\Psi(T) - \Psi(0))^2] \\ & + \frac{1}{\Gamma(\alpha)} \int_0^{\mathbf{t}} \Psi'(s)(\Psi(\mathbf{t}) - \Psi(s))^{\alpha-1} h(s) ds \\ & - \frac{b}{a+b} \frac{1}{\Gamma(\alpha)} \int_0^{\mathbf{T}} \Psi'(s)(\Psi(\mathbf{T}) - \Psi(s))^{\alpha-1} h(s) ds + \frac{c}{a+b} \Big\}. \end{aligned}$$

□

Now, for a real numbers a and b such that $a + b \neq 0$, we set

$$\begin{aligned} w = & \frac{|c_1|}{|a+b|} \Big\{ |\Psi(T)| + |b||\Psi(T)| + |a||\Psi(0)| \} + \frac{|c_2|}{|a+b|} \{ (|b|+1)(\Psi(T) - \Psi(0))^2 \Big\} \\ & + \frac{(P+Q)}{\Gamma(\alpha+1)} \Big(1 + |b|)(\Psi(T) - \Psi(0) \Big)^{\alpha} + \frac{|c|}{|a+b|}. \end{aligned}$$

Then, we have the following existence result:

Theorem 3.4. *Assume that hypothesis* $(\boldsymbol{H_0})$–$(\boldsymbol{H_2})$ *holds such that* $L \in \left(1, \frac{1}{w}\right)$ *and* $a + b \neq 0$, *then the hybrid fractional differential equation* (1.1) *has at least a solution defined on* J.

Proof. We define a subset S of X by

$$S = \{x \in X, ||x|| \leq N\},$$

where $N = \dfrac{F_0 w}{1 - Lw}$, and $F_0 = \sup_{t\in J}|f(t,0)|$. Via Lemma 3.3, it is easy to see that the problem (1.1) is equivalent to the nonlinear hybrid integral equation

$$\begin{aligned}
x(t) = [f(t,x(t)]\Big\{ & c_1[\Psi(t) - \frac{a}{a+b}\Psi(T) - \frac{a}{a+b}\Psi(0)] \\
& + c_2[(\Psi(t) - \Psi(0))^2 - \frac{b}{a+b}(\Psi(T) - \Psi(0))^2] \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} \Psi'(s)(\Psi(\mathbf{t}) - \Psi(s))^{\alpha-1} g(s,x(s))ds \\
& - \frac{b}{a+b}\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{T}} \Psi'(s)(\Psi(\mathbf{T}) - \Psi(s))^{\alpha-1} \times g(s,x(s))ds + \frac{c}{a+b}\Big\},\ t \in J.
\end{aligned}$$

Where $c_1 = \dfrac{1}{\Psi'(0)}\left(\dfrac{x(0)}{f(0,x(0))}\right)'$ and $c_2 = \dfrac{1}{2\Psi'(0)}\left(\dfrac{x(0)}{f(0,x(0))}\right)''$.
We define two operators $A\colon X \to X$ and $B\colon S \to X$ by:

$$Ax(t) = f(t,x(t)),\ t \in J, \tag{3.2}$$

and

$$\begin{aligned}
Bx(t) = {} & c_1[\Psi(t) - \frac{a}{a+b}\Psi(T) - \frac{a}{a+b}\Psi(0)] \\
& + c_2[(\Psi(t) - \Psi(0))^2 - \frac{b}{a+b}(\Psi(T) - \Psi(0))^2] \\
& + \frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{t}} \Psi'(s)(\Psi(\mathbf{t}) - \Psi(s))^{\alpha-1} g(s,x(s))ds \\
& - \frac{b}{a+b}\frac{1}{\Gamma(\alpha)}\int_0^{\mathbf{T}} \Psi'(s)(\Psi(\mathbf{T}) - \Psi(s))^{\alpha-1} \times g(s,x(s))ds + \frac{c}{a+b}.
\end{aligned}$$

Then the hybrid integral equation (3.2) is transformed into the operator equation as

$$x(t) = Ax(t)Bx(t),\ t \in J.$$

We proceed now to show that the operators A and B satisfy all the conditions of Theorem 3.2 in several steps.

