Rend. Mat. Appl. (7),
Volume 45 (2024), 281–292

Rendiconti di Matematica
e delle sue Applicazioni

Non-autonomous overdetermined problems for the normalized p-Laplacian

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Abstract. *We present existence and nonexistence results on the solution of an overdetermined problem for the normalized p-Laplacian in a bounded open set, with p ranging from* 1 *to infinity. More precisely we consider a non-constant Neumann condition at the boundary. The definitions and statements needed to understand the main results are recalled in detail.*

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a differentiable boundary $\partial\Omega$, whose outer normal we denote by ν. Choose $\bar{x} \in \Omega$ and define

$$R_1 = \min_{x \in \partial\Omega} |x - \bar{x}|, \quad R_2 = \max_{x \in \partial\Omega} |x - \bar{x}|. \tag{1.1}$$

In this paper we study overdetermined problems ruled by the normalized p-Laplacian for $p \in [1, \infty]$: more precisely, we study the problem

$$\begin{cases} -\Delta_p^N u = 1 & \text{in } \Omega; \\ u = 0, \ -\dfrac{\partial u}{\partial \nu} = q(|x - \bar{x}|) & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $q(r)$ is a real-valued function defined on $[R_1, R_2]$. For a smooth function u with nonvanishing gradient Du, the operator $\Delta_p^N u$ is given by

$$\Delta_p^N u = \begin{cases} \frac{1}{p} \, |Du|^{2-p} \operatorname{div}\left(|Du|^{p-2} \, Du\right), & p \in [1, \infty); \\ |Du|^{-2} \, \langle D^2 u \, Du, Du \rangle & p = \infty. \end{cases} \tag{1.3}$$

The relationship between the case when p is finite and the case $p = \infty$ is put into evidence by the equality

$$\Delta_p^N u = \frac{p-1}{p} \, \Delta_\infty^N u + \frac{1}{p} \, \Delta_1^N u \quad \text{for } p \in [1, \infty)$$

(see [2, (1.2)] or [11, (1.6)]): for a given function u and at a fixed point x such that $Du(x) \neq 0$, it follows that $\Delta_p^N u(x) \to \Delta_\infty^N u(x)$ when $p \to \infty$. The term

2020 Mathematics Subject Classification: 35N25, 35B06, 35D40.
Keywords: Overdetermined problems, Viscosity solutions, Normalized infinity-Laplacian.

"normalized" is used to make a distinction from the *classical* p-Laplace operator $\Delta_p u$ given by

$$\Delta_p u = \begin{cases} \operatorname{div}\left(|Du|^{p-2}\, Du\right), & p \in [1, \infty); \\ \langle D^2 u\, Du, Du \rangle, & p = \infty. \end{cases} \tag{1.4}$$

Solutions to (1.2) are intended in the viscosity sense for $p \in [1, \infty]$: the definition of a viscosity solution is recalled in Section 2, together with the meaning of $\partial u/\partial \nu$ in (1.2). In the special case when $p = 1$, we also consider solutions which are smooth near the boundary (see below). Concerning the usual Laplace operator $\Delta = \Delta_2$, in the fundamental work [17] Serrin showed, in particular, that if Ω is sufficiently smooth and there exists a solution $u \in C^2(\overline{\Omega})$ to the problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega; \\ u = 0,\ -\dfrac{\partial u}{\partial \nu} = q & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where q is any constant, then Ω is a ball centered at some $x_1 \in \mathbb{R}^n$ and u is given by $u(x) = \dfrac{r^2 - |x - x_1|^2}{2n}$, where r is the radius of Ω. Notice that problem (1.5) is invariant under translations, hence the point x_1 is arbitrary, while the radius r depends on the value of q. Buttazzo and Kawohl in [3] studied the corresponding overdetermined problem both for the infinity-Laplacian:

$$\begin{cases} -\Delta_\infty u = 1 & \text{in } \Omega; \\ u = 0,\ -\dfrac{\partial u}{\partial \nu} = q & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

and the normalized one:

$$\begin{cases} -\Delta_\infty^N u = 1 & \text{in } \Omega; \\ u = 0,\ -\dfrac{\partial u}{\partial \nu} = q & \text{on } \partial\Omega. \end{cases}$$

