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Existence of renormalized solutions for some non-coercive anisotropic elliptic problems with Neumann boundary condition

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Abstract. *The aim of this work is to prove the existence of renormalized solutions for the following anisotropic elliptic problem with Neumann boundary conditions*

$$\begin{cases} -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u) + H(x,u,\nabla u) + \alpha(x)|u|^{r-1}u = f & \text{in } \Omega, \\ \sum_{i=1}^{N} a_i(x,u,\nabla u).n_i = g & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, the data f belong to $L^1(\Omega)$ and $g \in L^1(\partial\Omega)$, and the Carathéodory functions $a_i(x,s,\xi)$ and $H(x,s,\xi)$ verify some nonstandard conditions.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N \geq 2)$, Boccardo et al. have studied in [10] the existence and uniqueness of solutions for the quasilinear elliptic equation

$$\begin{cases} -\mathrm{div}\,(a\,(x,u,\nabla u)) + \lambda|u|^{p-2}u = f & \text{in} \quad \Omega, \\ u \in W_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where $f \in W_0^{-1,p'}(\Omega)$, $\lambda \geq 0$. Moreover, the uniqueness result holds true under some additional conditions on p and λ, and fails for other conditions. Concerning the existence of renormalized solutions for elliptic equations with L^1-data, we refer the reader to [22], and for Radon measure-data to the paper [12].

The notion of renormalized solutions was introduced by Lions and DiPerna [13] for the study of Boltzmann equation. It was then adapted by Boccardo et al. [11], Lions and Murat [19] and Murat [22, 23] to nonlinear elliptic problems and by Lions [18] to evolution problems in fluid mechanics. At the same time the equivalent notion of entropy solutions have been developed independently by Bénilan et al. [7] for the study of nonlinear elliptic problems, we refer the reader to [9].

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In [5], Ben Cheikh Ali and Guibé have treated some degenerate quasilinear elliptic problem of the type

$$\begin{cases} \lambda(x,u) - \text{div } (a(x,\nabla u) + \Phi(x,u)) = f & \text{in} \quad \Omega, \\ (a(x,\nabla u) + \Phi(x,u)) \cdot \vec{n} = 0 & \text{on} \quad \Gamma_n, \\ u = \ 0 & \text{on} \quad \Gamma_d, \end{cases}$$

where $Au = -\text{div } (a(x,\nabla u))$ is a Leray–Lions type operator, $\lambda(x,s)$ and $\Phi(x,s)$ are two Carathéodory functions. The authors have proved existence and uniqueness of renormalized solution for this problem under some additional conditions.

Furthermore, Betta et al. have studied in [8] the existence of renormalized solution to the Neumann boundary value problem

$$\begin{cases} -\Delta_p u - div(c(x)|u|^{p-2}u) = f & \text{in} \ \ \Omega, \\ (|\nabla u|^{p-2}\nabla u + c(x)|u|^{p-2}u) \cdot \vec{n} = 0 & \text{on} \ \ \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$ with Lipschitz boundary, $1 < p \leq N$, and $\vec{n}$ is the outer unit normal to $\partial\Omega$, the datum f belong to $L^1(\Omega)$ and satisfies the compatibility condition $\displaystyle\int_\Omega f = 0$.

A significant interest and effort has been devoted in recent years to the study of the anisotropic elliptic and parabolic problems. This interest especially comes from their applications to the mathematical modeling of some physical processes in an anisotropic continuous medium (see [3, 24]).

In some sense, our paper is a natural continuation of the studies of these class of problems, in which we will study by using the variational method and some a priori estimates, the existence of at least one renormalized solutions to the following nonlinear and noncoercive elliptic problem

$$\begin{cases} Au + H(x,u,\nabla u) + \alpha(x)|u|^{r-1}u = f \quad \text{in } \Omega, \\ \displaystyle\sum_{i=1}^N a_i(x,u,\nabla u).n_i = g \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω be a regular bounded domain of $\mathbb{R}^N$, and $Au = -\displaystyle\sum_{i=1}^N a_i(x,u,\nabla u)$ is a Leray–Lions operator acted from $W^{1,\vec{p}}(\Omega)$ into its dual. The Carathéodory function $H(x,s,\xi)$ verifying only some growth condition, where the data f belong to $L^1(\Omega)$ and $g \in L^1(\partial\Omega)$.

The main difficulties in studying existence of solutions for our problem are due to the presence of a lower order term, the lower summability of the datum f and the boundary Neumann condition. To overcome these difficulties, we proved our results by considering the term $\alpha(x)|u|^{r-1}u$ for the equation (1.2) with $\alpha(.) \in L^\infty(\Omega)$ is a positive function, such that $\alpha(x) \geq \alpha_0$, $a.e \ in \ \Omega$ and $1 < r \leq \underline{p} - 1$.

This paper is organized as follows. In Section 2, we recall some definitions and properties concerning the anisotropic Sobolev spaces. In Section 3 we introduce the assumptions for which our problem has at least one solution. Section 4 is dedicated to study the existence of weak solutions for our equation with right-hand side $|F(x)| \leq C_0$ almost everywhere in Ω, and $G(x) \in L^\infty(\partial\Omega)$. In the last section, we establish the existence of renormalized solutions for the non-coercive elliptic equation (1.2) with the right-hand side $f(x) \in L^1(\Omega)$ and $g(x) \in L^1(\partial\Omega)$. Finally we will present some example.

2. Preliminaries

Let Ω be an open bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$. Let $p_1, \ldots, p_N$ be N real constants numbers, with $1 < p_i < \infty$ for $i = 1, \ldots, N$. We denote

$$\vec{p} = (1, p_1, \ldots, p_N), \qquad D^0 u = u \qquad \text{and} \qquad D^i u = \frac{\partial u}{\partial x_i} \qquad \text{for} \qquad i = 1, \ldots, N.$$

We set

$$\underline{p} = \min\{p_1, p_2, \ldots, p_N\} \qquad \text{and} \qquad \underline{p}^+ = \max\{p_1, p_2, \ldots, p_N\}.$$

We define the anisotropic Sobolev space $W^{1,\vec{p}}(\Omega)$ as follows:

$$W^{1,\vec{p}}(\Omega) = \left\{u \in W^{1,1}(\Omega) \quad \text{such that} \quad D^i u \in L^{p_i}(\Omega) \quad \text{for} \quad i = 1, 2, \ldots, N\right\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)}. \tag{2.1}$$

The space $\left(W^{1,\vec{p}}(\Omega), \|\cdot\|_{1,\vec{p}}\right)$ is a separable and reflexive Banach space (cf. [21]). Let us recall the Poincaré and Sobolev type inequalities in the anisotropic Sobolev space.

Proposition 2.1 (cf. [17], [25])**.** *Let $u \in W^{1,\vec{p}}(\Omega)$, we have*
(i) *Poincaré–Wirtinger inequality: there exists a constant $C_p > 0$, such that*

$$\|u - m(u)\|_{L^{p_i}(\Omega)} \leq C_p \sum_{i=1}^{N} \|D^i u\|_{L^{p_i}(\Omega)},$$

where

$$m(u) = \frac{1}{|\Omega|} \int_\Omega |u(x)|\, dx,$$

is the mean-value of u.
(ii) *Sobolev inequality: there exists an other constant $C_s > 0$ such that*

$$\|u - m(u)\|_q \leq \frac{C_s}{N} \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{p_i},$$

where

$$\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i} \qquad \textit{and} \qquad \begin{cases} q = \overline{p}^* = \dfrac{N\overline{p}}{N-\overline{p}} & \textit{if} \quad \overline{p} < N, \\ q \in [1, +\infty[& \textit{if} \quad \overline{p} \geq N. \end{cases}$$

Lemma 2.2. *Let* Ω *be a bounded open subset in* $\mathbb{R}^N$ $(N \geq 2)$, *we set*

$$s = \max(q,\ \underline{p}^+),$$

then, we have the following embedding :

- *if* $\overline{p} < N$ *then the embedding* $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *is compact for any* $r \in [1, s[$,
- *if* $\overline{p} = N$ *then the embedding* $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^r(\Omega)$ *is compact for any* $r \in [1, +\infty[$,
- *if* $\overline{p} > N$ *then the embedding* $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap C^0(\overline{\Omega})$ *is compact.*

The proof of this lemma follows from the Proposition 2.1.

Definition 2.3. Let $k > 0$, we consider the truncation function $T_k(\cdot)\colon \mathbb{R} \longmapsto \mathbb{R}$, given by

$$T_k(s) = \begin{cases} s & \text{if} \quad |s| \leq k, \\ k\dfrac{s}{|s|} & \text{if} \quad |s| > k, \end{cases}$$

and we define

$$T^{1,\vec{p}}(\Omega) := \{u\colon \Omega \mapsto \mathbb{R} \text{ measurable, such that } T_k(u) \in W^{1,\vec{p}}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.4. *Let* $u \in T^{1,\vec{p}}(\Omega)$. *For any* $i \in \{1, \dots, N\}$, *there exists a unique measurable function* $v_i\colon \Omega \mapsto \mathbb{R}$ *such that*

$$\forall k > 0 \qquad D^i T_k(u) = v_i . \chi_{\{|u|<k\}} \qquad a.e. \quad x \in \Omega,$$

where χ_A *denotes the characteristic function of a measurable set* A. *The functions* v_i *are called the weak partial derivatives of* u *and are still denoted* $D^i u$. *Moreover, if* u *belongs to* $W^{1,1}(\Omega)$, *then* v_i *coincides with the standard distributional derivative of* u, *that is,* $v_i = D^i u$.

The proof of the Proposition 2.4 follows the usual techniques developed in [7] for the case of Sobolev spaces. For more details concerning the anisotropic Sobolev spaces, we refer the reader to [2, 6, 14, 15].

Moreover, we introduce the set $T^{1,\vec{p}}_{tr}(\Omega)$ as a subset of $T^{1,\vec{p}}(\Omega)$ for which a generalized notion of trace may be defined (see also [1] for the case of constant exponent). More precisely, $T^{1,\vec{p}}_{tr}(\Omega)$ is the set of function u in $T^{1,\vec{p}}(\Omega)$, such that: there exists a sequence $(u_n)_n$ in $W^{1,\vec{p}}(\Omega)$ and a measurable function v on $\partial\Omega$ verifying

(a) $u_n \longrightarrow u$ a.e. in Ω,

(b) $D^i T_k(u_n) \longrightarrow D^i T_k(u)$ in $L^1(\Omega)$ for every $k > 0$.

(c) $u_n \longrightarrow v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [1].

Let $u \in W^{1,\vec{p}}(\Omega)$, the trace of u on $\partial\Omega$ will be denoted by $\tau(u)$. For any $u \in T^{1,\vec{p}}_{tr}(\Omega)$, the trace of u on $\partial\Omega$ will be denoted by $tr(u)$ or u, the operator $tr(\cdot)$ satisfied the following properties

(i) if $u \in T^{1,\vec{p}}_{tr}(\Omega)$, then $\tau(T_k(u)) = T_k(tr(u))$ for any $k > 0$.