Step 1: Let $x, y \in X$, then by hypothesis $(\boldsymbol{H_1})$,

$$|Ax(t) - Ay(t)| = |f(t,x(t)) - f(t,y(t))| \le L|x(t) - y(t)|,\ t \in J.$$

Step 2: Let $x_n, x \in S$ with $\lim_{n\to\infty}||x_n - x|| = 0$, it is easy to see that x_n is a bounded subset of S. As a result, we deduce that $||x|| \le N$. Via the continuity of g we see that $g(s,x_n(s)) \to g(s,x(s))$, as $n \to \infty$.

On the other hand, taking $\boldsymbol{(H_2)}$ into consideration, give us the following inequality:

$$\begin{aligned}&\frac{\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}}{\Gamma(\alpha)}||g(s,x_n(s)-g(s,x(s)||\\&\leq 2\frac{\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}}{\Gamma(\alpha)}(PN^p+Q).\end{aligned}$$

Notice that the functions

$$s\mapsto 2\frac{\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}}{\Gamma(\alpha)}(PN^p+Q)$$

and

$$s\mapsto 2\frac{\Psi'(s)(\Psi(T)-\Psi(s))^{\alpha-1}}{\Gamma(\alpha)}(PN^p+Q)$$

are Lebesgue integrable respectively over $[0,t]$ and $[0,T]$, this fact, together with the Lebesgue dominated convergence Theorem, implies that

$$\int_0^t \frac{\Psi'(s)(\Psi(t)-\Psi(s))^{\alpha-1}}{\Gamma(\alpha)}||g(s,x_n(s)-g(s,x(s))|| \underset{n\to\infty}{\longrightarrow} 0,$$

and

$$\int_0^{\mathbf{T}} \frac{\Psi'(s)(\Psi(\mathbf{T})-\Psi(s))^{\alpha-1}}{\Gamma(\alpha)}||g(s,x_n(s)-g(s,x(s))|| \underset{n\to\infty}{\longrightarrow} 0.$$

It follows that $||Bx_n-Bx||\to 0$ as $n\to\infty$. Thus B is a continuous operator. To show that B is compact we take $x\in S$, thus, we have

$$\begin{aligned}|Bx(t)|\leq{}&|c_1|\left\{|\Psi(T)|+\frac{|b|}{|a+b|}|\Psi(T)|+\frac{|a|}{|a+b|}|\Psi(0)|\right\}\\&+|c_2|\left\{(\Psi(T)-\Psi(0))^2+\frac{|b|}{|a+b|}(\Psi(T)-\Psi(0))^2\right\}\\&+\frac{|b|}{|a+b|}\frac{(P||x||^p+Q)}{\Gamma(\alpha+1)}(\Psi(T)-\Psi(0))^\alpha+\frac{|c|}{|a+b|}.\end{aligned}$$

Therefore,

$$\begin{aligned}||Bx||\leq{}&|c_1|\left\{|\Psi(T)|+\frac{|b|}{|a+b|}|\Psi(T)|+\frac{|a|}{|a+b|}|\Psi(0)|\right\}\\&+|c_2|\left\{(\Psi(T)-\Psi(0))^2+\frac{|b|}{|a+b|}(\Psi(T)-\Psi(0))^2\right\}\\&+\frac{|b|}{|a+b|}\frac{(PN^p+Q)}{\Gamma(\alpha+1)}(\Psi(T)-\Psi(0))^\alpha+\frac{|c|}{|a+b|},\end{aligned}$$

hence, B is a uniformly bounded operator. Now, for equicontinuity of B, we take $t_1, t_2 \in J$ such that $t_1 < t_2$, and $x \in S$, thus, we get

$$\begin{aligned}|Bx(t_2) - Bx(t_1)| \leq |c_1|(\Psi(t_2) - \Psi(t_1)) \\ &+ |c_2|(\Psi(t_1) - \Psi(0))^2 - (\Psi(t_2) - \Psi(0))^2 \\ &+ \frac{(PN^p + Q)}{\Gamma(\alpha+1)}(\Psi(t_2) - \Psi(0))^\alpha - (\Psi(t_1) - \Psi(0))^\alpha.\end{aligned}$$

Consequently, we get $||Bx(t_2) - Bx(t_1)|| \to 0$ when $t_2 \to t_1$. Therefore, B is equicontinuous, thus, by the Arzelà–Ascoli Theorem [30], the operator B is compact.