Banerjee and Kawohl [2], instead, considered the corresponding problem for the normalized p-Laplacian with $p \in (1, \infty)$. They proved that if $u \in C(\overline{\Omega})$ is a viscosity solution to

$$\begin{cases} -\Delta_p^N u = 1 & \text{in } \Omega; \\ u = 0,\ -\dfrac{\partial u}{\partial \nu} = q & \text{on } \partial\Omega, \end{cases}$$

then Ω is a ball. The result is also true if $p = 1$ provided that u is smooth near the boundary [11, Remark 4.3]. However, for $p = \infty$ it is generally false [3, Theorem 2]. Problem (1.6), where Ω contains the origin and u satisfies a non-constant Neumann condition at the boundary $\partial\Omega$, given by

$$-\frac{\partial u}{\partial \nu} = q(|x|), \tag{1.7}$$

has been studied in [8]. The corresponding problem for the classical p-Laplacian with finite $p > 1$, namely

$$\begin{cases} -\Delta_p u = 1 & \text{in } \Omega; \\ u = 0, \ -\dfrac{\partial u}{\partial \nu} = q(|x|) & \text{on } \partial\Omega, \end{cases}$$

was considered in [7, 6, 9]. The main results were basically focused on the geometry of Ω. This paper deals with a similar problem related to the normalized p-Laplacian, $p \in [1, \infty]$. Most of the notations we use are standard. By c_p we denote the constant

$$c_p = \begin{cases} \frac{p}{p+n-2}, & p \in [1, \infty); \\ 1, & p = \infty. \end{cases} \tag{1.8}$$

Our first result is the following:

Theorem 1.1. *Let $p \in [1, \infty]$ and let c_p be as above.*

1. *Suppose that the equation $q(r) - c_p r = 0$ possesses a unique solution $R \in [R_1, R_2]$, and $(q(r) - c_p r)(r - R) > 0$ for all $r \in [R_1, R_2] \setminus \{R\}$. Then problem (1.2) has a viscosity solution only in the special case when $R_1 = R = R_2$ (i.e., $\Omega = B(\bar{x}, R)$). If, instead, $R_1 < R_2$, then problem (1.2) has no viscosity solution.*

2. *Suppose that the function $\rho(r) = \frac{q(r)}{r}$ is strictly increasing. Then problem (1.2) has a viscosity solution only if Ω is a ball centered at $\bar{x}$.*

3. *If q is continuous, and if the equation $q(r) - c_p r = 0$ does not possess any solution, then problem (1.2) has no viscosity solution.*

The theorem is proved in Section 4 by means of a comparison argument. The result shows that the behavior of the normalized p-Laplacian with respect to the overdetermined problem (1.2) enjoys a continuity property at infinity: more precisely, since $c_\infty = 1 = \lim_{p\to\infty} c_p$, the statement for $p = \infty$ is readily obtained from the case when p is finite by just letting $p \to \infty$. By contrast, the classical (not normalized) p-Laplacian (1.4) exhibits a different behavior: indeed, a result similar to Theorem 1.1 valid for $p \in (1, \infty)$ has been proved in [9, Corollary 1.2]. There, the ratio $q(r)/r^{\frac{1}{p-1}}$ (which is obtained by letting $\varepsilon_0 = p - 1$) is required to be nondecreasing: such a ratio tends to $q(r)$ when $p \to \infty$. However, the corresponding result for the infinity-Laplacian which is found in [8, Theorem 1.1] requires monotonicity of $q(r)/r^{1/3}$. When q is constant, counterexamples are known: see [3, p. 241].

