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Bounded solutions for Dirichlet problems with degenerate coercivity and a quadratic gradient term

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Abstract. *We give existence results for weak solutions of Dirichlet problems for elliptic equations having degenerate coercivity and a first order term which has quadratic growth with respect to the gradient. The proof is based on the use of test functions having exponential growth.*

1. Introduction

In this paper we will prove some existence and boundedness results for boundary value problems of the form

$$\begin{cases} -\operatorname{div}(a(x,u)\nabla u) + u = H(x,u,\nabla u) + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

whose simplest example is

$$\begin{cases} -\operatorname{div}\left(\dfrac{\nabla u}{(A(x)+|u|)^\gamma}\right) + u = h(x)|\nabla u|^2 + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $A(x)$ is a measurable function on Ω such that $0 < \lambda \le A(x) \le \mu$ (λ, $\mu \in \mathbb{R}_+$), γ is a positive constant, h is a measurable bounded function on Ω which may change sign and $f \in L^\infty(\Omega)$.

Here Ω is a bounded, open subset of $\mathbb{R}^N$, $N \ge 2$. We now give the assumptions for problem (1.1):

$$\begin{cases} a(x,s)\colon \Omega \times \mathbb{R} \to \mathbb{R} \\ \text{is measurable with respect to } x \text{ for every } s \in \mathbb{R} \\ \text{and continuous with respect to } s \text{ for almost every } x \in \Omega, \end{cases} \tag{1.3}$$

and satisfies

$$\frac{\alpha}{(1+|s|)^\gamma} \le a(x,s) \le \beta \quad \text{(degenerate coercivity)}, \tag{1.4}$$

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for α, β, $\gamma \in \mathbb{R}_+$;

$$\begin{cases} H(x,s,\xi)\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R} \\ \text{is measurable with respect to } x \text{ for every } (s,\xi) \in \mathbb{R} \times \mathbb{R}^N \\ \text{and continuous with respect to } (s,\xi) \text{ for almost every } x \in \Omega, \end{cases} \tag{1.5}$$

such that, for some $R > 0$,

$$|H(x,s,\xi)| \le R\,|\xi|^2. \tag{1.6}$$

As far as the datum f is concerned, we will always assume that

$$f(x) \in L^\infty(\Omega). \tag{1.7}$$

This class of problems presents some features and difficulties which we are now going to describe briefly.

The first one is the fact that, due to hypothesis (1.4), the coefficient $a(x,u)$ of the principal part may go to zero when u goes to $\pm\infty$. This means that the differential operator $A(v) = -\operatorname{div}(a(x,v)\nabla v)$, though well defined between $W_0^{1,2}(\Omega)$ and its dual, is not coercive on $W_0^{1,2}(\Omega)$: for instance, if we take $v_n(x) = T_n(|x|^{1-N/2} - 1)$ in the unit ball $B_1 \subset \mathbb{R}^N$ $(N \ge 3)$, then

$$\|v_n\|_{W_0^{1,2}(B_1)} \to \infty, \quad \text{but} \quad \int_{B_1} \frac{|\nabla v_n|^2}{(1+v_n)^\gamma} \le C \quad \text{for every } \gamma > 0.$$

This implies that the classical methods used to prove the existence of a solution of the simple problem

$$\begin{cases} -\operatorname{div}(a(x,u)\nabla u) = f(x) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

cannot be applied, even if the datum f is very regular. In the papers [5] and [4] the boundary value problem (1.8) was studied for $\gamma \in (0,1]$, obtaining existence results for weak solutions of (1.8), which have different regularity according to the integrability of the right-hand side f.

The operator $A(v) = -\operatorname{div}(a(x,v)\nabla v)$ satisfies a condition similar to operators with nonstandard growth conditions, or p-q growth conditions (see [13]), since, in the case $\gamma = 1$ and $a(x,v) = 1/(1+v)$, it is possible to prove that

$$C\big(\|\nabla v\|_{\frac{N}{N-1}} - 1\big) \le \langle A(v), v\rangle = \int_\Omega \frac{|\nabla v|^2}{1+|v|} \le \|\nabla v\|_2^2,\ \forall\, v \in W_0^{1,2}(\Omega), \tag{1.9}$$

where C is a positive constant depending on $N \ge 2$ and $|\Omega|$. The second inequality in (1.9) is trivial; the first one can be proved using Hölder's and Sobolev's inequalities.