(ii) if $\varphi \in W^{1,\vec{p}}(\Omega)$, then, for any $u \in T^{1,\vec{p}}_{tr}(\Omega)$, we have $u - \varphi \in T^{1,\vec{p}}_{tr}(\Omega)$ and $tr(u-\varphi) = tr(u) - \tau(\varphi)$.

In the case where $u \in W^{1,\vec{p}}(\Omega)$, $tr(u)$ coincides with $\tau(u)$. Obviously, we have

$$W^{1,\vec{p}}(\Omega) \subset T^{1,\vec{p}}_{tr}(\Omega) \subset T^{1,\vec{p}}(\Omega).$$

Lemma 2.5 (see [16, Theorem 13.47])**.** *Let $(u_n)_n$ be a sequence in $L^1(\Omega)$ and $u \in L^1(\Omega)$ such that:*

(i) *$u_n \to u$ a.e. in Ω,*

(ii) *$u_n \geq 0$ and $u \geq 0$ a.e. in Ω,*

(iii) $\displaystyle\int_\Omega u_n\, dx \to \int_\Omega u\, dx,$

then $u_n \to u$ in $L^1(\Omega)$.

3. Essential assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$. We consider the strongly nonlinear anisotropic elliptic problem

$$\begin{cases} Au + H(x,u,\nabla u) + \alpha(x)|u|^{r-1}u = f & \text{in } \Omega, \\ a(x,u,\nabla u).\overrightarrow{n} = g & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

with $0 < r \leq \underline{p} - 1$, and the data $f(\cdot)$ is assumed to be a measurable function in $L^1(\Omega)$ and $g \in L^1(\partial\Omega)$, and the positive function $\alpha(\cdot) \in L^\infty(\Omega)$ such that $\alpha(x) \geq \alpha_0 > 0$ a. e. in Ω.

The Leray–Lions operator A acted from $W^{1,\vec{p}}(\Omega)$ into its dual, defined by

$$Au = -\sum_{i=1}^{N} D^i a_i(x,u,\nabla u),$$

where $a_i\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \longmapsto \mathbb{R}$ are Carathéodory functions for $i = 1, \dots, N$ (measurable with respect to x in Ω for every (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$, and continuous with respect to (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$ for almost every x in Ω), which satisfy the following conditions:

$$|a_i(x, s, \xi)| \leq \beta(K_i(x) + |s|^{p_i - 1} + |\xi_i|^{p_i - 1}) \qquad \text{for} \quad i = 1, \dots, N, \tag{3.2}$$

where the nonnegative functions $K_i(\cdot)$ are assumed to be in $L^{p_i'}(\Omega)$ for $i = 1, \dots, N$, with $\beta > 0$.

$$a_i(x, s, \xi)\xi_i \geq b(|s|)|\xi_i|^{p_i} \quad \text{with} \quad \frac{b_0}{(1 + |s|)^\lambda} \leq b(|s|) \quad \text{for any} \quad s \in \mathbb{R}, \tag{3.3}$$

such that $b(|\cdot|)\colon \mathbb{R}^+ \mapsto \mathbb{R}^+$ is a decreasing function, with $b_0 > 0$ and $0 \leq \lambda < \underline{p} - 1$.

$$\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi_i - \xi_i') > 0 \qquad \text{for} \quad \xi_i \neq \xi_i', \tag{3.4}$$

for almost every $x \in \Omega$ and all (s, ξ) in $\mathbb{R} \times \mathbb{R}^N$.

The strongly nonlinear term $H(x, s, \xi)$ is a Carathéodory function that verifying the growth condition:

$$|H(x, s, \xi)| \leq f_0(x) + \sum_{i=1}^{N} d(|s|)|\xi_i|^{p_i}, \tag{3.5}$$

where $f_0(\cdot) \in L^1(\Omega)$, and $d(|\cdot|)\colon \mathbb{R} \mapsto \mathbb{R}^+$ is a continuous decreasing function, such that $\dfrac{d(|\cdot|)}{b(|\cdot|)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. As a consequence of (3.3) and the continuity of the function $a_i(x, s, \cdot)$ with respect to ξ, we have

$$a_i(x, s, 0) = 0.$$

We are going now to recall the following technical Lemma, useful to prove our main results.

Lemma 3.1 (see [4]). *Let $k > 0$, assuming that (3.2)–(3.4) hold true, and let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $W^{1,\vec{p}}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,\vec{p}}(\Omega)$ and*

$$\begin{aligned} &\int_\Omega (|u_n|^{\underline{p}-2} u_n - |u|^{\underline{p}-2} u)(u_n - u)\, dx \\ &\quad + \sum_{i=1}^{N} \int_\Omega (a_i(x, T_k(u_n), \nabla u_n) - a_i(x, T_k(u_n), \nabla u))(D^i u_n - D^i u)\, dx \to 0, \end{aligned} \tag{3.6}$$

then $u_n \to u$ strongly in $W^{1,\vec{p}}(\Omega)$ for a subsequence.

4. Existence of weak solutions for L^∞-data

We consider the strongly nonlinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x, T_n(u), \nabla u) + H_n(x, u, \nabla u) + \alpha(x)|u|^{r-1}u = F(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x, T_n(u), \nabla u)\cdot n_i = G(x) & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where

$$G(x) \in L^\infty(\partial\Omega) \qquad \text{and} \qquad |F(x)| \le C_0, \quad \text{for any} \quad x \in \Omega, \tag{4.2}$$

with C_0 is a positive constant.

Definition 4.1. A measurable function u is called a weak solution for the strongly nonlinear anisotropic elliptic equation (4.1), if $u \in W^{1,\vec{p}}(\Omega)$ and $|u|^{r+1} \in L^1(\Omega)$, such that u verifies the following equality

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u) D^i v \, \mathrm{d}x + \int_\Omega H_n(x, u, \nabla u) v \, \mathrm{d}x + \int_\Omega \alpha(x)|u|^{r-1}uv \, \mathrm{d}x \\ = \int_\Omega F \; v \, \mathrm{d}x + \int_{\partial\Omega} G \; v \, \mathrm{d}\sigma \end{aligned} \tag{4.3}$$

for any $v \in W^{1,\vec{p}}(\Omega)$.

Theorem 4.2. *Assuming that (3.2)–(3.4) and (4.2) hold true. Then there exists at least one weak solution $u \in W^{1,\vec{p}}(\Omega)$ for the strongly nonlinear elliptic equation (4.1).*

Proof of Theorem 4.2

Step 1: Approximate problem

We consider the following approximate problem for the strongly nonlinear elliptic equation (4.1), giving by

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x, T_n(u_m), \nabla u_m) + H_n(x, u_m, \nabla u_m) & \\ \quad +\alpha(x)|T_m(u_m)|^{r-1}T_m(u_m) + \frac{1}{m}|u_m|^{\underline{p}-2}u_m = F(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x, T_n(u_m), \nabla u_m)\cdot n_i = G(x) & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We define the operator A_m acted from $W^{1,\vec{p}}(\Omega)$ into its dual $(W^{1,\vec{p}}(\Omega))'$ giving by:

$$\begin{aligned} \langle A_m u, v\rangle &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i v\,\mathrm{d}x + \frac{1}{m}\int_\Omega |u|^{\underline{p}-2}uv\,\mathrm{d}x \\ &\quad -\int_{\partial\Omega} G(x)v\,\mathrm{d}\sigma. \end{aligned} \tag{4.5}$$

We consider the operator $R_m\colon W^{1,\vec{p}}(\Omega)\mapsto (W^{1,\vec{p}}(\Omega))'$ given by

$$\langle R_m u, v\rangle = \int_\Omega H_n(x,u,\nabla u)v\,\mathrm{d}x + \int_\Omega \alpha(x)|T_m(u)|^{r-1}T_m(u)v\,\mathrm{d}x, \tag{4.6}$$

for any $u, v\in W^{1,\vec{p}}(\Omega)$. In view of Hölder's type inequality we have

$$\begin{aligned} |\langle R_m u, v\rangle| &= \int_\Omega \alpha(x)|T_m(u)|^r|v|\,\mathrm{d}x + \int_\Omega |H_n(x,u,\nabla u)||v|\,\mathrm{d}x \\ &\le \|\alpha\|_{L^\infty(\Omega)}m^r\int_\Omega |v|\,\mathrm{d}x + n\int_\Omega |v|\,\mathrm{d}x \\ &\le C_1\|v\|_{1,\vec{p}}. \end{aligned} \tag{4.7}$$

Lemma 4.3. *The bounded operator $B_m = A_m + R_m$ acting from $W^{1,\vec{p}}(\Omega)$ into $(W^{1,\vec{p}}(\Omega))'$ is a pseudo-monotone operator. Moreover, B_m is coercive in the following sense:*

$$\frac{\langle B_m v, v\rangle}{\|v\|_{1,\vec{p}}}\longrightarrow\infty \qquad \textit{as} \quad \|v\|_{1,\vec{p}}\longrightarrow\infty, \tag{4.8}$$

for any $v\in W^{1,\vec{p}}(\Omega)$.

Indeed, in view of Hölder's inequality and (3.2), it is easy to see that the operator A_m is bounded, and by (4.7) we conclude that B_m is bounded.
For the coercivity, for any $u\in W^{1,\vec{p}}(\Omega)$ we have

$$\begin{aligned} \langle B_m u, u\rangle &= \langle A_m u, u\rangle + \langle R_m u, u\rangle \\ &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^i u\,\mathrm{d}x + \int_\Omega H_n(x,u,\nabla u)u\,\mathrm{d}x \\ &\quad + \int_\Omega \alpha(x)|T_m(u)|^r|u|\,\mathrm{d}x + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,\mathrm{d}x + \int_{\partial\Omega} G(x)u\,\mathrm{d}\sigma. \\ &\ge \sum_{i=1}^{N}\int_\Omega \frac{b_0|D^i u|^{p_i}}{(1+|T_n(u)|)^\lambda}\,\mathrm{d}x - n\int_\Omega |u|\,\mathrm{d}x + \alpha_0\int_\Omega |T_m(u)|^{r+1}\,\mathrm{d}x \\ &\quad + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,\mathrm{d}x - \|G(x)\|_{L^\infty(\partial\Omega)}\|u\|_{L^1(\partial\Omega)} \\ &\ge \frac{b_0}{(1+n)^\lambda}\sum_{i=1}^{N}\int_\Omega |D^i u|^{p_i}\,\mathrm{d}x + \frac{1}{m}\int_\Omega |u|^{\underline{p}}\,\mathrm{d}x - n\int_\Omega |u|\,\mathrm{d}x \\ &\quad - \|G(x)\|_{L^\infty(\partial\Omega)}\|u\|_{L^1(\partial\Omega)} \\ &\ge C_3\|u\|_{1,\vec{p}}^{\underline{p}} - C_2\|u\|_{1,\vec{p}}, \end{aligned} \tag{4.9}$$

with $\underline{p} > 1$. It follows that

$$\frac{\langle B_m u, u\rangle}{\|u\|_{1,\vec{p}}} \longrightarrow \infty \qquad \text{as} \quad \|u\|_{1,\vec{p}} \longrightarrow \infty. \tag{4.10}$$