Unlike [9, Corollary 1.2], our Theorem 1.1 also applies to the special case when $p = 1$. In such a case, as mentioned before, we focus on solutions u which are smooth near the boundary. To this purpose, we adopt the following definition:

Definition 1.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class C^2, and for $\varepsilon > 0$ define $\Omega_\varepsilon = \{\, x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < \varepsilon \,\}$. We say that a viscosity solution to (1.2) is a *smooth solution near the boundary* if:

1. $u \in C^2(\overline{\Omega}_\varepsilon)$ for some $\varepsilon > 0$;
2. $Du \neq 0$ in $\overline{\Omega}_\varepsilon$;
3. the equation $-\Delta_p^N u = 1$ is satisfied in the classical sense in $\overline{\Omega}_\varepsilon$;
4. the boundary conditions in (1.2) hold pointwise.

When $p = 1$, a solution u which is smooth near the boundary satisfies the equation

$$(n-1)\,|Du|\,H(x) = 1 \quad \text{in } \overline{\Omega}_\varepsilon, \tag{1.9}$$

where $H(x)$ is the mean curvature of the level surface $u =$ constant passing through the point x (see [5, (14.102)] and [11, Remark 4.3]). Note that if (1.2) has a solution u which is smooth near the boundary, then the surface $\partial\Omega$ must have a positive mean curvature $H(x)$ as a consequence of (1.9). In such a case we may prescribe the Neumann condition by means of a function $q(|x-\bar{x}|,\, H(x))$ that not only depends on the distance from $x \in \partial\Omega$ to some fixed point $\bar{x} \in \Omega$, but is also allowed to depend on the mean curvature $H(x)$ of $\partial\Omega$. More precisely, we consider the overdetermined problem

$$\begin{cases} -\Delta_1^N u = 1 & \text{in } \Omega; \\ u = 0, \ -\dfrac{\partial u}{\partial \nu} = q(|x-\bar{x}|,\, H(x)) & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where $q \colon [R_1, R_2] \times (0, \infty) \to (0, \infty)$ is a prescribed, positive function. In the case when $q(r, h)$ is independent of h, problem (1.10) clearly reduces to (1.2). We have:

Theorem 1.3. *Let Ω be a bounded open set of class C^2. Choose $\bar{x} \in \Omega$ and define R_1, R_2 as in (1.1). Consider a positive function $q(r, h)$ such that*

1. *$q(r, h)$ is monotone non-decreasing in h for every $r \in [R_1, R_2]$;*
2. *the ratio $\frac{q(r,1/r)}{r}$ is strictly increasing.*

If problem (1.10) has a solution u which is smooth near the boundary, then $R_1 = R_2$ (i.e., $\Omega = B(\bar{x}, R_1)$).

Example 1.4. If the function q has the special form $q(r, h) = r^\alpha\, h^\beta$, then $q(r,\, 1/r) = r^{\alpha-\beta}$ and the assumptions in the theorem are satisfied provided that $\alpha - 1 > \beta \geq 0$.

Remark 1.5. Several symmetry results were obtained in [8] for overdetermined problems related to the equation $\Delta_\infty u = 0$. As mentioned in [1, Remark 2.2, p. 599], "there is no difference between the two resulting equations (in the viscosity sense) when the right-hand side $f \equiv 0$". Hence all results in [8] concerning solutions to $\Delta_\infty u = 0$ also hold for normalized infinity-harmonic functions, i.e., for solutions of $\Delta_\infty^N u = 0$.

The paper is organized as follows: in Section 2 we define viscosity solutions, and in Section 3 we recall some preliminary lemmas which will be used to prove our main results in Section 4.

2. Viscosity Solutions

Let $p \in [1, \infty]$. As usual (see for instance [2, 12, 16]), if $u \in C(\Omega)$ is twice differentiable at $x_0 \in \Omega$ and if $Du(x_0) \neq 0$, we define the upper and lower normalized p-Laplacian of u at x_0, respectively by $\Delta_p^+ u(x_0) = \Delta_p^- u(x_0) = \Delta_p u(x_0)$, where $\Delta_p u$ is given by (1.3). If, instead, $Du(x_0) = 0$, we denote by $\lambda_{\min} = \lambda_1 \leq \ldots \leq \lambda_n = \lambda_{\max}$ the eigenvalues of the Hessian matrix $D^2u(x_0)$ and define