Step 3: Let $x \in X$ and $y \in S$ such that $x = AxBy$, then

$$\begin{aligned}|x(t)| &\leq |Ax(t)By(t)| \leq (L|x(t)| + F_0)|By(t)| \\ &\leq \frac{F_0|By(t)|}{1 - L|By(t)|}.\end{aligned}$$

Without any loss of generality, we take $p = 0$, then

$$\begin{aligned}|By(y)| \leq &\frac{|c_1|}{|a+b|}\{|\Psi(T)| + |b||\Psi(T)| + |a||\Psi(0)|\} \\ &+ \frac{|c_2|}{|a+b|}\{(|b|+1)(\Psi(T) - \Psi(0))^2\} \\ &+ \frac{(P+Q)}{\Gamma(\alpha+1)}(1+|b|)(\Psi(T) - \Psi(0))^\alpha + \frac{|c|}{|a+b|}.\end{aligned}$$

Therefore,

$$||x|| \leq \frac{F_0 w}{1 - Lw},$$

where

$$\begin{aligned}w = &\frac{|c_1|}{|a+b|}\{|\Psi(T)| + |b||\Psi(T)| + |a||\Psi(0)|\} \\ &+ \frac{|c_2|}{|a+b|}\{(|b|+1)(\Psi(T) - \Psi(0))^2\} \\ &+ \frac{(P+Q)}{\Gamma(\alpha+1)}(1+|b|)(\Psi(T) - \Psi(0))^\alpha + \frac{|c|}{|a+b|}.\end{aligned}$$

Step 4: Finally, it remains to show that $LM < 1$ with $M = ||B(s)||$. We have

$$M = ||B(s)|| = \sup\{||Bx||, x \in S\} \leq w.$$

Thus,

$$LM \leq Lw < 1.$$

From the above stepes, we can see that all the conditions of Theorem 3.2 are satisfied, thus the operator $AxBx$ has fixed point in S. As a result, the problem (1.1) has at least a solution defined on J. □

4. Ψ-fractional hybrid differential inequalities

In this section, we discuss a fundamental results relative to strict inequalities for the problem (1.1). We begin with the definition of the class $C_p([0,T],\mathbb{R})$.

Defintion 4.1 ([26]). Let $q \in (0,1)$ and $p > 0$ such that $p+q=1$, we set

$$C_p([0,T],\mathbb{R}) = \left\{ m \in C([0,T],\mathbb{R}) \ \text{ such that } \ t^p m \in C([0,T],\mathbb{R}) \right\}$$

Remark 4.2.

- For the rest of the paper, we consider $q \in (0,1)$ and $p > 0$ such that $p+q=1$.
- For all $m \in C_p([0,T],\mathbb{R})$, the Ψ-Riemann–Liouville derivative defined in Definition 2.1 reduced to

$$D^{q;\Psi} m(t) = \frac{1}{\Gamma(p)} \frac{1}{\Psi'(t)} \frac{d}{dt} \int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{p-1} m(s) ds,$$

Lemma 4.3. *Let $m \in C_p([0,T],\mathbb{R})$. Suppose that for all $t_1 \in [0,T]$, and $t \in (0,t_1)$ we have*

$$m(t) < 0,$$

then,

$$D^{q;\Psi} m(t_1) \geq 0, \ q > 0.$$

Proof. We consider $m \in C_p([0,T],\mathbb{R})$, suppose that $t_1 \in [0,T]$, we have $m(t) < 0$ with $0 < t < t_1$. Then, $m(t)$ and $t^p m(t)$ are continuous over $[0,T]$.
Since $m(t)$ is continuous on $[0,T]$, let $t_1 \in [0,T]$, then, there exist a constant $h > 0$ and a continuous function k such that

$$-k(t_1) \leq m(s) \leq k(t_1), \ \text{where} \ \ 0 < t_1 - h \leq s \leq t_1 + h < T. \tag{4.1}$$