The other relevant feature of problem (1.1) is the presence of a first-order term which has quadratic growth with respect to the gradient. Elliptic equations of the form

$$-\mathrm{div}(a(x,u)\nabla u) = h(x,u)|\nabla u|^2 + f(x) \tag{1.10}$$

occur in several contexts, for instance in Calculus of Variations: the Euler–Lagrange equations of integral functionals of the form

$$J[v] = \int_\Omega B(x,v)|\nabla v|^2 - \int_\Omega f(x)\,v$$

are of this type. Moreover similar problems appear as the Hamilton–Jacobi–Bellman equations in stochastic control. Finally, equations with quadratic gradient terms modelize stationary solutions for rough surfaces which grow by particle deposition (see [12]).

Existence, regularity and uniqueness (also nonuniqueness) results for elliptic equations with quadratic first-order terms have been widely studied in the last decades (for instance [6, 7, 1]). In particular, equations of the form (1.10) have been studied in [11, 10, 15, 14], where it is shown that in order to have existence of a solution, there must be a condition of "smallness" on the datum f or some relation between the functions $a(x,u)$ and $h(x,u)$.

As pointed out in some articles (see for instance [6, 8, 3]), the presence of a zero-order term u in the left-hand side of (1.1) has a regularizing effect, which may allow to relax these conditions. In particular, adapting the approach of [6] (see also [7]), we are able to prove the following result for problem (1.1).

Theorem 1.1. *Under the assumptions (1.3)–(1.7), there exists a bounded weak solution of the Dirichlet problem (1.1), that is, a function $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that*

$$\int_\Omega a(x,u)\nabla u\,\nabla v + \int_\Omega u\,v = \int_\Omega H(x,u,\nabla u)\,v + \int_\Omega f\,v$$

for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Subsequently we extend the results to a wider class of boundary value problems, of the form

$$\begin{cases} -\mathrm{div}(a(x,u)\nabla u) + b(x,u) = H(x,u,\nabla u) + f(x), & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

where the functions $a(x,s)$, $b(x,s)$ and $H(x,s,\xi)$ satisfy (1.3), (1.5) (and a similar one for b) and the following general growth assumptions, for a.e. $x \in \Omega$, for every $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$:

$$\begin{cases} 0 < \alpha(|s|) \le a(x,s) \le \beta \in \mathbb{R}\,, \\ \text{with } \alpha : [0,+\infty) \to (0,+\infty) \text{ continuous and decreasing,} \end{cases} \tag{1.12}$$

$$\begin{cases} |H(x,s,\xi)| \le \delta(|s|)\,|\xi|^2, \\ \text{with } \delta : [0,+\infty) \to [0,+\infty) \text{ continuous and increasing,} \end{cases} \tag{1.13}$$

$$\begin{cases} b(x,s)\,\mathrm{sign}(s) \geq \varphi(|s|), \text{with } \varphi : [0,+\infty) \to [0,+\infty) \\ \text{continuous and increasing such that } \lim_{t\to+\infty} \varphi(t) = +\infty\,. \end{cases} \tag{1.14}$$

These assumptions may include operators having extremely degenerate coercivity, for example $-\mathrm{div}\big(\frac{\nabla u}{1+e^{|u|}}\big)$, and first order terms which may have *any* growth with respect to u. Moreover, the term $b(x,u)$, whose presence is essential for our results, may also have a very slow growth with respect to u (for instance $b(u) = \log(1 + |u|)\mathrm{sign}(u)$. Under these hypotheses we prove the following result:

Theorem 1.2. *Under the assumptions* (1.12)–(1.14) *and* (1.7)*, there exists a bounded weak solution of problem* (1.11)*, that is, a function* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ *such that*

$$\int_\Omega a(x,u)\nabla u\,\nabla v + \int_\Omega b(x,u)\,v = \int_\Omega H(x,u,\nabla u)\,v + \int_\Omega f\,v$$

for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

In the following section, we will prove Theorem 1.1. The next brief section is devoted to some extensions. The last part of this paper is devoted to the proof of Theorem 1.2.