$$\Delta_p^+ u(x_0) = \begin{cases} \frac{p-1}{p}\lambda_1 + \frac{1}{p}\sum_{i=2}^{n}\lambda_i, & p \in [1,2]; \\ \frac{p-1}{p}\lambda_n + \frac{1}{p}\sum_{i=1}^{n-1}\lambda_i, & p \in (2,\infty); \\ \lambda_n, & p = \infty, \end{cases}$$

and

$$\Delta_p^- u(x_0) = \begin{cases} \frac{p-1}{p}\lambda_n + \frac{1}{p}\sum_{i=1}^{n-1}\lambda_i, & p \in [1,2]; \\ \frac{p-1}{p}\lambda_1 + \frac{1}{p}\sum_{i=2}^{n}\lambda_i, & p \in (2,\infty); \\ \lambda_1, & p = \infty. \end{cases}$$

In the case when $p < \infty$, the definitions above may equivalently be rewritten as follows (cf. [12, p. 177]):

$$\Delta_p^+ u(x_0) = \begin{cases} \frac{p-2}{p}\lambda_{\min} + \frac{1}{p}\Delta u, & p \in [1,2]; \\ \frac{p-2}{p}\lambda_{\max} + \frac{1}{p}\Delta u, & p \in (2,\infty). \end{cases}$$

and

$$\Delta_p^- u(x_0) = \begin{cases} \frac{p-2}{p}\lambda_{\max} + \frac{1}{p}\Delta u, & p \in [1,2]; \\ \frac{p-2}{p}\lambda_{\min} + \frac{1}{p}\Delta u, & p \in (2,\infty). \end{cases}$$

For $f\colon \Omega \to \mathbb{R}$ we will give the definition of viscosity solution to the PDE

$$-\Delta_p^N u(x) = f(x) \text{ in } \Omega. \tag{2.1}$$

We denote by $USC(\Omega)$ and $LSC(\Omega)$, respectively, the spaces of *upper semicontinuous* and *lower semicontinuous* real-valued functions on Ω. Furthermore, for any $x_0 \in \Omega$ and $\varphi \in C^2$ in a neighborhood of x_0 we write $u \prec_{x_0} \varphi$ (respectively, $u \succ_{x_0} \varphi$) if the difference $u - \varphi$ has a local maximum (minimum) at x_0. The notation extends in an obvious way to the case when u is also defined at some $x_0 \in \partial\Omega$.

Definition 2.1.

1. $u \in USC(\Omega)$ is called a viscosity subsolution (or simply subsolution) of the PDE (2.1) in Ω if for every $x_0 \in \Omega$, and for every $\varphi \in C^2(\Omega)$ satisfying $u \prec_{x_0} \varphi$, we have
$$-\Delta_p^+ \varphi(x_0) \le f(x_0).$$
In this case we write $-\Delta_p^N u(x) \le f(x)$ in Ω.

2. $u \in LSC(\Omega)$ is called a viscosity supersolution (or simply supersolution) of the PDE (2.1) in Ω if for every $x_0 \in \Omega$, and for every $\varphi \in C^2(\Omega)$ satisfying $u \succ_{x_0} \varphi$, we have
$$-\Delta_p^- \varphi(x_0) \ge f(x_0).$$
In this case we write $-\Delta_p^N u(x) \ge f(x)$ in Ω.

3. $u \in C(\Omega)$ is called viscosity solution (or simply solution) of the PDE (2.1) in Ω, if u is both a subsolution and a supersolution.

We now consider a boundary datum $g(x)$ and define a viscosity solution of the Dirichlet problem
$$\begin{cases} -\Delta_p^N u &= f, \quad \text{in } \Omega \\ \qquad u &= g, \quad \text{on } \partial\Omega \end{cases} \tag{2.2}$$
as follows. We also give a meaning to the boundary condition (1.7) (see [2, Remark 1.2]).

Definition 2.2.

1. $u \in USC(\overline{\Omega})$ is a subsolution of (2.2) if u is a subsolution of $-\Delta_p^N u = f$ in Ω and satisfies $u \le g$ on $\partial\Omega$.