It remain to show that B_m is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W^{1,\vec{p}}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u & \text{in} \quad W^{1,\vec{p}}(\Omega), \\ B_m u_k \rightharpoonup \chi_m & \text{in} \quad (W^{1,\vec{p}}(\Omega))', \\ \limsup\limits_{k\to\infty} \langle B_m u_k, u_k\rangle \leq \langle \chi_m, u\rangle. \end{cases} \tag{4.11}$$

We will show that

$$\chi_m = B_m u \quad \text{ and } \quad \langle B_m u_k, u_k\rangle \longrightarrow \langle \chi_m, u\rangle \quad \text{as} \quad k \to +\infty.$$

In view of the compact embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow\hookrightarrow L^{\underline{p}}(\Omega)$ and the continuous embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow L^1(\partial\Omega)$, then there exists a subsequence still denoted $(u_k)_{k\in\mathbb{N}^*}$ such that $u_k \to u$ strongly in $L^{\underline{p}}(\Omega)$ and $u_k \rightharpoonup u$ weakly in $L^1(\partial\Omega)$. As $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W^{1,\vec{p}}(\Omega)$, using the growth condition (3.2), it's clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k\in\mathbb{N}^*}$ is bounded in $L^{p_i'}(\Omega)$, and there exists a measurable function $\varphi_i \in L^{p_i'}(\Omega)$ such that

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \qquad \text{weakly in} \quad L^{p_i'}(\Omega) \quad \text{as} \quad k \to \infty. \tag{4.12}$$

Moreover, since $u_k \to u$ a.e. in Ω, and in view of Lebesgue's dominated convergence theorem we conclude that

$$|T_m(u_k)|^{r-1} T_m(u_k) \rightharpoonup |T_m(u)|^{r-1} T_m(u) \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{4.13}$$

Similarly, there exists $\psi_n \in L^{\underline{p}'}(\Omega)$ such that

$$H_n(x, u_k, \nabla u_k) \longrightarrow \psi_n \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega), \tag{4.14}$$

and since $u_n \to u$ strongly in $L^{\underline{p}}(\Omega)$, then

$$\frac{1}{m}|u_k|^{\underline{p}-2} u_k \longrightarrow \frac{1}{m}|u|^{\underline{p}-2} u \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega), \tag{4.15}$$

Thus, for any $v \in W^{1,\vec{p}}(\Omega)$ we have

$$
\begin{aligned}
\langle \chi_m, v\rangle &= \lim_{k\to\infty} \langle B_m u_k, v\rangle \\
&= \lim_{k\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v\, dx + \lim_{k\to\infty} \int_\Omega H_n(x, u_k, \nabla u_k) v\, dx \\
&\quad + \lim_{k\to\infty} \int_\Omega \alpha(x)|T_m(u_k)|^{r-1} T_m(u_k) v\, dx + \lim_{k\to\infty} \frac{1}{m} \int_\Omega |u_k|^{\underline{p}-2} u_k v\, dx \\
&\quad - \lim_{k\to\infty} \int_{\partial\Omega} G\, v\, d\sigma \\
&= \sum_{i=1}^{N} \int_\Omega \varphi_i\, D^i v\, dx + \int_\Omega \psi_n v\, dx + \int_\Omega \alpha(x)|T_m(u)|^{r-1} T_m(u) v\, dx \\
&\quad + \frac{1}{m} \int_\Omega |u|^{\underline{p}-2} u v\, dx - \int_{\partial\Omega} G\, v\, d\sigma.
\end{aligned} \tag{4.16}
$$

In view of (4.11) and (4.16), we conclude that

$$
\begin{aligned}
\limsup_{k\to\infty} \langle B_m(u_k), u_k\rangle &= \limsup_{k\to\infty} \Big(\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k\, dx \\
&\quad + \int_\Omega H_n(x, u_k, \nabla u_k) u_k\, dx + \int_\Omega \alpha(x)|T_m(u_k)|^{r} |u_k|\, dx \\
&\quad + \frac{1}{m} \int_\Omega |u_k|^{\underline{p}}\, dx - \int_{\partial\Omega} G\, u_k\, d\sigma \Big) \\
&\le \sum_{i=1}^{N} \int_\Omega \varphi_i\, D^i u\, dx + \int_\Omega \psi_n u\, dx + \int_\Omega \alpha(x)|T_m(u)|^{r} |u|\, dx \\
&\quad + \frac{1}{m} \int_\Omega |u|^{\underline{p}}\, dx - \int_{\partial\Omega} G\, u\, d\sigma.
\end{aligned} \tag{4.17}
$$

Thanks to (4.13) and (4.15) we have

$$
\int_\Omega \alpha(x)|T_m(u_k)|^{r}|u_k|\, dx + \frac{1}{m}\int_\Omega |u_k|^{\underline{p}} dx \longrightarrow \int_\Omega \alpha(x)|T_m(u)|^{r}|u|\, dx + \frac{1}{m}\int_\Omega |u|^{\underline{p}} dx, \tag{4.18}
$$

as k tends to infinity. Having in mind that the embedding $W^{1,\vec{p}}(\Omega) \hookrightarrow L^1(\partial\Omega)$ is continuous, then $u_k \rightharpoonup u$ weakly in $L^1(\partial\Omega)$, and since $G \in L^\infty(\partial\Omega)$ then

$$
\int_{\partial\Omega} G\, u_k\, d\sigma \longrightarrow \int_{\partial\Omega} G u\, d\sigma \qquad \text{as} \quad k \to \infty. \tag{4.19}
$$

Moreover, thanks to (4.14) we have

$$
\int_\Omega H_n(x, u_k, \nabla u_k) u_k\, \mathrm{d}x \longrightarrow \int_\Omega \psi_n u\, \mathrm{d}x \quad \text{as} \quad k \to \infty, \tag{4.20}
$$

It follows that

$$\limsup_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx\leq\sum_{i=1}^{N}\int_{\Omega}\varphi_i\;D^iu\,dx. \tag{4.21}$$

On the other hand, in view of (3.4) we have

$$\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx\geq 0, \tag{4.22}$$

then

$$\begin{aligned}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx&\geq\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu\,dx\\&+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u)(D^iu_k-D^iu)\,dx.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k)\to T_n(u)$ strongly in $L^{p_i}(\Omega)$, thus $a_i(x,T_n(u_k),\nabla u)\to a_i(x,T_n(u),\nabla u)$ strongly in $L^{p_i'}(\Omega)$, then

$$\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u)(D^iu_k-D^iu)\,dx\to 0\qquad\text{as}\quad k\to\infty.$$

In view of (4.12) we get

$$\liminf_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx\geq\sum_{i=1}^{N}\int_{\Omega}\varphi_i\;D^iu\,dx. \tag{4.23}$$

Having in mind (4.21), we conclude that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)D^iu_k\,dx=\sum_{i=1}^{N}\int_{\Omega}\varphi_i\;D^iu\,dx. \tag{4.24}$$

Therefore, by combining (4.16)–(4.19) and 4.24 we obtain

$$\langle B_mu_k,u_k\rangle\longrightarrow\langle\chi_m,u\rangle\qquad\text{as}\qquad k\to\infty. \tag{4.25}$$

On the other hand, thanks to (4.24) we can show that

$$\lim_{k\to\infty}\sum_{i=1}^{N}\int_{\Omega}(a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u))(D^iu_k-D^iu)\,dx=0.$$

Having in mind that $u_k\to u$ strongly in $L^{\underline{p}}(\Omega)$, it follows that

$$\begin{aligned}&\int_{\Omega}(|u_n|^{\underline{p}-2}u_n-|u|^{\underline{p}-2}u)(u_k-u)\,dx\\&\quad+\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_k),\nabla u_k)-a_i(x,T_n(u_k),\nabla u)(D^iu_k-D^iu)\,dx\longrightarrow 0,\end{aligned} \tag{4.26}$$

as k tends to infinity. In view of Lemma 3.1, we conclude that

$$u_k \to u \quad \text{in} \quad W^{1,\vec{p}}(\Omega) \qquad \text{and} \qquad D^i u_k \to D^i u \quad \text{a.e. in} \quad \Omega,$$

and since $(a_i(x, T_n(u_k), \nabla u_k))$ is bounded in $L^{p_i'}(\Omega)$, and $a_i(x, T_n(u_k), \nabla u_k) \to a_i(x, T_n(u), \nabla u)$ a.e in Ω, then

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \quad \text{weakly in} \quad L^{p_i'}(\Omega) \quad \text{for} \quad i = 1, \dots, N.$$

Similarly, we have

$$H_n(x, u_k, \nabla u_k) \to H_n(x, u, \nabla u) \quad \text{strongly in} \quad L^{\underline{p}'}(\Omega).$$

Having in mind (4.13)–(4.15) we obtain $\chi_m = B_m u$. Thus, the proof of the Lemma 4.3 is concluded.

In view of Lemma 4.3 (cf. [17, Theorem 8.2]) there exists at least one weak solution $u_m \in W^{1,\vec{p}}(\Omega)$ for the approximate problem (4.4), i.e.

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i v \, dx + \int_\Omega H_n(x, u_m, \nabla u_m) v \, dx \\
&\qquad + \int_\Omega \alpha(x) |T_m(u_m)|^{r-1} T_m(u_m) v \, dx + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}-2} u_m v \, dx \\
&= \int_\Omega F(x) v \, dx + \int_{\partial\Omega} G(x) v \, d\sigma,
\end{aligned} \tag{4.27}$$

for any $v \in W^{1,\vec{p}}(\Omega)$.