If we set $H(t) = \displaystyle\int_0^t \Psi'(s)(\Psi(t)-\Psi(s))^{p-1} m(s) ds$, then we have

$$\begin{aligned} H(t_1+h) - H(t_1) = & \int_0^{t_1} [(\Psi(t+h)-\Psi(s))^{p-1} - (\Psi(t_1)-\Psi(s))^{p-1}] m(s) ds \\ & + \int_0^{t_1+h} \Psi'(s)(\Psi(t_1+h)-\Psi(s))^{p-1} m(s) ds = I_1 + I_2, \end{aligned}$$

where

$$I_1 = \int_0^{t_1} [(\Psi(t+h) - \Psi(s))^{p-1} - (\Psi(t_1) - \Psi(s))^{p-1}] m(s) ds,$$

$$I_2 = \int_0^{t_1+h} \Psi'(s)(\Psi(t_1+h) - \Psi(s))^{p-1} m(s) ds.$$

Since $t_1 + h > t_1$, $p - 1 < 0$ and Ψ an increasing function, then we have

$$(\Psi(t_1+h) - \Psi(s))^{p-1} < (\Psi(t_1) - \Psi(s))^{p-1}.$$

This, with the fact that $m(t) < 0$ for all $t \in (0, t_1)$, implies that $I_1 \geq 0$.
Using (4.1), we see that

$$I_2 \geq \frac{-k(t_1)}{p}(\Psi(t_1+h) - \Psi(t_1))^p.$$

Hence,

$$H(t_1+h) - H(t_1) + \frac{k(t_1)}{p}(\Psi(t_1+h) - \Psi(t_1))^p \geq 0.$$

Then dividing both sides by h and taking limits as $h \to 0$, we obtain

$$\lim_{h\to 0} \left\{ \frac{H(t_1+h) - H(t_1)}{h} + \frac{k(t_1)}{p} \left(\frac{(\Psi(t_1+h) - \Psi(t_1))}{h} \right)^p h^{p-1} \right\} \geq 0.$$

Since $p \in (0,1)$, thus $\frac{dH(t_1)}{dt} \geq 0$, which implies that $D^{q;\Psi}(t_1) \geq 0$. □

Theorem 4.4. *Assume that hypothesis $(\boldsymbol{H_0})$ hold and suppose that there exist two functions y, z in $C_p([0,T], \mathbb{R})$ satisfying the following inequalities*

$$D^{\alpha;\Psi}\left(\frac{y(t)}{f(t, y(t))}\right) \leq g(t, y(t)), \ \forall t \in J, \tag{4.2}$$

and

$$D^{\alpha;\Psi}\left(\frac{z(t)}{f(t, z(t))}\right) \geq g(t, z(t)), \ \forall t \in J, \tag{4.3}$$

such that one of the inequalities is strict, then

$$y^0 < z^0 \quad \textit{implies that} \quad y(t) < z(t) \quad \textit{for all} \quad t \in J,$$

where $y^0 = t^{1-\alpha} y(t)|_{t=0}$ and $z^0 = t^{1-\alpha} z(t)|_{t=0}$.

Proof. We assume that this claim is false, since $y^0 < z^0$, $t^{\alpha-1}y(t)$ and $t^{\alpha-1}z(t)$ are continuous functions, then there exists t_1 such that $t_1 \in (0,T)$ where

$$y(t_1) = z(t_1) \ \ \textit{and} \ \ y(t) < z(t), \ \ \textit{for all} \ \ t \in (0, t_1).$$