2. $u \in LSC(\overline{\Omega})$ is a supersolution of (2.2) if u is a supersolution of $-\Delta_p^N u = f$ in Ω and satisfies $u \ge g$ on $\partial\Omega$.

3. $u \in C(\overline{\Omega})$ is a solution of (2.2) if u is both a subsolution and supersolution of (2.2).

4. A solution u of (2.2) satisfies the boundary condition (1.7) if for every $x_0 \in \partial\Omega$ and every $\varphi \in C^2$ in a neighborhood of x_0 we have: if $u \prec_{x_0} \varphi$ then $-\frac{\partial\varphi}{\partial\nu} \ge q(|x_0|)$; if, instead, $u \succ_{x_0} \varphi$ then $-\frac{\partial\varphi}{\partial\nu} \le q(|x_0|)$.

Remark 2.3. (i) A smooth function u with $Du \neq 0$ in Ω satisfying (2.2) in the classical sense is also a viscosity solution. (ii) The normalized p-Laplacian is a nonlinear operator for $p \neq 2$. Nevertheless, if $u \in USC(\Omega)$ is a subsolution of (2.2), then $v = -u \in LSC(\Omega)$ and v is a supersolution of the Dirichlet problem
$$\begin{cases} -\Delta_p^N v &= -f, \quad \text{in } \Omega \\ \qquad v &= -g, \quad \text{on } \partial\Omega \end{cases} \tag{2.3}$$
Similarly, if $u \in LSC(\Omega)$ is a supersolution of (2.2), then $v = -u \in USC(\Omega)$ and v is a subsolution of (2.3).

3. Well-posedness, comparison principle, radial solutions

The proof of Theorem 1.1 is based on the comparison principle and the explicit expression of the radial solutions which are recalled in this section.

Lemma 3.1 (Comparison principle). *Let $p \in [1, \infty]$, let $\Omega \subset \mathbb{R}^n$ be a bounded (possibly disconnected) open set, and $f \in C(\Omega)$. We assume that $f \neq 0$ in Ω and does not change sign. Let $u, v \in C(\overline{\Omega})$ satisfy*

$$-\Delta_p^N u \leq f(x) \quad \textit{and} \quad -\Delta_p^N v \geq f(x), \quad x \in \Omega.$$

If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω. The result also holds if $p = \infty$ and $f \equiv 0$ in Ω.

Proof. The first claim follows from [12, Theorem 5], taking Remark 2.3 (*ii*) into account. The case when $p = \infty$ and $f \equiv 0$ follows from Jensen's fundamental result [10, Theorem 3.11] by virtue of the equivalence between infinity-harmonicity and normalized infinity-harmonicity (Remark 1.5). It is also a special case of [15, Theorem 2.5]. □

Uniqueness for problem (2.2) is a consequence of the comparison principle stated above. Recall that uniqueness lacks in the case when $p = 1$ and $f \equiv 0$, and a famous example was given in [18, Section 3.6] (see Fig. 1 below and [11, Fig. 2]).

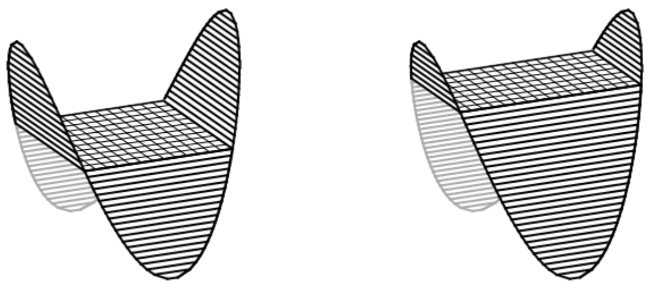

Figure 1: A least-gradient and another solution of $\Delta_1^N u = 0$

Lemma 3.2 (Radial solution). *Let $p \in [1, \infty]$ and let c_p be as in (1.8). For every $R > 0$ the function*

$$u_R(x) = \frac{c_p}{2} \left(R^2 - |x - \bar{x}|^2\right) \tag{3.1}$$

is the unique solution of the problem

$$\begin{cases} -\Delta_p^N u = 1, & x \in B(\bar{x}, R) \\ u = 0, & x \in \partial B(\bar{x}, R) \end{cases} \tag{3.2}$$

Moreover $-\dfrac{\partial u_R}{\partial \nu} = c_p\, R$, *where* $\nu = R^{-1}\,(x - \bar{x})$ *is the outer normal at* $x \in \partial B(\bar{x}, R)$.