Step 2: Weak convergence of the sequence $(u_m)_m$

Let $m \geq n \geq 1$, by taking $v = u_m$ as a test function for the approximate problem (4.4), we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_m), \nabla u_m) D^i u_m \, dx + \int_\Omega H_n(x, u_m, \nabla u_m) u_m \, dx \\
&\qquad + \int_\Omega \alpha(x) |T_m(u_m)|^{r} \; |u_m| \, dx + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}} \, dx \\
&= \int_\Omega F(x) u_m \, dx + \int_{\partial\Omega} G(x) u_m \, d\sigma.
\end{aligned} \tag{4.28}$$

Thus, in view of (3.3) and (4.2) we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega b(|T_n(u_m)|) |D^i u_m|^{p_i(x)} \, dx + \int_\Omega H_n(x, u_m, \nabla u_m) u_m \, dx \\
&\qquad + \alpha_0 \int_\Omega |T_m(u_m)|^{r} \; |u_m| \, dx + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}} \, dx \\
&\leq C_0 \int_\Omega |u_m| \, dx + \|G\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u_m| \, d\sigma.
\end{aligned} \tag{4.29}$$

For the first term on the right-hand side of (4.29), by applying Young's inequality we have

$$\begin{aligned} C_0 \int_\Omega |u_m|\,dx &\le C_0 \int_{\{|u_m|\le C_1\}} |u_m|\,dx + C_0 \int_{\{|u_m|>C_1\}} |u_m|\,dx \\ &\le C_2 + \frac{\alpha_0}{4} \int_{\{|u_m|>C_1\}} |T_m(u_m)|^r\ |u_m|\,dx \\ &\le C_2 + \frac{\alpha_0}{4} \int_\Omega |T_m(u_m)|^r\ |u_m|\,dx, \end{aligned} \tag{4.30}$$

with $C_1 = \left(\frac{2}{\alpha_0} C_0\right)^{\frac{1}{r}} + 1$. Similarly, we show that

$$\begin{aligned} \int_\Omega H_n(x, T_n(u_m), \nabla u_m) u_m\,dx &\le n \int_\Omega |u_m|\,dx \\ &\le C_3 + \frac{\alpha_0}{4} \int_\Omega |T_m(u_m)|^r |u_m|\,dx \end{aligned} \tag{4.31}$$

Concerning the second term on the right-hand of (4.29), we show that

$$\begin{aligned} &\|G\|_{L^\infty(\partial\Omega)} \int_{\partial\Omega} |u_m|\,d\sigma \le C_4 \|u_m\|_{1,1} \\ &\quad = C_4(\|u_m\|_{L^1(\Omega)} + \sum_{i=1}^N \|D^i u_m\|_{L^1(\Omega)}) \\ &\quad \le C_5 + \frac{\alpha_0}{4} \int_{\partial\Omega} |T_m(u_m)|^r\ |u_m|\,dx + \frac{b}{2(1+n)^\lambda} \int_\Omega |D^i u_m|^{p_i}\,dx. \end{aligned} \tag{4.32}$$

By combining (4.29) and (4.30)–(4.32) we conclude that

$$\frac{b_0}{2(1+n)^\lambda} \sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,dx + \frac{\alpha_0}{4} \int_\Omega |T_m(u_m)|^r\ |u_m|\,dx + \frac{1}{m} \int_\Omega |u_m|^{\underline{p}}\,dx \le C_6. \tag{4.33}$$

Moreover, we deduce that

$$\begin{aligned} \|u_m\|_{1,\vec{p}} &= \|u_m\|_{L^1(\Omega)} + \sum_{i=1}^N \|D^i u_m\|_{L^1(\Omega)} + \sum_{i=1}^N \|D^i u_m\|_{L^{p_i}(\Omega)} \\ &\le \int_\Omega |u_m|\,dx + 2\sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,dx + N(\text{meas}(\Omega)+1) \\ &\le \int_\Omega |T_m(u_m)|^r |u_m|\,dx + 2\sum_{i=1}^N \int_\Omega |D^i u_m|^{p_i}\,dx + C_7 \\ &\le C_8. \end{aligned} \tag{4.34}$$

with C_8 is a constant that doesn't depend on m. Thus, the sequence $(u_m)_m$ is uniformly bounded in $W^{1,\vec{p}}(\Omega)$, and there exists a subsequence still denoted $(u_m)_m$

such that

$$\begin{cases} u_m \rightharpoonup u & \text{weakly in } W^{1,\vec{p}}(\Omega), \\ u_m \longrightarrow u & \text{strongly in } \ L^{\underline{p}}(\Omega) \quad \text{and} \quad \text{a.e. in} \quad \Omega, \\ u_m \rightharpoonup u & \text{weakly in } \ L^1(\partial\Omega). \end{cases} \tag{4.35}$$

It follows that

$$\frac{1}{m}|u_m|^{\underline{p}-2}u_m \longrightarrow 0 \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{4.36}$$

Moreover, in view of (4.33) we conclude that $(T_m(u_m))_m$ is uniformly bounded in $L^{r+1}(\Omega)$, and since $T_m(u_m) \to u$ almost everywhere in Ω, we get

$$T_m(u_m) \rightharpoonup u \quad \text{weakly in} \quad L^{r+1}(\Omega). \tag{4.37}$$

Step 3: The convergence almost everywhere of the gradient

By taking $v = u_m - u$ as a test function for the approximated problem (4.1) we obtain

$$\begin{aligned} &\sum_{i=1}^{N}\int_\Omega a_i(x, T_n(u_m), \nabla u_m)(D^i u_m - D^i u)\,dx + \int_\Omega H_n(x, u_m, \nabla u_m)\ (u_m - u)\,dx \\ &\quad + \int_\Omega \alpha(x)|T_m(u_m)|^{r-1}T_m(u_m)\ (u_m - u)\,dx + \frac{1}{m}\int_\Omega |u_m|^{\underline{p}-2}u_m(u_m - u)\,dx \\ &= \int_\Omega F(x)\ (u_m - u)\,dx + \int_{\partial\Omega} G(x)\ (u_m - u)\,d\sigma, \end{aligned} \tag{4.38}$$

it follows that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega}\left(a_i(x,T_n(u_m),\nabla u_m)-a_i(x,T_n(u_m),\nabla u)\right)(D^i u_m-D^i u)\,dx\\
&\quad+\int_{\Omega}H_n(x,u_m,\nabla u_m)\,(u_m-u)\,dx\\
&\quad+\int_{\Omega}\alpha(x)\left(|T_m(u_m)|^{r-1}T_m(u_m)-|T_m(u)|^{r-1}T_m(u)\right)(u_m-u)\,dx\\
&=-\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_n(u_m),\nabla u)(D^i u_m-D^i u)\,dx\\
&\quad-\int_{\Omega}\alpha(x)|T_m(u)|^{r-1}T_m(u)(u_m-u)\,dx\\
&\quad-\frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-2}u_m(u_m-u)\,dx+\int_{\Omega}F(x)(u_m-u)\,dx+\int_{\partial\Omega}G(x)(u_m-u)\,d\sigma\\
&\le\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_n(u_m),\nabla u)|\,|D^i u_m-D^i u|\,dx+\int_{\Omega}\alpha(x)|T_m(u)|^{r}\,|u_m-u|\,dx\\
&\quad+\frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-1}\,|u_m-u|\,dx+\int_{\Omega}|F(x)|\,|u_m-u|\,dx+\int_{\partial\Omega}|G(x)|\,|u_m-u|\,d\sigma.
\end{aligned}
\tag{4.39}
$$

For the first term on the right-hand side of (4.39), we have $T_n(u_m)\to T_n(u)$ strongly in $L^{p_i}(\Omega)$ then $|a_i(x,T_n(u_m),\nabla u)|\longrightarrow|a_i(x,T_n(u),\nabla u)|$ strongly in $L^{p_i'}(\Omega)$, and since $D^i u_m\to D^i u$ weakly in $L^{p_i}(\Omega)$, it follows that

$$
\sum_{i=1}^{N}\int_{\Omega}|a_i(x,T_n(u_m),\nabla u)|\,|D^i u_m-D^i u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.40}
$$

Concerning the second term on the right hand side of (4.39), we have $|T_m(u)|^r\in L^{\frac{r+1}{r}}(\Omega)$ and since $u_m\rightharpoonup u$ weakly in $L^{r+1}(\Omega)$, it follows that

$$
\int_{\Omega}\alpha(x)|T_m(u)|^r|u_m-u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.41}
$$

Moreover, in view of (4.36) and (4.2), we deduce that

$$
\frac{1}{m}\int_{\Omega}|u_m|^{\underline{p}-1}\,|u_m-u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty,\tag{4.42}
$$

and

$$
\int_{\Omega}|F(x)|\,|u_m-u|\,dx\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.43}
$$

Furthermore, we have $G(x)$ belongs to $L^{\infty}(\partial\Omega)$, and since $u_m\rightharpoonup u$ weakly in $L^1(\partial\Omega)$ it follows that

$$
\int_{\partial\Omega}|G(x)|\,|u_m-u|\,d\sigma\longrightarrow 0\qquad\text{as}\quad m\to\infty.\tag{4.44}
$$

By combining (4.39) and (4.40)–(4.43) we conclude that

$$\lim_{m\to\infty}\Big(\sum_{i=1}^{N}\int_\Omega (a_i(x,T_n(u_m),\nabla u_m)-a_i(x,T_n(u_m),\nabla u))\,(D^iu_m-D^iu)\,dx \\ +\alpha_0\int_\Omega \big(|u_m|^{\underline{p}-2}u_m-|u|^{\underline{p}-2}u\big)\,(u_m-u)\,dx\Big)=0. \tag{4.45}$$

In view of Lemma 3.1, we conclude that

$$\begin{cases} u_m\to u \quad \text{strongly in} \quad W^{1,\vec{p}}(\Omega),\\ D^iu_m\to D^iu \qquad \text{a.e. in} \quad \Omega \quad \text{for} \quad i=1,\ldots,N. \end{cases} \tag{4.46}$$

Thus, $a_i(x,T_n(u_m),\nabla u_m)\to a_i(x,T_n(u),\nabla u)$ almost everywhere in Ω, and since $(a_i(x,T_n(u_m),\nabla u_m))_m$ is uniformly bounded in $L^{p_i'}(\Omega)$, it follows that

$$a_i(x,T_n(u_m),\nabla u_m)\rightharpoonup a_i(x,T_n(u),\nabla u) \qquad \text{weakly in} \quad L^{p_i'}(\Omega), \tag{4.47}$$

for $i=1,\ldots,N$. Moreover, in view of Lebesgue dominated convergence theorem, we obtain

$$H_n(x,u_m,\nabla u_m)\to H_n(x,u,\nabla u) \qquad \text{strongly in} \quad L^{\underline{p}'}(\Omega). \tag{4.48}$$

Step 4: Passage to the limit

By taking $v\in W^{1,\vec{p}}(\Omega)$ as a test function for the approximate problem (4.1) we have

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_m),\nabla u_m)D^iv\,dx+\int_\Omega H_n(x,u_m,\nabla u_m)v\,dx \\ +\int_\Omega \alpha(x)|T_m(u_m)|^{r-1}T_m(u_m)v\,dx+\frac{1}{m}\int_\Omega |u_m|^{\underline{p}-1}u_mv\,dx \\ =\int_\Omega F(x)\;v\,dx+\int_{\partial\Omega} G(x)\;v\,d\sigma. \tag{4.49}$$

In view of (4.36)–(4.37), (4.47) and (4.48), then letting m tends to infinity we conclude that

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u),\nabla u)D^iv\,dx+\int_\Omega H_n(x,u,\nabla u)v\,dx+\int_\Omega \alpha(x)|u|^{r-1}uv\,dx \\ =\int_\Omega F(x)\;v\,dx+\int_{\partial\Omega} G(x)\;v\,d\sigma. \tag{4.50}$$

Thus, the proof of the Theorem 4.2 is concluded.