2. Proof of Theorem 1.1

In this section, we assume (1.3)–(1.7) and we define

$$T_k(s) = \max\{-k, \min\{k,s\}\}.$$

For $n \in \mathbb{N}$, we consider the following Dirichlet problem

$$\begin{cases} -\mathrm{div}(a(x,T_n(u_n))\nabla u_n) + u_n = \dfrac{H(x,u_n,\nabla u_n)}{1+\frac{1}{n}|\nabla u_n|^2} + f(x) \\ u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,, \end{cases} \tag{2.1}$$

that is, $\forall\, v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x,T_n(u_n))\nabla u_n\,\nabla v + \int_\Omega u_n\,v = \int_\Omega \frac{H(x,u_n,\nabla u_n)}{1+\frac{1}{n}|\nabla u_n|^2}\,v + \int_\Omega f\,v. \tag{2.2}$$

Note that, by condition (1.13),

$$\frac{|H(x,u_n,\nabla u_n)|}{1+\frac{1}{n}|\nabla u_n|^2} \leq R\min\left\{n,\ |\nabla u_n^2|\right\}. \tag{2.3}$$

For any fixed n, the existence is a consequence of the fact that the function $a(x,T_n(s))$ is bounded from below by a positive constant. Since the right-hand side of (2.1) is bounded by $nR+\|f\|_\infty$, existence of a weak solution of (2.1) follows from an application of Schauder's fixed point theorem.

Lemma 2.1. *The sequence $\{u_n\}$ is bounded in $L^\infty(\Omega)$.*

Proof. For $k > 0$, we take $v = [g(|u_n|) - g(k)]_+ \mathrm{sign}(u_n)$ in (2.2), where

$$g(t) = e^{\frac{R(1+t)^{\gamma+1}}{\alpha(\gamma+1)}}, \tag{2.4}$$

obtaining, by (2.3),

$$\int_{\{|u_n|>k\}} |\nabla u_n|^2 \Big(\underbrace{\frac{\alpha g'(|u_n|)}{(1+|u_n|)^\gamma} - R\, g(|u_n|)}_{=0} + R\, g(k) \Big) \\ + \int_{\{|u_n|>k\}} \big(|u_n| - \|f\|_\infty\big) \big(g(|u_n|) - g(k)\big) \le 0.$$

Since the first integral is positive, taking $k = \|f\|_\infty$, the above inequality implies

$$|u_n| \le \|f\|_\infty . \tag{2.5}$$

$\square$

Corollary 2.2. *For every $n \ge \|f\|_\infty$, $a(x, T_n(u_n)) = a(x, u_n)$, therefore $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of the Dirichlet problem*

$$-\mathrm{div}(a(x, u_n)\nabla u_n) + u_n = \frac{H(x, u_n, \nabla u_n)}{1 + \frac{1}{n}|\nabla u_n|^2} + f(x),$$

that is, $\forall\, v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x, u_n)\nabla u_n \nabla v + \int_\Omega u_n\, v = \int_\Omega \frac{H(x, u_n, \nabla u_n)}{1 + \frac{1}{n}|\nabla u_n|^2}\, v + \int_\Omega f\, v. \tag{2.6}$$

Lemma 2.3. *The sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$.*

Proof. Note that, by (2.5),

$$\frac{\alpha}{(1 + \|f\|_\infty)^\gamma} \le \frac{\alpha}{(1 + |u_n|)^\gamma} \le a(x, u_n) \le \beta.$$

Then, for every C^1 odd increasing function $q(s)\colon \mathbb{R} \to \mathbb{R}$, the choice $v = q(u_n)$ in (2.6) yields

$$\frac{\alpha}{(1 + \|f\|_\infty)^\gamma} \int_\Omega |\nabla u_n|^2 q'(u_n) \le R \int_\Omega |\nabla u_n|^2 q(|u_n|) + q(\|f\|_\infty) \int_\Omega |f|.$$

In particular, choosing

$$q(t) = \big(e^{\lambda |t|} - 1\big)\mathrm{sign}(t), \tag{2.7}$$

with $\lambda\,\alpha \geq R(1+\|f\|_{\infty})^{\gamma}$, we obtain

$$\frac{\alpha\lambda}{(1+\|f\|_{\infty})^{\gamma}}\int_\Omega |\nabla u_n|^2 e^{\lambda\,|u_n|} \leq R\int_\Omega |\nabla u_n|^2[e^{\lambda\,|u_n|}-1] + q(\|f\|_{\infty})\int_\Omega |f|$$

and therefore

$$R\int_\Omega |\nabla u_n|^2 \leq q(\|f\|_{\infty})\int_\Omega |f|.$$

□

As a consequence of the previous lemma and (2.5), passing to a subsequence if necessary, we may assume that the sequence $\{u_n\}$ converges to a function $u \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ weakly in $W_0^{1,2}(\Omega)$ and a. e.