Proof. The solution is unique by Lemma 3.1. For $p \in (1, \infty)$, the representation (3.1) is found in [11, p. 20]. If $p = \infty$, the result follows from [3, pag. 243] with the observation that $d(x, \partial\Omega) = R - |x - \bar{x}|$ and by letting $a = R$. We give details for the case when $p = 1$, with reference to Definition 2.2 and Remark 2.3 (i). By differentiation of (3.1), $Du_R(x) = -\frac{1}{n-1}(x - \bar{x})$ and $D^2 u_R(x) = -\frac{1}{n-1}\, I$, where I denotes the identity matrix. Hence $\lambda_{\min} = \lambda_{\max} = -\frac{1}{n-1}$. If $x \neq \bar{x}$, using (1.3) we see that the equation is satisfied in the classical sense. Otherwise $Du_R(\bar{x}) = 0$ and we have $\Delta_1^+\, u_R(\bar{x}) = \Delta_1^-\, u_R(\bar{x}) = \Delta u_R - \lambda_{\min} = -1$. Thus, $-\Delta_1^N\, u_R(x) = 1$ in all of $B(\bar{x}, R)$ and, of course, $u_R(x) = 0$ on $\partial B(\bar{x}, R)$. □

In order to prove Theorem 1.1, we also need to establish the positivity of the solution to the following Dirichlet problem:

$$\begin{cases} -\Delta_p^N u &= 1, \quad x \in \Omega \\ \quad\;\; u &= 0, \quad x \in \partial\Omega \end{cases} \tag{3.3}$$

Lemma 3.3. *Let* $p \in [1, \infty]$. *Any solution of* (3.3) *is positive in* Ω.

Proof. Notice that $v = 0$ satisfies $-\Delta_p^N v \leq 1$ in Ω and $v = 0$ on $\partial\Omega$. As v is a subsolution, by the comparison principle (Lemma 3.1) we have $0 \leq u$ in Ω. Consider $B(x_0, R)$ contained in Ω, and consider the solution u_R of problem (3.2) in $B(x_0, R)$. Since $u_R(x) = 0$ on $\partial B(x_0, R)$ and $u \geq 0$ on $\partial B(x_0, R)$, Lemma 3.1 implies $0 < u_R(x) \leq u(x)$ in $B(x_0, R)$. Since x_0 is arbitrary in Ω, we have $u > 0$ in Ω. □

By choosing $B(x_0, R) \subset \Omega$ so that $\partial B(x_0, R) \cap \partial\Omega \neq \emptyset$, we immediately obtain the following boundary-point lemma (see also [2, Lemma 2.3]):

Lemma 3.4 (Hopf)**.** *Let* $p \in [1, \infty]$. *Suppose* Ω *satisfies an interior sphere condition at every boundary point, and let* $u \in C(\overline{\Omega})$ *be a viscosity solution of* (3.3). *Then for all* $x \in \partial\Omega$ *we have*

$$\limsup_{t \to 0^+} \frac{u(x) - u(x - \nu t)}{t} < 0.$$

We conclude this section by quoting some existence and regularity results.

Lemma 3.5 (Existence)**.** *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded (possibly disconnected) open set, and let* $f \in C(\Omega)$ *and* $g \in C(\partial\Omega)$. *The Dirichlet problem* (2.2) *has a viscosity solution provided that one of the following conditions hold:*

1. $p \in (n, \infty]$, *and* f *does not vanish and does not change sign in* Ω.

2. $p = \infty$ *and* f *is bounded in* Ω.