5. Main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, with smooth boundary $\partial\Omega$.

Definition 5.1. A measurable function u is called renormalized solution to the strongly nonlinear anisotropic elliptic equation (3.1), If $u \in T^{1,\vec{p}}_{tr}(\Omega)$, $H(x,u,\nabla u) \in L^1(\Omega)$, $|u|^{r-1}u \in L^1(\Omega)$, and

$$\lim_{h\to\infty} \frac{1}{h} \sum_{i=1}^{N} \int_{\{|u|\leq h\}} a_i(x,u,\nabla u) D^i u \, dx = 0, \tag{5.1}$$

such that u verifies the following equality

$$\begin{aligned} &\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)(S^{'}(u)\varphi D^i u + S(u) D^i \varphi)\, dx + \int_\Omega H(x,u,\nabla u) S(u)\varphi \, dx \\ &\quad + \int_\Omega \alpha(x)|u|^{r-1}u S(u)\varphi \, dx = \int_\Omega f\ S(u)\varphi\, dx + \int_{\partial\Omega} g\ S(u)\varphi\, d\sigma, \end{aligned} \tag{5.2}$$

for every $\varphi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ and any smooth function $S(\cdot) \in W^{1,\infty}(\mathbb{R})$ with a compact support.

Theorem 5.2. *Let $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$. Assuming that (3.2)–(3.5) hold true, then there exists at least one renormalized solution u for the strongly nonlinear anisotropic elliptic Neumann problem (3.1).*

Remark 5.3. Note that the uniqueness of renormalized solution for the problem (3.1) can be proved in the case of $H(x,s,\xi) \equiv 0$ and $a_i(x,s,\xi) = a_i(x,s)|\xi_i|^{p_i-2}\xi_i$. For more details, we refer the reader to [9, 14] and [15].

6. Proof of Theorem 5.2

Step 1: Approximate problems

We set $f_n(\cdot) = T_n(f(\cdot))$ and $g_n(\cdot) = T_n(g(\cdot))$, then $f_n(\cdot)$ is bounded in $L^\infty(\Omega) \cap L^1(\Omega)$, and g_n is bounded in $L^\infty(\partial\Omega) \cap L^1(\partial\Omega)$ such that:

$$f_n \to f \qquad \text{strongly in} \quad L^1(\Omega) \qquad \text{and} \qquad g_n \to g \qquad \text{strongly in} \quad L^1(\partial\Omega).$$

We consider the approximate problem:

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i a_i(x,T_n(u_n),\nabla u_n) + H_n(x,u_n,\nabla u_n) + \alpha(x)|u_n|^{r-1}u_n = f_n(x) & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} a_i(x,T_n(u_n),\nabla u_n).n_i = g_n(x) & \text{on } \partial\Omega, \end{cases} \tag{6.1}$$

where $H_n(x,s,\xi) = T_n(H(x,s,\xi))$.

In view of Theorem 4.2, there exists at least one weak solution $u_n \in W^{1,\vec{p}}(\Omega)$ for the strongly nonlinear elliptic problem (6.1), i.e.,

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i v\,dx + \int_\Omega H_n(x,u_n,\nabla u_n)v\,dx$$
$$+\int_\Omega \alpha(x)|u_n|^{r-1}u_n v\,dx = \int_\Omega f_n v\,dx + \int_{\partial\Omega} g_n v\,d\sigma \qquad \text{for any} \quad v \in W^{1,\vec{p}}(\Omega). \tag{6.2}$$

Step 2: Weak convergence of truncations.

Let $n \in \mathbb{N}$ be large enough $(n \geq k > 1)$, we define

$$B(s) = 2\int_0^s \frac{d(|\tau|)}{b(|\tau|)}\,d\tau.$$

Note that, since the function $\dfrac{d(|\tau|)}{b(|\tau|)}$ is integrable on $\mathbb{R}$, then

$$0 \leq B(\infty) := 2\int_0^{+\infty} \frac{d(|\tau|)}{b(|\tau|)}\,dt$$

is a finite real number.

By taking $v = T_k(u_n)e^{B(|u_n|)} \in W^{1,\vec{p}}(\Omega)$ as a test function for the approximate problem (6.2), we have

$$\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i T_k(u_n)e^{B(|u_n|)}\,dx$$
$$+2\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^i u_n \frac{d(|u_n|)}{b(|u_n|)}|T_k(u_n)|e^{B(|u_n|)}\,dx$$
$$+\int_\Omega H_n(x,u_n,\nabla u_n)T_k(u_n)e^{B(|u_n|)}\,dx + \int_\Omega \alpha(x)|u_n|^{r-1}u_n T_k(u_n)e^{B(|u_n|)}\,dx$$
$$= \int_\Omega f_n T_k(u_n)e^{B(|u_n|)}\,dx + \int_{\partial\Omega} g_n T_k(u_n)e^{B(|u_n|)}\,d\sigma. \tag{6.3}$$

In view of (3.3), (4.2) and (3.5), we obtain

$$
\begin{aligned}
& b_0 \sum_{i=1}^{N} \int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{(1+|u_n|)^\lambda}\, dx + 2 \sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i} |T_k(u_n)| d(|u_n|) e^{B(|u_n|)} dx \\
& \quad + \int_\Omega \alpha(x) |u_n|^r |T_k(u_n)|\, dx \\
& \le \int_\Omega |H_n(x,u_n,\nabla u_n)|\ |T_k(u_n)| e^{B(|u_n|)} dx + \int_\Omega |f_n||T_k(u_n)| e^{B(|u_n|)} dx \\
& \quad + \int_{\partial\Omega} |g_n||T_k(u_n)| e^{B(|u_n|)}\, d\sigma \\
& \le k e^{B(\infty)} (\|f\|_{L^1(\Omega)} + \|f_0\|_{L^1(\Omega)} + \|g\|_{L^1(\partial\Omega)}) \\
& \quad + \sum_{i=1}^{N} \int_\Omega d(|u_n|) |D^i u_n|^{p_i} |T_k(u_n)| e^{B(|u_n|)} dx.
\end{aligned}
$$

Thus, for any $k \ge 1$ we have

$$
\begin{aligned}
b_0 \sum_{i=1}^{N} \int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{(1+|u_n|)^\lambda}\, dx + \sum_{i=1}^{N} \int_\Omega |D^i u_n|^{p_i} |T_k(u_n)| d(|u_n|) dx \\
+\alpha_0 \int_\Omega |u_n|^r |T_k(u_n)|\, dx \ \le\ C_1 k.
\end{aligned}
\tag{6.4}
$$

It follows that

$$
\frac{b_0}{(1+k)^\lambda} \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i}\, dx \le b_0 \sum_{i=1}^{N} \int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{(1+|u_n|)^\lambda}\, dx \ \le\ C_1 k. \tag{6.5}
$$

Therefore, we conclude that

$$
\begin{aligned}
& \|T_k(u_n)\|_{1,\vec{p}} \\
& = \|T_k(u_n)\|_{1,1} + \sum_{i=1}^{N} \|D^i T_k(u_n)\|_{p_i} \\
& = \int_\Omega |T_k(u_n)|\, dx + \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|\, dx + \sum_{i=1}^{N} \Big(\int_\Omega |D^i T_k(u_n)|^{p_i}\, dx \Big)^{\frac{1}{p_i}} \\
& \le k \cdot \mathrm{meas}(\Omega) + 2 \sum_{i=1}^{N} \int_\Omega |D^i T_k(u_n)|^{p_i}\, dx + N + N \cdot |\Omega| \\
& \le C_2 k^{1+\lambda},
\end{aligned}
\tag{6.6}
$$

where C_2 is a positive constant that does not depend on k and n. Thus $(T_k(u_n))_n$ is bounded in $W^{1,\vec{p}}(\Omega)$ uniformly in n, and there exists a subsequence still denoted $(T_k(u_n))_n$ and a measurable function $v_k \in W^{1,\vec{p}}(\Omega)$ such that

$$
\begin{cases}
T_k(u_n) \rightharpoonup v_k & \text{weakly in} \quad W^{1,\vec{p}}(\Omega), \\
T_k(u_n) \to v_k & \text{strongly in} \quad L^1(\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega.
\end{cases}
\tag{6.7}
$$

Moreover, in view of (6.4) we conclude that

$$\sum_{i=1}^{N}\int_{\{|u_n|>k\}}|D^i u_n|^{p_i} d(|u_n|)dx \ \leq\ C_1, \tag{6.8}$$

and

$$\alpha_0\int_\Omega |u_n|^r\,dx \leq \alpha_0\int_{\{|u_n|\geq 1\}}|u_n|^r|T_k(u_n)|\,dx+\alpha_0|\Omega|\ \leq\ C_3. \tag{6.9}$$

Thus, we obtain

$$k^r\cdot \text{meas}(\{k<|u_n|\})\leq \int_{\{|u_n|>k\}}|u_n|^r\,dx\leq \int_\Omega |u_n|^r\,dx\leq \frac{C_3}{\alpha_0},$$

it follows that

$$\limsup_{n\to\infty}\ \text{meas}(\{k<|u_n|\})\leq \frac{C_1}{\alpha_0 k^r}\longrightarrow 0 \quad \text{as} \quad k\to\infty. \tag{6.10}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure. For all $\delta>0$, we have

$$\begin{aligned}\text{meas}\{|u_n-u_m|>\delta\} \ &\leq \text{meas}\{|u_n|>k\}+\ \text{meas}\{|u_m|>k\}\\ &\qquad +\ \text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}.\end{aligned}$$

Let $\varepsilon>0$, using (6.10) we may choose $k=k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n|>k\}\leq\frac{\varepsilon}{3} \qquad \text{and} \qquad \text{meas}\{|u_m|>k\}\leq\frac{\varepsilon}{3}. \tag{6.11}$$

On the other hand, thanks to (6.7) we have $T_k(u_n)\to v_k$ strongly in $L^1(\Omega)$ and a.e. in Ω. Thus, we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure, and for all $k>0$ and $\varepsilon,\delta>0$, there exists $n_0=n_0(k,\varepsilon,\delta)$ such that

$$\text{meas}\{|T_k(u_n)-T_k(u_m)|>\delta\}\leq\frac{\varepsilon}{3} \qquad \text{for all } m,n\geq n_0(k,\varepsilon,\delta). \tag{6.12}$$

By combining (6.11)–(6.12), we conclude that

$$\forall\varepsilon,\delta>0 \quad \text{there exists} \quad n_0=n_0(\varepsilon,\delta) \quad \text{such that} \quad \text{meas}\{|u_n-u_m|>\delta\}\leq\varepsilon,$$

for any $n,m\geq n_0(\varepsilon,\delta)$. It follows that $(u_n)_n$ is a Cauchy sequence in measure, then converges almost everywhere, for a subsequence, to some measurable function u. Consequently, we have

$$\begin{cases} T_k(u_n)\rightharpoonup T_k(u) & \text{weakly in} \quad W^{1,\vec{p}}(\Omega),\\ T_k(u_n)\to T_k(u) & \text{strongly in} \quad L^1(\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega,\\ T_k(u_n)\to T_k(u) & \text{strongly in} \quad L^1(\partial\Omega) \quad \text{and} \quad \text{a.e in} \quad \Omega.\end{cases} \tag{6.13}$$

In view of Lebesgue's dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \quad \text{strongly in } L^{p_i}(\Omega) \text{ and a.e. in } \Omega \quad \text{for } i = 1, \ldots, N. \tag{6.14}$$