We set

$$Y(t) = \frac{y(t)}{f(t,y(t))} \text{ and } Z(t) = \frac{z(t)}{f(t,z(t))}.$$

Then we have $Y(t_1) = Z(t_1)$, and via $(\boldsymbol{H_0})$ we get $Y(t) < Z(t)$ for all $t \in (0,t_1)$. If we set $m(t) = Y(t) - Z(t)$ for all $t \in (0,t_1)$, then we have $m(t) < 0, t \in (0,t_1)$ and $m(t_1) = 0$, since $m \in C_p([0,T],\mathbb{R})$, using Lemma 4.3, we obtain

$$g(t_1, y(t_1)) > D^{\alpha;\Psi}(Y(t_1)) > D^{\alpha;\Psi}(Z(t_1)) \geq g(t_1, z(t_1)).$$

Which is contradictory with the fact that $y(t_1) = z(t_1)$, hence the proof is finished. □

Theorem 4.5. *Assume that hypothesis* $(\boldsymbol{H_0})$ *holds and* a, b, c *are real constants with* $a + b \neq 0$. *Suppose that there exist functions* $y, z \in C_p([0,T],\mathbb{R})$, *such that*

$$D^{\alpha;\Psi}\left(\frac{y(t)}{f(t,y(t))}\right) \leq g(t,y(t)), \ \forall t \in J, \tag{4.4}$$

and

$$D^{\alpha;\Psi}\left(\frac{z(t)}{f(t,z(t))}\right) \geq g(t,z(t)), \ \forall t \in J, \tag{4.5}$$

with one of the inequalities being strict. If $a > 0, b < 0$ *and* $y(T) < z(T)$, *then*

$$a\frac{y(0)}{f(0,y(0))} + b\frac{y(T)}{f(T,y(T))} < a\frac{z(0)}{f(0,z(0))} + b\frac{z(T)}{f(T,z(T))} \ \Rightarrow \ y(t) < z(t) \text{ for all } t \in J.$$

Proof. We have

$$a\left(\frac{y(0)}{f(0,y(0))} - \frac{z(0)}{f(0,z(0))}\right) < b\left(\frac{z(T)}{f(T,z(T))} - \frac{y(T)}{f(T,y(T))}\right),$$

Since $a > 0, b < 0$ and $y(T) < z(T)$ by $(\boldsymbol{H_0})$, we have $y(0) < z(0)$.
Hence, as application of Theorem 4.4 we get $y(t) < z(t)$. □

Theorem 4.6. *Assume that the conditions of Theorem 4.4 hold. Suppose that there exists a real number* $M > 0$ *such that*

$$g(t,x_1) - g(t,x_2) \leq M\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right), \ t \in J \text{ and } x_1, x_2 \in \mathbb{R} \text{ with } x_1 \geq x_2.$$

Then

$$a\frac{y(0)}{f(0,y(0))} + b\frac{y(T)}{f(T,y(T))} < a\frac{z(0)}{f(0,z(0))} + b\frac{z(T)}{f(T,z(T))},$$

implies that

$$y(t) < z(t) \text{ for all } t \in J.$$

Where $\frac{1}{\Gamma(p)}(\Psi(T) - \Psi(0))^{p-1} \geq M, p \in (0,1)$

Proof. We set

$$\frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))} = \frac{z(t)}{f(t,z(t))} + \varepsilon,\ \varepsilon > 0,$$

and let

$$Z_\varepsilon(t) = \frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))} \text{ and } Z(t) = \frac{z(t)}{f(t,z(t))},\ t \in J.$$

So, we have

$$Z_\varepsilon(t) > Z(t) \Rightarrow z_\varepsilon(t) > z(t).$$

Since

$$g(t,x_1) - g(t,x_2) \leq M\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right)$$

and

$$D^{\alpha;\Psi}\left(\frac{y(t)}{f(t,y(t))}\right) \leq g(t,y(t)),\ t \in J,$$

hence

$$\begin{aligned} D^{\alpha;\Psi}(Z_\varepsilon(t)) &= D^{\alpha;\Psi}(Z(t)) + D^{\alpha;\Psi}(\varepsilon) \\ &= D^{\alpha;\Psi}(Z(t)) + \frac{\varepsilon}{\Gamma(p)}(\Psi(T)-\Psi(0))^{p-1},\ p \in (0,1) \\ &\geq g(t,z_\varepsilon(t)) + \varepsilon\left(\frac{1}{\Gamma(p)}(\Psi(T)-\Psi(0))^{p-1} - M\right) \\ &> g(t,z_\varepsilon(t)). \end{aligned}$$