Proof. If $p = \infty$, Claim (1) follows from [14, Theorem 1.8] using Remark 2.3 *(ii)*. The claim was extended to $p > n$ in [13, Corollary 4.5]: indeed, assumption (3.19) of [13] reduces to $p > n$. Claim (2) is a special case of [16, Theorem 6.1] corresponding to $F(x) = |x|$. □

Remark 3.6. Concerning regularity, global $C^{1,\beta}$-regularity is proved in [2, Theorem 4.2] for $p \in (1, \infty)$. In the case when $p = \infty$ it is known that the viscosity solution to (3.3) is locally Lipschitz continuous in Ω: see, for instance, [19, Lemma 5.3] with $F(x) = |x|$. See also [4] for further details.

Let us point out that the existence, uniqueness and regularity results recalled above allow to construct the following counterexample, which mimics the one in [8, p. 242]. The example is valid for $p \in (n, \infty)$ and shows that if we let the function q in (1.2) be arbitrary, i.e., if we drop every assumption on q, then problem (1.2) may well be solvable even though the domain Ω is not a ball.

Example 3.7. Let $\Omega \subset \mathbb{R}^2$ be an ellipse in canonical position, with semi-axes $a < b$, and let $p \in (2, \infty)$. Thus, there exists a unique solution $u = u_0$ of the Dirichlet problem (3.3). Note that the problem is invariant under reflection with respect to each axis: i.e., if we define $v(x_1, x_2) = u_0(\pm x_1, \pm x_2)$ for whatever choice of the signs $\pm$, we always find $\Delta_p^N v = \Delta_p^N u_0$. But since the solution of problem (3.3) is unique, we must have $u_0 \equiv v$, hence $u_0(x_1, x_2) = u_0(\pm x_1, \pm x_2)$. Furthermore, since u_0 is differentiable up to the boundary, the last equality implies that $|\nabla u_0(x_1, x_2)| = |\nabla u_0(\pm x_1, \pm x_2)|$ for every $(x_1, x_2) \in \partial\Omega$. Now observe that for every $r \in [a, b]$ the set F_r of all $x \in \partial\Omega$ such that $|x| = r$ is invariant under reflection with respect to each axis, and therefore it is legitimate to define $q(r) = |\nabla u_0(x)|$ by choosing any $x = (x_1, x_2) \in F_r$ (because the value of $q(r)$ is independent of the choice of $x \in F_r$). Then, with this particular function q, problem (3.3) is solvable (and has the solution u_0) although Ω is not a disc.

4. Existence and nonexistence of solutions

In this section we prove our main results.

Proof of Theorem 1.1. We follow the same guidelines as in [8]. (1) Let $\Omega = B(\bar{x}, R)$, where R is the solution of $q(r) - c_p r = 0$. By Lemma 3.2, $u_R(x) = \frac{c_p}{2}(R^2 - |x - \bar{x}|^2)$ is the solution to the problem (1.2) in $B(\bar{x}, R)$. Therefore, the solution to (1.2) exists in $B(\bar{x}, R)$.
On the other hand, assume u is the solution to (1.2). Define $u_i(x) = \frac{c_p}{2}(R_i^2 - |x - \bar{x}|^2)$ for $i = 1, 2$. Then u_i is the solution to (3.3) in the ball $B(\bar{x}, R_i)$. Since $u \geq 0$ on $\partial B(\bar{x}, R_1)$ (see Lemma 3.3) and $u_1 = 0$ on $\partial B(\bar{x}, R_1)$, we have $u_1 \leq u$ on $\partial B(\bar{x}, R_1)$. By Lemma 3.1, $u_1 \leq u$ in $B(\bar{x}, R_1)$. Since $u_2 \geq 0$ on $\partial\Omega$ and $u = 0$ on $\partial\Omega$, we have $u = 0 \leq u_2$ on $\partial\Omega$ and hence $u \leq u_2$ in Ω by Lemma 3.1. Let $P_1 \in \partial B(\bar{x}, R_1) \cap \partial\Omega$. Then the outer normal ν to $\partial\Omega$ at P_1 equals $\dfrac{P_1 - \bar{x}}{R_1}$, the

outer normal to $B(\bar{x}, R_1)$. Since u_1 is a smooth function satisfying $u \succ_{P_1} u_1$, by Definition 2.2 (4) we may write