Moreover, thanks to (6.5) it's clear that: for any $i = 1, \ldots, N$

$$\int_\Omega \frac{|D^i T_k(u_n)|^{p_i}}{k^{p_i}}\, dx \;\leq\; \frac{C_1 k(1+k)^\lambda}{b_0 k^{p_i}} \longrightarrow 0 \qquad \text{as} \quad k \to \infty,$$

and in view of (6.10) we have $\Big\| \dfrac{T_k(u_n)}{k} \Big\|_{L^1(\Omega)} \longrightarrow 0$ as k tends to infinity, then

$$\begin{aligned}
\lim_{k\to\infty} \Big\| \frac{T_k(u_n)}{k} \Big\|_{L^1(\partial\Omega)} &\leq \lim_{k\to\infty} C \Big\| \frac{T_k(u_n)}{k} \Big\|_{W^{1,1}(\Omega)} \\
&\leq C \lim_{k\to\infty} \Big\| \frac{T_k(u_n)}{k} \Big\|_{L^1(\Omega)} + C \lim_{k\to\infty} \sum_{i=1}^N \Big\| \frac{D^i T_k(u_n)}{k} \Big\|_{L^1(\Omega)} \\
&\leq C \lim_{k\to\infty} \Big\| \frac{T_k(u_n)}{k} \Big\|_{L^1(\Omega)} + C' \lim_{k\to\infty} \sum_{i=1}^N \Big\| \frac{D^i T_k(u_n)}{k} \Big\|_{L^{p_i}(\Omega)} \\
&= 0.
\end{aligned}$$

We conclude that

$$\frac{T_k(u_n)}{k} \longrightarrow 0 \quad \text{weak} - * \quad \text{in } L^\infty(\partial\Omega). \tag{6.15}$$

Step 3: Some a priori estimates.

In this section, we will show that:

$$\lim_{h\to\infty} \limsup_{n\to\infty} \sum_{i=1}^N \frac{1}{h} \int_{\{|u_n|\leq h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx = 0.$$

By taking $v = \dfrac{T_h(u_n)}{h} e^{B(|u_n|)}$ as a test function in the approximate problem (6.2), we obtain

$$\begin{aligned}
&\frac{1}{h} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i (T_h(u_n) e^{B(|u_n|)})\, dx \\
&\quad + \int_\Omega H_n(x, u_n, \nabla u_n) \frac{T_h(u_n)}{h} e^{B(|u_n|)}\, dx + \int_\Omega \alpha(x) |u_n|^{r-1} u_n \frac{T_h(u_n)}{h} e^{B(|u_n|)}\, dx \\
&= \int_\Omega f_n \frac{T_h(u_n)}{h} e^{B(|u_n|)}\, dx + \int_{\partial\Omega} g_n \frac{T_h(u_n)}{h} e^{B(|u_n|)}\, d\sigma,
\end{aligned} \tag{6.16}$$

using (3.3), (4.2) and (3.5), it follows that

$$
\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^iT_h(u_n)e^{B(|u_n|)}dx\\
&+\frac{2}{h}\sum_{i=1}^{N}\int_{\Omega}|D^iu_n|^{p_i}d(|u_n|)|T_h(u_n)|e^{B(|u_n|)}dx+\int_{\Omega}\alpha(x)|u_n|^{r}\frac{|T_h(u_n)|}{h}e^{B(|u_n|)}\,dx\\
&\le \int_{\Omega}|f_n|\frac{|T_h(u_n)|}{h}e^{B(|u_n|)}dx+\int_{\partial\Omega}|g_n|\frac{|T_h(u_n)|}{h}e^{B(|u_n|)}\,d\sigma\\
&+\frac{1}{h}\int_{\Omega}|H_n(x,u_n,\nabla u_n)|\;|T_h(u_n)|e^{B(|u_n|)}dx\\
&\le e^{B(\infty)}\int_{\Omega}(|f|+|f_0|)\frac{|T_h(u_n)|}{h}\,dx+e^{B(\infty)}\int_{\partial\Omega}|g_n|\frac{|T_h(u_n)|}{h}\,d\sigma\\
&+\frac{1}{h}\sum_{i=1}^{N}\int_{\Omega}|D^iu_n|^{p_i}d(|u_n|)|T_h(u_n)|e^{B(|u_n|)}dx.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
&\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\,dx+\frac{1}{h}\sum_{i=1}^{N}\int_{\Omega}d(|u_n|)|D^iu_n|^{p_i}|T_h(u_n)|\,dx\\
&\quad+\int_{\Omega}\alpha(x)|u_n|^{r}\frac{|T_h(u_n)|}{h}\,dx\\
&\le e^{B(\infty)}\int_{\Omega}(|f|+|f_0|)\frac{|T_h(u_n)|}{h}dx+e^{B(\infty)}\int_{\partial\Omega}|g|\frac{|T_h(u_n)|}{h}\,d\sigma.
\end{aligned}
\tag{6.17}
$$

Thanks to (6.10) we have: meas $\{|u_n|>h\}\to 0$ as h tends to infinity, thus $\dfrac{|T_h(u_n)|}{h}\rightharpoonup 0$ weak$-*$ in $L^\infty(\Omega)$. Thanks to Lebesgue's dominated convergence theorem, we obtain

$$
\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\Omega}(|f|+|f_0|)\frac{|T_h(u_n)|}{h}\,dx=0. \tag{6.18}
$$

Similarly, thanks to (6.15) we have $\dfrac{|T_h(u_n)|}{h}\rightharpoonup 0$ weak$-*$ in $L^\infty(\partial\Omega)$, and since $g_n\to g$ strongly in $L^1(\partial\Omega)$ we get

$$
\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\partial\Omega}|g|\frac{|T_h(u_n)|}{h}\,d\sigma=0. \tag{6.19}
$$

Thus, by letting h tends to infinity in (6.17) we conclude that

$$
\lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\{|u_n|\le h\}} a(x,T_n(u_n),\nabla u_n)D^iu_n\,dx=0. \tag{6.20}
$$

Moreover, we obtain

$$
\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{|u_n|>h\}} d(|u_n|)|D^iu_n|^{p_i}\,dx=0, \tag{6.21}
$$

and

$$\lim_{h\to\infty}\limsup_{n\to\infty}\int_{\{|u_n|>h\}}\alpha(x)|u_n|^r\,dx=0. \tag{6.22}$$

Step 4: Strong convergence of truncations.

In the sequel, we denote by $\varepsilon_i(n)$, $i=1,2,\ldots,$ various real-valued functions of real variables that converges to 0 as n tends to infinity. Similarly, we define $\varepsilon_i(h)$, and $\varepsilon_i(n,h)$.

In this step, we will show the convergence of the sequence $(D^iu_n)_n$ almost everywhere in Ω to D^iu for any $i=1,\ldots,N$. We set

$$S_h(\tau)=1-\frac{|T_{2h}(\tau)-T_h(\tau)|}{h}\qquad\text{and}\qquad\varphi(s)=s.\exp(\frac{\gamma^2s^2}{2}),$$

where $\gamma=3\Big\|\dfrac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(\mathbb{R})}$, note that $\varphi'(s)-\gamma|\varphi(s)|\geq\dfrac{1}{2}$ for any $s\in\mathbb{R}$.

By taking $v=\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}$ as a test function in the approximate problem (6.2), we have

$$\begin{aligned}
&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx\\
&-\sum_{i=1}^N\frac{1}{h}\int_{\{h<|u_n|\leq 2h\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n|\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\,dx\\
&\quad+2\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}\mathrm{sign}(u_n)\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}dx\\
&\quad+\int_\Omega\alpha(x)|u_n|^{r-1}u_n\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}dx\\
&\quad+\int_\Omega H_n(x,u_n,\nabla u_n)\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}dx\\
&=\int_\Omega f_n\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\,dx\\
&\quad+\int_{\partial\Omega}g_n\varphi(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}\,d\sigma.
\end{aligned}$$

We have $a_i(x,r,0)=0$, and $S_h(u_n)=1$ on the set $\{|u_n|\leq h\}$. Moreover, $\varphi(T_k(u_n)-T_k(u))$ have the same sign as u_n on the set $\{|u_n|>k\}$. By using (3.3) and (3.5) we obtain

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}dx\\
&-\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iT_k(u)\varphi'(T_k(u_n)-T_k(u))S_h(u_n)e^{B(|u_n|)}dx\\
&-2\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\frac{d(|u_n|)}{b(|u_n|)}\ |\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\ dx\\
&+2\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}} |D^iu_n|^{p_i}d(|u_n|)\ |\varphi(T_k(u_n)-T_k(u))|S_h(u_n)e^{B(|u_n|)}\ dx\\
&+\int_{\{k<|u_n|\le 2h\}} \alpha(x)|u_n|^{r}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)e^{B(|u_n|)}dx\\
&\le e^{B(\infty)}\int_{\Omega}(|f_n|+|f_0|)\ |\varphi(T_k(u_n)-T_k(u))|\ dx\\
&+e^{B(\infty)}\int_{\partial\Omega}|g_n|\ |\varphi(T_k(u_n)-T_k(u))|\ d\sigma\\
&+e^{B(\infty)}\|\alpha(\cdot)\|_{L^\infty(\Omega)}\int_{\{|u_n|\le k\}}|T_k(u_n)|^{r}|\varphi(T_k(u_n)-T_k(u))|\ dx\\
&+\sum_{i=1}^{N}\int_{\Omega} d(|u_n|)|D^iu_n|^{p_i}|\varphi(T_k(u_n)-T_k(u))|S_h(u_n)e^{B(|u_n|)}\ dx\\
&+\frac{\varphi(2k)e^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\ dx.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u_n))(D^iT_k(u_n)-D^iT_k(u))\varphi'(T_k(u_n)-T_k(u))e^{B(|u_n|)}\ dx\\
&\quad -e^{B(\infty)}\varphi'(2k)\sum_{i=1}^{N}\int_{\{k<|u_n|\le 2h\}}|a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))|\ |D^iT_k(u)|\ dx\\
&\quad -3\sum_{i=1}^{N}\int_{\{|u_n|\le k\}} a_i(x,T_k(u_n),\nabla T_k(u_n))D^iu_n\frac{d(|u_n|)}{b(|u_n|)}\ |\varphi(T_k(u_n)-T_k(u))|e^{B(|u_n|)}\ dx\\
&\le e^{B(\infty)}\int_{\Omega}(|f|+|f_0|)\ |\varphi(T_k(u_n)-T_k(u))|\ dx+e^{B(\infty)}\int_{\partial\Omega}|g|\ |\varphi(T_k(u_n)-T_k(u))|\ d\sigma\\
&\quad +e^{B(\infty)}\|\alpha(\cdot)\|_{L^\infty(\Omega)}\int_{\{|u_n|\le k\}}|T_k(u_n)|^{r}|\varphi(T_k(u_n)-T_k(u))|dx\\
&\quad +\frac{\varphi(2k)e^{B(\infty)}}{h}\sum_{i=1}^{N}\int_{\{h<|u_n|\le 2h\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n\ dx.
\end{aligned}
\tag{6.23}
$$