Lemma 2.4. *The sequence $\{u_n\}$ converges strongly to u in $W_0^{1,2}(\Omega)$.*

Proof. Here, following [6, 7], we use $q(u_n-u)$ as test function, where $q(t)$ is defined as in (2.7) and $\lambda\,\alpha > 2R(1+\|f\|_{\infty})^{\gamma}$. Then

$$\begin{aligned}\int_\Omega a(x,u_n)\nabla u_n[\nabla u_n-\nabla u]q'(u_n-u) &+ \int_\Omega u_n\, q(u_n-u)\\ &\leq R\int_\Omega |\nabla u_n|^2|q(u_n-u)| + \int_\Omega |f||q(u_n-u)|,\end{aligned}$$

that is, since $|\nabla u_n|^2 \leq 2|\nabla u_n-\nabla u|^2 + 2|\nabla u|^2$,

$$\begin{aligned}&\frac{\alpha}{(1+\|f\|_{\infty})^{\gamma}}\int_\Omega |\nabla u_n-\nabla u|^2 q'(u_n-u) + \int_\Omega u_n\, q(u_n-u)\\ &\quad\leq 2R\int_\Omega |\nabla u_n-\nabla u|^2|q(u_n-u)| + 2R\int_\Omega |\nabla u|^2|q(u_n-u)|\\ &\qquad + \int_\Omega |f||q(u_n-u)| - \int_\Omega a(x,u_n)\nabla u[\nabla u_n-\nabla u]q'(u_n-u),\end{aligned}$$

which implies

$$\begin{aligned}&\left(\frac{\alpha\lambda}{(1+\|f\|_{\infty})^{\gamma}}-2R\right)\int_\Omega |\nabla u_n-\nabla u|^2\\ &\quad\leq -\int_\Omega u_n\, q(u_n-u) + 2R\int_\Omega |\nabla u|^2|q(u_n-u)| \qquad (2.8)\\ &\qquad + \int_\Omega |f||q(u_n-u)| - \int_\Omega a(x,u_n)\nabla u[\nabla u_n-\nabla u]q'(u_n-u)\,.\end{aligned}$$

Using the a.e. convergence and boundedness of u_n and the weak-L^2 convergence of ∇u_n, it is now easy to show that for $n\to\infty$ all the integrals of the right-hand side of (2.8) go to zero. □

Corollary 2.5. *The properties* (1.5), (1.6) *and the strong convergence of the sequence* $\{u_n\}$ *to* u *in* $W_0^{1,2}(\Omega)$ *imply the strong convergence of the sequence* $\left\{\dfrac{H(x,u_n,\nabla u_n)}{1+\frac{1}{n}|\nabla u_n|^2}\right\}$ *to* $H(x,u,\nabla u)$ *in* $L^1(\Omega)$.

Conclusion of the proof of Theorem 1.1. Using the properties of the sequence $\{u_n\}$ proved above (in particular, in the previous corollary), it is possible to pass to the limit in every term of (2.6) in order to prove the statement. □

3. Extensions

In the following two subsections, we present some developments, possible thanks to the properties of our method.

3.1. Q-condition

In this subsection, instead of (1.7) we assume that there exists a constant $Q > 0$ such that

$$|f(x)| \leq Q\, a(x), \quad a \in L^1(\Omega), \tag{3.1}$$

and we consider the Dirichlet problems

$$\begin{cases} u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ -\mathrm{div}(a(x,T_n(u_n))\nabla u_n) + a(x)\, u_n = H(x,u_n,\nabla u_n) + f(x); \end{cases} \tag{3.2}$$

that is, $\forall\, v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x,T_n(u_n))\nabla u_n \nabla v + \int_\Omega a(x)\, u_n\, v = \int_\Omega H(x,u_n,\nabla u_n)\, v + \int_\Omega f(x)\, v. \tag{3.3}$$

In [2] the existence of a solution u_n to problem (3.3) is proven; moreover it is shown, despite the assumption $f \in L^1(\Omega)$, that

$$\|u_n\|_\infty \leq Q$$

and the above estimate does not depend on the principal part.

Thus, for u_n solution of (3.2), we are in the same position of Lemma 2.1; then, it is possible to prove the following existence result concerning bounded weak solutions despite the very poor assumption $f \in L^1(\Omega)$, thanks to (3.1).

Theorem 3.1. *Under the assumptions* (1.3)–(1.6) *and* (3.1), *there exists a bounded weak solution of*

$$u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \ -\mathrm{div}(a(x,u)\nabla u) + a(x)\, u = H(x,u,\nabla u) + f(x)$$

that is, $\forall\, v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega a(x,u)\nabla u \nabla v + \int_\Omega a(x)\, u\, v = \int_\Omega H(x,u,\nabla u)\, v + \int_\Omega f\, v,$$

with $\|u\|_\infty \leq Q$, *where* Q *is the constant appearing in* (3.1).