$$c_p\, R_1 = -\frac{\partial u_1}{\partial \nu}(P_1) \leq q(R_1). \tag{4.1}$$

Let $P_2 \in \partial B(\bar{x}, R_2) \cap \partial\Omega$. Then the outer normal ν to $\partial\Omega$ at P_2 equals $\dfrac{P_2 - \bar{x}}{R_2}$, the outer normal to $B(\bar{x}, R_2)$. Furthermore $u \prec_{P_2} u_2$. Hence

$$q(R_2) \leq -\frac{\partial u_2}{\partial \nu}(P_2) = c_p\, R_2. \tag{4.2}$$

Inequalities (4.1) and (4.2) may be rephrased as

$$q(R_1) - c_p\, R_1 \geq 0 \text{ and } q(R_2) - c_p\, R_2 \leq 0. \tag{4.3}$$

Since the equation $q(r) - c_p\, r = 0$ has the unique solution R in $[R_1, R_2]$ and $q(r) - c_p\, r < 0$ for $r < R$, we have $R = R_1$ and again since $q(r) - c_p\, r > 0$ for $r > R$, we have $R = R_2$. Therefore, $R_1 = R = R_2$, which is $\Omega = B(\bar{x}, R)$.

(2) If (1.2) has a solution, then we obtain (4.1) and (4.2), hence $\rho(r)$ satisfies

$$\rho(R_2) = \frac{q(R_2)}{R_2} \leq c_p \leq \frac{q(R_1)}{R_1} = \rho(R_1).$$

Since ρ is strictly increasing, we must have $R_1 = R_2$ and the result follows.

(3) Suppose that (1.2) has a solution. Then, by (4.3), and since q is continuous, we have $q(R) - c_p\, R = 0$ at some point $R \in [R_1, R_2]$, contradicting the assumption. Therefore problem (1.2) must be unsolvable. □

Proof of Theorem 1.3. The result follows by exploiting (1.9). As mentioned in the Introduction, we have $H(x) > 0$ for every $x \in \partial\Omega$. Take $P_i \in \partial B(\bar{x}, R_i) \cap \partial\Omega$, $i = 1, 2$ as in the proof of Theorem 1.1, and recall that the mean curvature of the sphere $\partial B(\bar{x}, R_i)$ is $1/R_i$. Hence we may write $H(P_1) \leq 1/R_1$ and $H(P_2) \geq 1/R_2$. This and (1.9) imply

$$q(R_1,\, 1/R_1) \geq q(R_1, H(P_1)) = |Du(P_1)| = \frac{1}{(n-1)\, H(P_1)} \geq \frac{R_1}{n-1} = c_1\, R_1$$
$$q(R_2,\, 1/R_2) \leq q(R_2, H(P_2)) = |Du(P_2)| = \frac{1}{(n-1)\, H(P_2)} \leq \frac{R_2}{n-1} = c_1\, R_2$$

because $q(r, h)$ is monotone non-decreasing in h. The inequalities above imply

$$\frac{q(R_1,\, 1/R_1)}{R_1} \geq c_1 \geq \frac{q(R_2,\, 1/R_2)}{R_2}.$$

Then, since the ratio $\frac{q(r,1/r)}{r}$ is strictly increasing in r, we must have $R_1 = R_2$ as claimed. □

Acknowledgements. The authors are partially supported by the research project *Analysis of PDEs in connection with real phenomena*, CUP F73C22001130007, funded by Fondazione di Sardegna (annuity 2021). L. Cadeddu and A. Greco are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). L. Cadeddu's research has been accomplished within the UMI Group TAA *Approximation Theory and Applications*. The second author is particularly grateful to Juan Manfredi for having driven his attention to the relevant papers [12, 18] and for the helpful hints given on the occasion of the meeting "Recent Progress in Geometric Analysis and PDEs" held in Bologna, Italy, from September 16 to 18, 2021, to celebrate the 70$^{\text{th}}$ birthday of Bruno Franchi.

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Received: 6 March 2024/Accepted: 11 March 2024/Published online: 27 March 2024

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