Concerning the second term on the left-hand side of (6.23), it's clear that the sequence $(a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n)))_n$ is bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\vartheta_i\in L^{p_i'}(\Omega)$ such that $a_i(x,T_{2h}(u_n),\nabla T_{2h}(u_n))\rightharpoonup\vartheta_i$ in

$L^{p_i'}(\Omega)$ for any $i = 1, \dots, N$, we conclude that

$$\begin{aligned}\varepsilon_1(n) &= \sum_{i=1}^{N} \int_{\{k<|u_n|\leq 2h\}} |a_i(x, T_{2h}(u_n), \nabla T_{2h}(u_n))| \, |D^i T_k(u)| \, dx \\ &\longrightarrow \sum_{i=1}^{N} \int_{\{k<|u|\leq 2h\}} |\vartheta_i| \, |D^i T_k(u)| \, dx = 0 \qquad \text{as} \quad n \to \infty.\end{aligned} \tag{6.24}$$

For the terms on the right-hand side of (6.23), we have $|f(x)|$ and $|f_0(x)|$ belongs to $L^1(\Omega)$, and since $\varphi(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weak$-\star$ in $L^\infty(\Omega)$, it follows that

$$\varepsilon_2(n) = \int_\Omega (|f(x)| + |f_0(x)|) \, |\varphi(T_k(u_n) - T_k(u))| \, dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{6.25}$$

Also, thanks to Lebesgue Dominated Convergence theorem, we have $|T_k(u_n)|^r \to |T_k(u)|^r$ strongly in $L^1(\Omega)$, it follows that

$$\varepsilon_3(n) = \int_{\{|u_n|\leq k\}} |T_k(u_n)|^r |\varphi(T_k(u_n) - T_k(u))| dx \longrightarrow 0 \qquad \text{as} \quad n \to \infty. \tag{6.26}$$

Similarly, we have $\varphi(T_k(u_n) - T_k(u)) \rightharpoonup 0$ weak$-\star$ in $L^\infty(\partial\Omega)$, then

$$\varepsilon_4(n) = \int_{\partial\Omega} |g(x)| \, |\varphi(T_k(u_n) - T_k(u))| \, d\sigma \longrightarrow 0 \quad \text{as} \quad n \to \infty. \tag{6.27}$$

Moreover, thanks to (6.20) we have

$$\varepsilon_5(h) = \frac{\varphi(2k) e^{B(\infty)}}{h} \sum_{i=1}^{N} \int_{\{h<|u_n|\leq 2h\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n dx \longrightarrow 0 \text{ as } h \to \infty. \tag{6.28}$$

By combining (6.23) and (6.24)–(6.28) we conclude that

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n))(D^i T_k(u_n) - D^i T_k(u)) \varphi'(T_k(u_n) - T_k(u)) e^{B(|u_n|)} dx \\ &-3 \sum_{i=1}^{N} \int_\Omega a_i(x, T_k(u_n), \nabla T_k(u_n)) D^i T_k(u_n) \frac{d(|u_n|)}{b(|u_n|)} \, |\varphi(T_k(u_n) - T_k(u))| e^{B(|u_n|)} dx \\ &\leq \varepsilon_6(n, h).\end{aligned} \tag{6.29}$$

It follows that

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\\
&\qquad\qquad\times\Big(\varphi'(T_k(u_n)-T_k(u)) - 3\frac{d(|u_n|)}{b(|u_n|)}\,|\varphi(T_k(u_n)-T_k(u))|\Big)e^{B(|u_n|)}\,dx\\
&-e^{B(\infty)}\Big(\varphi'(2k)+3\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_{L^\infty(I\!R)}\varphi(2k)\Big)\\
&\qquad\qquad\times\sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&-3e^{B(\infty)}\Big\|\frac{d(|\cdot|)}{b(|\cdot|)}\Big\|_\infty\sum_{i=1}^{N}\int_\Omega |a_i(x,T_k(u_n),\nabla T_k(u_n))|\,|D^iT_k(u)||\varphi(T_k(u_n)-T_k(u))|\,dx\\
&\le \varepsilon_6(n,h).
\end{aligned}\tag{6.30}$$

For the second term on the left-hand side of (6.30), in view of (6.14) we have $T_k(u_n)\to T_k(u)$ strongly in $L^{p_i}(\Omega)$, then

$$a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))\qquad\text{strongly in}\quad L^{p_i'}(\Omega),$$

and since $D^iT_k(u_n)$ tends to $D^iT_k(u)$ weakly in $L^{p_i}(\Omega)$, we obtain

$$\begin{aligned}
\varepsilon_7(n) &= \sum_{i=1}^{N}\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx\\
&\le \sum_{i=1}^{N}\Big|\int_\Omega a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))\,dx\Big| \to 0\quad\text{as } n\to\infty.
\end{aligned}\tag{6.31}$$

Concerning the last term on the left-hand side of (6.30), we have $(|a_i(x,T_k(u_n),\nabla T_k(u_n))|)_n$ is bounded in $L^{p_i'}(\Omega)$, then there exists a measurable function $\nu_i\in L^{p_i'}(\Omega)$ such that $|a_i(x,T_k(u_n),\nabla T_k(u_n))|\rightharpoonup\nu_i$ weakly in $L^{p_i'}(\Omega)$, and we have $|D^iT_k(u)|\,|\varphi(T_k(u_n)-T_k(u))|$ tends strongly to 0 in $L^{p_i}(\Omega)$ for any $i=1,\dots,N$, it follows that

$$\varepsilon_9(n)=\sum_{i=1}^{N}\int_\Omega |a_i(x,T_k(u_n),\nabla T_k(u_n))||D^iT_k(u)|\,|\varphi(T_k(u_n)-T_k(u))|dx\to 0,\tag{6.32}$$

as n tends to infinity. By combining (6.31)–(6.32) we conclude that

$$\begin{aligned}
0&\le \frac12\sum_{i=1}^{N}\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))dx\\
&\le \sum_{i=1}^{N}\int_\Omega \Big(a_i(x,T_k(u_n),\nabla T_k(u_n)) - a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))\\
&\qquad\qquad\times\Big(\varphi'(T_k(u_n)-T_k(u)) - 3\frac{d(|u_n|)}{b(|u_n|)}\,|\varphi(T_k(u_n)-T_k(u))|\Big)e^{B(|u_n|)}\,dx\\
&\le \varepsilon_{10}(n,h)\longrightarrow 0\qquad\text{as}\qquad n,h\to 0.
\end{aligned}\tag{6.33}$$

In view of Lebesgue dominated convergence theorem, we have $T_k(u_n) \to T_k(u)$ strongly in $L^{\underline{p}}(\Omega)$. Thus, by letting n then h tend to infinity we deduce that

$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))dx \\ +\int_{\Omega}(|T_k(u_n)|^{r-1}T_k(u_n)-|T_k(u)|^{r-1}T_k(u))(T_k(u_n)-T_k(u))\,dx \to 0 \tag{6.34}$$

as $n \to \infty$.

Thanks to Lemma 3.1, we conclude that

$$\begin{cases} T_k(u_n) \to T_k(u) & \text{strongly in} \quad W^{1,\vec{p}}(\Omega), \\ D^iu_n \to D^iu & \text{a.e. in} \quad \Omega \quad \text{for} \quad i=1,\dots,N. \end{cases} \tag{6.35}$$

Moreover, we have $a_i(x,T_n(u_n),\nabla u_n)D^iu_n$ tends to $a_i(x,u,\nabla u)D^iu$ almost everywhere in Ω, and in view of Fatou's lemma and (6.20) we conclude that

$$\begin{aligned} &\lim_{h\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_h(u),\nabla T_h(u))D^iT_h(u)\,dx \\ &\le \lim_{h\to\infty}\liminf_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)\,dx \\ &\le \lim_{h\to\infty}\limsup_{n\to\infty}\frac{1}{h}\sum_{i=1}^{N}\int_{\Omega}a_i(x,T_h(u_n),\nabla T_h(u_n))D^iT_h(u_n)\,dx = 0, \end{aligned} \tag{6.36}$$

which prove (5.1).

Step 4: The equi-integrability of the sequences $(H_n(x,u_n,\nabla u_n))_n$ and $\Big(\alpha(x)|u_n|^{r-1}u_n\Big)_n$.

In order to pass to the limit in the approximate problem (6.2), we shall show that

$$H_n(x,u_n,\nabla u_n) \longrightarrow H(x,u,\nabla u) \quad \text{and} \quad \alpha(x)|u_n|^{r-1}u_n \longrightarrow \alpha(x)|u|^{r-1}u \tag{6.37}$$

strongly in $L^1(\Omega)$.

Thanks to (6.35) we have $H_n(x,u_n,\nabla u_n) \to H(x,u,\nabla u)$ and $|u_n|^{r-1}u_n \to |u|^{r-1}u$ a.e. in Ω. Then, in view of Vitali's theorem, it suffices to prove that the sequences $(H_n(x,u_n,\nabla u_n))_n$ and $(\alpha(x)|u_n|^{r-1}u_n)_n$ are uniformly equi-integrable.

For any measurable subset $E \subseteq \Omega$ we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_E d(|u_n|)|D^i u_n|^{p_i}\,dx + \int_E \alpha(x)|u_n|^r\,dx \\
&\le \sum_{i=1}^{N} \int_E d(|T_{h(\eta)}(u_n)|)|D^i T_{h(\eta)}(u_n)|^{p_i}\,dx + \int_E \alpha(x)|T_{h(\eta)}(u_n)|^r\,dx \\
&\quad + \sum_{i=1}^{N} \int_{\{h(\eta)<|u_n|\}} d(|u_n|)|D^i u_n|^{p_i}\,dx + \int_{\{h(\eta)<|u_n|\}} \alpha(x)|u_n|^r\,dx.
\end{aligned} \tag{6.38}$$

Thanks to (6.35), there exists $\beta(\eta) > 0$ such that

$$\sum_{i=1}^{N} \int_E d(|T_{h(\eta)}(u_n)|)|D^i T_{h(\eta)}(u_n)|^{p_i}\,dx + \int_E \alpha(x)|T_{h(\eta)}(u_n)|^r\,dx \le \frac{\eta}{2}, \tag{6.39}$$

for any $E \subset \Omega$ with $\text{meas}(E) \le \beta(\eta)$. Moreover, in view of (6.21) and (6.22) we obtain: for all $\eta > 0$, there exists $h(\eta) > 0$ such that

$$\sum_{i=1}^{N} \int_{\{h(\eta)<|u_n|\}} d(|u_n|)|D^i u_n|^{p_i}\,dx + \int_{\{h(\eta)<|u_n|\}} \alpha(x)|u_n|^r\,dx \le \frac{\eta}{2} \quad \text{for all } \ h \ge h(\eta). \tag{6.40}$$

By combining (6.38), (6.39) and (6.40), one easily has

$$\sum_{i=1}^{N} \int_E d(|u_n|)|D^i u_n|^{p_i}\,dx + \int_E \alpha(x)|u_n|^r\,dx \le \eta \tag{6.41}$$

for all E such that $\text{meas}(E) \le \beta(\eta)$.