We have $z_\varepsilon(0) > z(0) \geq y(0)$, hence, by virtue of Theorem 4.5 we get

$$y(t) < z_\varepsilon(t) \textit{ for all } t \in J.$$

By taking $\varepsilon \to 0$, we get $y(t) \leq z(t)$, for all $t \in J$. □

5. An illustrative example

In this section, we give an example to illustrate our main result. Consider the following hybrid fractional differential equation:

$$\begin{cases} {}^C D_{0^+}^{\frac{5}{2},t}\left(\dfrac{x(t)}{\Phi(t,x(t))}\right) = \varphi(t,x(t)), & t \in \Delta = [0,1], \\ x'(0) = x'(1),\quad \dfrac{x(0)}{\sqrt{\frac{1}{10}|x(0)|+1}} + \dfrac{x(1)}{\sqrt{\frac{1}{10}|x(1)|+1}} = 0. \end{cases} \tag{5.1}$$

where $\alpha = \frac{3}{2}$, $T = 1$, $a = b = 1$, $c = 0$, $\Psi(t) = t$,

$$\varphi(t,x(t)) = \frac{\sqrt{|x(t)| + e^{-|x(t)|}}}{t^2+2t+3}\sin^2(x(t)),\ \text{and } \Phi(t,x(t)) = \sqrt{\frac{1}{10}|x(t)|+1}.$$

It is clear that the assumption (H_0) is satisfied. We proceed to prove that the assumption (H_1) hold, let $t \in \Delta$ and $x, y \in \mathcal{C}(\Delta, \mathbb{R})$, then we have

$$\begin{aligned}|\Phi(t,x(t)) - \Phi(t,y(t))| &= \left|\sqrt{\frac{1}{10}|x(t)|+1} - \sqrt{\frac{1}{10}|x(t)|+1}\right| \\ &\leq \frac{1}{10}\frac{\Big|x(t)-y(t)\Big|}{\sqrt{\frac{1}{10}|x(t)|+1}+\sqrt{\frac{1}{10}|x(t)|+1}} \\ &\leq \frac{1}{10}\left|x(t)-y(t)\right|.\end{aligned}$$

Thus, the assumption (H_1) is true where $L = \dfrac{1}{10}$.
It remains to verify that the assumption (H_2) hold, let $t \in \Delta$ and $u \in \mathbb{R}$, then we have

$$\begin{aligned}|\varphi(t,x(t))| &= \left|\frac{\sqrt{|x(t)|+1}}{t^2+2t+3}\right| \\ &\leq \frac{1}{3}\sqrt{|x(t)|}+\frac{1}{3},\end{aligned}$$

Thus the assumption (H_2) is verified where $P = Q = \dfrac{1}{3}$ and $p = \dfrac{1}{2}$.
Hence, as a consequence of Theorem 3.2, we deduce that the problem (5.1) has at least solution on $[0, 1]$. To guarantee the uniqueness of this solution, it is enough to take the constants c_1 and c_2 such that

$$\frac{|c_1|+|c_2|}{10} + \frac{1}{3\Gamma\left(\frac{5}{2}\right)} < 1.$$

6. Conclusion

In the work, we consider a nonlocal fractional hybrid boundary value problem involving Ψ-Caputo fractional derivative of order $\alpha \in (2, 3]$, by the Krasnoselskii fixed point Theorem, we prove that the main problem has at least a solution. In addition, we prove some fundamental fractional differential inequalities for a specific class of continuous functions. Finally, as an application, a nontrivial example is presented to illustrate our theoretical results.

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Fatima Ezzahra Bourhim
Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco
fati.zahra.bourhim98@gmail.com

Ali El Mfadel
Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco
a.elmfadel@usms.ma

M'hamed Elomari
Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco
m.elomari@usms.ma

Naoufel Hatime
Laboratory of Applied Mathematics and Scientific Computing, Sultan Moulay Slimane University, Beni Mellal, Morocco
hatime.naoufel@usms.ma