3.2. Function $H(x,s,\xi)$ unbounded with respect to s

If we assume (instead of (1.5))

$$|H(x,s,\xi)| \leq R\,(1+|s|)^{\lambda}\,|\xi|^2, \tag{3.4}$$

where $\lambda \in \mathbb{R}_+$, we need to modify the real function $g(t)$ in (2.4) by changing γ with $\gamma+\lambda$.

4. Proof of Theorem 1.2

We assume here the hypotheses (1.12)–(1.14) and (1.7). We consider the problem

$$\begin{cases} u_n \in W_0^{1,2}(\Omega) : \\ -\operatorname{div}(a(x,T_n(u_n))\nabla u_n) + b_n(x,u_n) = T_n(H(x,u_n,\nabla u_n)) + f, \end{cases} \tag{4.1}$$

where

$$b_n(x,s) = \begin{cases} \min\{b(x,s), ns\} & \text{for } s \geq 0 \\ \max\{b(x,s), -ns\} & \text{for } s < 0. \end{cases}$$

Note that, if we set $\varphi_n(t) = \min\{\varphi(t), nt\}$ for $t > 0$, then

$$b_n(x,s)\,\operatorname{sign}(s) \geq \varphi_n(|s|) \geq \varphi_1(|s|).$$

Existence of a solution u_n of problem (4.1), that is, of a function $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ such that

$$\int_\Omega a(x,T_n(u_n))\nabla u_n\,\nabla v + \int_\Omega b_n(x,u_n)\,v = \int_\Omega T_n(H(x,u_n,\nabla u_n))\,v + \int_\Omega f(x)\,v, \tag{4.2}$$

for every $v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, follows again from Schauder's fixed point theorem. We now define

$$B(t) = \int_0^t \frac{\delta(\sigma)}{\alpha(\sigma)}\,\mathrm{d}\sigma \qquad \text{for } t \geq 0.$$

For $k > 0$ to be chosen below, we take

$$v = \left(e^{B(|u_n|)} - e^{B(k)}\right)_+ \operatorname{sign}(u_n)$$

in (4.2). Using the assumptions on the functions appearing in the equation, we obtain

$$\begin{aligned} &\int_{\{|u_n|>k\}} \delta(|u_n|)e^{B(|u_n|)}|\nabla u_n|^2 + \int_{\{|u_n|>k\}} \varphi_n(|u_n|)\left(e^{B(|u_n|)} - e^{B(k)}\right) \\ &\leq \int_{\{|u_n|>k\}} \delta(|u_n|)\left(e^{B(|u_n|)} - e^{B(k)}\right)|\nabla u_n|^2 + \int_{\{|u_n|>k\}} |f|\left(e^{B(|u_n|)} - e^{B(k)}\right), \end{aligned}$$

and therefore

$$e^{B(k)} \int_{\{|u_n|>k\}} \delta(|u_n|)|\nabla u_n|^2 + \int_{\{|u_n|>k\}} \big(\varphi_n(|u_n|) - \|f\|_\infty\big)\big(e^{B(|u_n|)} - e^{B(k)}\big) \leq 0.$$

Since the first integral is nonnegative, by taking $k = k_n$ such that $\varphi_n(k) = \|f\|_\infty$ we obtain

$$\|u_n\|_\infty \leq \varphi_n^{-1}(\|f\|_\infty). \tag{4.3}$$

In particular, since $\varphi_n(s) \geq \varphi_1(s)$, this implies a uniform L^∞-estimate for the sequence $\{u_n\}$:

$$\|u_n\|_\infty \leq \varphi_1^{-1}(\|f\|_\infty). \tag{4.4}$$

From now on, the proof is very similar to the one is presented in the previous section, since the functions u_n are weak solutions of the equation

$$-\mathrm{div}(a(x,u_n)\nabla u_n) + b_n(x,u_n) = T_n(H(x,u_n,\nabla u_n)) + f,$$

where

$$a(x,u_n) \geq \alpha(\varphi_1^{-1}(\|f\|_\infty), \quad |T_n(H(x,u_n,\nabla u_n))| \leq \delta(\varphi_1^{-1}(\|f\|_\infty)|\nabla u_n|^2.$$

Note that from the estimate (4.3) it follows that the limit solution u satisfies

$$\|u\|_\infty \leq \varphi^{-1}(\|f\|_\infty).$$

□

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