Then, the sequences $\big(\alpha(x)|u_n|^{r-1}u_n\big)_n$ is uniformly equi-integrable. Moreover, thanks to (3.5) we conclude that the sequence $(H_n(x,u_n,\nabla u_n))_n$ is also equi-integrable. In view of Vitali's theorem, the convergence (6.37) is concluded.

Step 5: Passage to the limit

Let $\varphi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$, and choosing $S(\cdot)$ a smooth function in $W^{1,\infty}(I\!R)$ such that supp $(S(\cdot)) \subseteq [-M, M]$ for some $M \ge 0$. By choosing $S(u_n)\varphi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$ as a test function in the approximate problem (6.2), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n) D^i \varphi)\,dx \\
&\quad + \int_\Omega H_n(x,u_n,\nabla u_n) S(u_n)\varphi\,dx + \int_\Omega \alpha(x)|u_n|^{s-1}u_n S(u_n)\varphi\,dx \\
&= \int_\Omega f_n S(u_n)\varphi\,dx + \int_{\partial\Omega} g_n S(u_n)\varphi\,d\sigma.
\end{aligned} \tag{6.42}$$

In view of (6.35), we have $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p_i'}(\Omega)$, and since $a_i(x, T_M(u_n), \nabla T_M(u_n))$ tends to $a_i(x, T_M(u), \nabla T_M(u))$ almost everywhere in Ω, it follows that

$$a_i(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a_i(x, T_M(u), \nabla T_M(u)) \quad \text{weakly in} \quad L^{p_i'}(\Omega),$$

and since $S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n))D^i\varphi$ tends strongly to $S'(u)\varphi D^i T_M(u) + S(T_M(u))D^i\varphi$ in $L^{p_i}(\Omega)$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(S'(u_n)\varphi D^i u_n + S(u_n)D^i\varphi)dx \\
&= \lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \left(S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n))D^i\varphi\right)dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u), \nabla T_M(u)) \left(S'(u)\varphi D^i T_M(u) + S(T_M(u))D^i\varphi\right) dx \\
&= \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \left(S'(u)\varphi D^i u + S(u)D^i\varphi\right) dx.
\end{aligned} \tag{6.43}$$

Concerning the second and third terms on the left-hand side of (6.42), we have $S(u_n)\varphi \rightharpoonup S(u)\varphi$ weak$-*$ in $L^\infty(\Omega)$, and thanks to (6.37) we deduce that

$$\lim_{n\to\infty} \int_\Omega H_n(x, u_n, \nabla u_n)S(u_n)\varphi\, dx = \int_\Omega H(x, u, \nabla u)S(u)\varphi\, dx, \tag{6.44}$$

and

$$\lim_{n\to\infty} \int_\Omega \alpha(x)|u_n|^{r-1}u_n S(u_n)\varphi\, dx = \int_\Omega \alpha(x)|u|^{r-1}uS(u)\varphi\, dx. \tag{6.45}$$

Moreover, we have

$$\lim_{n\to\infty} \int_\Omega f_n S(u_n)\varphi\, dx = \int_\Omega f S(u)\varphi\, dx. \tag{6.46}$$

Similarly, we have $S(u_n)\varphi \rightharpoonup S(u)\varphi$ weak$-*$ in $L^\infty(\partial\Omega)$ then

$$\lim_{n\to\infty} \int_{\partial\Omega} g_n S(u_n)\varphi\, d\sigma = \int_{\partial\Omega} gS(u)\varphi\, d\sigma. \tag{6.47}$$

Hence, putting all the terms (6.42) and (6.43)–(6.47) together, we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u) \left(S'(u)\varphi D^i u + S(u)D^i\varphi\right)\, dx + \int_\Omega H(x, u, \nabla u)S(u)\varphi\, dx \\
&\quad + \int_\Omega \alpha(x)|u|^{r-1}uS(u)\varphi\, dx = \int_\Omega fS(u)\varphi\, dx + \int_{\partial\Omega} gS(u)\varphi\, d\sigma,
\end{aligned} \tag{6.48}$$

which conclude the proof of Theorem 5.2.

Example:

Let $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$, we consider the following Carathéodory functions

$$a_i(x,u,\nabla u) = \frac{\left|D^i u\right|^{p_i-2} D^i u}{(1+|u|)^{\lambda}} \qquad \text{and} \qquad H(x,u,\nabla u) = -\sum_{i=1}^{N} \frac{|D^i u|^{p_i}}{(1+|u|)^{p_i+\lambda}}.$$

It is clear that the functions $a_i(x,u,\nabla u)$ and $H(x,u,\nabla u)$ verify the conditions (3.2)–(3.4) and (3.5) respectively. In view of the Theorem 4.2, the strongly nonlinear elliptic problem

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} D^i \left(\frac{\left|D^i u\right|^{p_i-2} D^i u}{(1+|u|)^{\lambda}} \right) + |u|^{\underline{p}-2} u = f + \sum_{i=1}^{N} \frac{|D^i u|^{p_i}}{(1+|u|)^{p_i+\lambda}} & \text{in } \Omega, \\ \displaystyle\sum_{i=1}^{N} \frac{\left|D^i u\right|^{p_i-2} D^i u}{(1+|u|)^{\lambda}} \cdot n_i = g & \text{on } \partial\Omega, \end{cases} \tag{6.49}$$

has at least one renormalized solution $u \in T_{tr}^{1,\vec{p}}(\Omega)$, i.e.

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega} \frac{\left|D^i u\right|^{p_i-2} D^i u}{(1+|u|)^{\lambda}} \left(S'(u)\varphi D^i u + S(u) D^i \varphi\right) \, dx \\ &\qquad - \sum_{i=1}^{N} \int_{\Omega} \frac{|D^i u|^{p_i}}{(1+|u|)^{p_i+\lambda}} S(u)\varphi \, dx + \int_{\Omega} |u|^{\underline{p}-2} u S(u)\varphi \, dx \\ &= \int_{\Omega} f \; S(u)\varphi \, dx + \int_{\partial\Omega} g \; S(u)\varphi \, d\sigma, \end{aligned} \tag{6.50}$$

for every $\varphi \in W^{1,\vec{p}}(\Omega) \cap L^{\infty}(\Omega)$ and any smooth function $S(\cdot) \in W^{1,\infty}(\Omega)$ with a compact support.

References

[1] F. Andreu, J. M. Mazón, S. Segura de León, J. Toledo: Quasi-linear elliptic and parabolic equations in L^1 with non-linear boundary conditions, Adv. Math. Sci. Appl. 7 (1997), pp. 183–213.

[2] S. Antontsev, M. Chipot: Anisotropic equations: uniqueness and existence results, J. Differential and Integral Equations Volume 21, Numbers 5-6 (2008), 401–419.

[3] S. N. Antontsev, J. F. Rodrigues: On stationary thermorheological viscous flows, Ann. Univ. Ferrara Sez. VII Sci. Mat. 52 (2006), 19–36.

[4] M. B. Benboubker, H. Hijaj, S. Ouaro: Entropy solutions to nonlinear elliptic anisotropic problem with variable exponent. J. Appl. Anal. Comput. 4 (2014), no. 3, 245–270.

[5] M. Ben Cheikh Ali, O. Guibé: Nonlinear and non-coercive elliptic problems with integrable data. Adv. Math. Sci. Appl. 16 (2006), no. 1, 275–297.

[6] M. Bendahmane, M. Chrif, S. El Manouni: An Approximation Result in Generalized Anisotropic Sobolev Spaces and Application. Z. Anal. Anwend. **30** (2011), no. 3, 341–353.

[7] P. Bénilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, J. L . Vázquez: An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Ann. Scuola Norm. Sup. Pisa Cl. Sci. 4, (1995), 241–273.

[8] M. F. Betta, O. Guibé, A. Mercaldo: Neumann problems for nonlinear elliptic equations with L^1-data. J. Differential Equations, 259(3) (2015), 898–924.
[9] D. Blanchard, F. Murat, H. Redwane: Existence and Uniqueness of a Renormalized Solution for a Fairly General Class of Nonlinear Parabolic Problems, J. Differential Equations, 177 (2001), 331–374.
[10] L. Boccardo, T. Gallouêt, F. Murat: Unicité de la solution de certaines équations elliptiques non linéaires, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), 1159–1164.
[11] L. Boccardo, D. Giachetti, J.-I. Diaz, F. Murat: Existence and regularity of renormalized solutions for some elliptic problems involving derivations of nonlinear terms, J. Differential Equations 106 (1993), 215–237.
[12] G. Dal Maso, F. Murat, L. Orsina, A. Prignet: Renormalized solutions of elliptic equations with general measure data. Ann. Scuola Norm. Sup. Pisa Cl. Sci. Vol. **28 (4)** (1999), 741–808.
[13] R.-J. DiPerna, P.-L. Lions: On the Cauchy problem for Boltzmann equations: Global existence and weak stability, Ann. Math. 130 (1989), 321–366.
[14] R. Di Nardo, F. Feo: Existence and uniqueness for nonlinear anisotropic elliptic equations. Archiv der Mathematik, 2014, 102(2), 141–153.
[15] R. Di Nardo, F. Feo, O. Guibé: Uniqueness result for nonlinear anisotropic elliptic equations. Adv. Differential Equations 18 (2013), no. 5-6, 433–458.
[16] E. Hewitt, K. Stromberg: Real and abstract analysis. Springer-Verlag, Berlin Heidelberg New York, 1965.
[17] J. L. Lions: Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod et Gauthiers-Villars, Paris 1969.
[18] P.-L. Lions: Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible models, Oxford Univ. Press, Oxford, 1996.
[19] P.-L. Lions, F. Murat: Solutions renormalisées d'équations elliptiques, Manuscript for C.R.A.S. (juin 1992).
[20] J. Leray, J.-L. Lions: Quelques résultats de Višik sur les problèmes elliptiques non linéaires par les méthodes de Minty–Browder, Bull. Soc. Math. France 93 (1965), 97–107.
[21] M. Mihailescu, P. Pucci, V. Radulescu: Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent. J. Math. Anal. Appl., **340** (2008), 687–698.
[22] F. Murat: Soluciones normalizadas de EDP elipticas non lineales, Tech. Report R93023, Laboratoire d'Analyse Numérique, Paris VI, 1993, Cours à l'Université de Séville.
[23] F. Murat: Équations elliptiques non linéaires avec second membre L^1 ou mesure, in Comptes Rendus du 26ème Congrès National d'Analyse Numérique, Les Karellis, pp. A12–A24, 1994.
[24] M. Růžička: Electrorheological fluids: modeling and mathematical theory. Lecture Notes in Mathematics, 1748. Springer-Verlag, Berlin, 2000.
[25] L. Tartar: Imbedding theorems of Sobolev spaces into Lorentz spaces, BUMI Serie 8 (1988), 479–500.
[26] V. V. Zhikov: Averaging of functionals in the calculus of variations and elasticity, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1986), no. 4, 675–710.

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