Rend. Mat. Appl. (7),
Online first (2024), 29 pp.

Rendiconti di Matematica
e delle sue Applicazioni

Global Kodaira–Spencer class and Massey products

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Abstract. *We define a new notion of supported global deformation class for a semistable family of complex varieties over a curve $f\colon X \to B$. We use this notion to study when X, possibly up to a finite covering, has a generically finite morphism onto a product $B \times Y$ with Y of general type.*

1. Introduction

Let $f\colon X \to B$ be a semistable family of complex varieties over a complex curve B and with smooth $(n-1)$-dimensional general member denoted by X_b, $b \in B$. The Kodaira–Spencer map at b identifies a vector subspace inside the space of infinitesimal deformations of X_b. It is a natural question to study $f\colon X \to B$ in terms of these infinitesimal deformations. In particular, the importance of *supported deformations* in the theory of curves and of fibrations on a surface is well-known; see [1, p. 2 and section 6] and [23].

In this paper we take a step forward and we construct a supported *global* deformation class $\rho(\xi)$ naturally given by f; see Definition 1.1, [26] and also [13] in the case of a fibered surface. The technical core of the paper is to find the relation between $\rho(\xi)$ and the theory of relative Massey products. Here we recall that the notion of Massey products in algebraic geometry has been introduced in [10] and [23] and then applied in [21, 9, 13, 3, 22, 28, 29, 30, 31, 8] and [27]. We refer to these sources for a complete discussion and to Section 2 for a brief review.

1.1. Main results

The sheaf on B whose elements are the holomorphic 1-forms on the fibers of f which are liftable to *closed* holomorphic forms of X is a local system which we denote by $\mathbb{D}^1$. Let $L \leq \Gamma(A, \mathbb{D}^1)$ be a vector space of $\dim L = l \geq n$ on an open set $A \subseteq B$. Denote by s_i, $i = 1, \dots, l$, a choice of closed forms on X which are liftings of the elements of a basis of L. We define $\mathcal{D}^A$ as the divisor in $f^{-1}(A)$ given by the common zeroes of the sections $s_{i_1} \wedge \cdots \wedge s_{i_{n-1}} \wedge \sigma$ where the s_{i_j} run among the liftings above and σ runs over the local sections of ω_B. We denote by $\mathcal{D}$ the divisor obtained by the horizontal components of $\mathcal{D}^A$ and with D_b the restriction of $\mathcal{D}$ to the general fiber X_b. We call $\mathcal{D}$ the horizontal divisor associated to L.

2020 Mathematics Subject Classification: 14D06, 14D05, 14E20, 14J40.
Keywords: Semistable Fibrations, Global Kodaira–Spencer class, Supported deformations, Massey products, Local systems.

Finally, we say that L is *strict* if the morphism $\bigwedge^{n-1} L\otimes\omega_{B_{|A}}\to f_*\omega_{X_{|A}}$ is an injection of vector bundles, see Definition 2.6.

The two main applications of our point of view are as follows. The first one is strictly related to the class $\rho(\xi)$ and the second one is related to the Albanese morphism.

Theorem [A]. *Let $f\colon X\to B$ be a semistable fibration. Assume that there exist a strict subspace $L\leq\Gamma(A,\mathbb{D}^1)$, $\dim L\geq n$, such that $f_*\mathcal{O}_X(\mathcal{D})$ is a line bundle and that the deformation class $\rho(\xi)$ is supported on $\mathcal{D}$, where $\mathcal{D}$ is the horizontal divisor associated to L. Then up to a finite étale covering $\widetilde{B}\to B$ the associated base change $\widetilde{X}$ has a generically finite surjective morphism $\widetilde{X}\to\widetilde{B}\times Y$, where Y is an $(n-1)$-dimensional variety of general type.*

See Corollary 4.21. We will show that this result is based on the possibility to pass from local conditions on A to some global conditions on the finite covering $\widetilde{B}$.

We also give an analogue of the notion of strictness in the case of $(n-1)$-forms, instead of 1-forms. This allows to show the following result on the Albanese morphism:

Theorem [B]. *Let X be a smooth n-dimensional variety and $\alpha\colon X\to A:=$ Alb(X) its Albanese morphism. Assume that $\mathcal{L}:=\operatorname{Im}(\alpha^*\Omega^{n-1}_A\to\Omega^{n-1}_X)$ is a line bundle on X, then the global sections of $\mathcal{L}$ define a rational map $h\colon X\dashrightarrow Y$ to a variety Y of general type. Furthermore if $H^0(X,\mathcal{L})$ is strict, we can take h to be a morphism and Y is the Stein factorization of $X\to Z$ where $Z:=\alpha(X)$. Finally if the restrictions of the Albanese map to the fibers X_b have degree 1, then these fibers are birational to Y.*

See Theorem 6.1 and following Corollary.

1.2. Global Kodaira–Spencer class

We recall that all the fibers X_b are $n-1$-dimensional and either smooth or reduced and normal crossing divisors. The open set of B corresponding to smooth fibers will be denoted by B^0 and its complement $B\setminus B^0$ is the image of the singular fibers.

The exact sequence defining the sheaf of relative differential forms

$$0\to f^*\omega_B\to\Omega^1_X\to\Omega^1_{X/B}\to 0 \tag{1.1}$$

gives the associated extension class $\xi\in\operatorname{Ext}^1(\Omega^1_{X/B},f^*\omega_B)$. By restriction of Sequence (1.1) to the general fiber we get the sequence

$$0\to\mathcal{O}_{X_b}\otimes T^\vee_{B,b}\to\Omega^1_{X|X_b}\to\Omega^1_{X_b}\to 0 \tag{1.2}$$

and we construct the classes

$$\xi_b\in\operatorname{Ext}^1(\Omega^1_{X_b},\mathcal{O}_{X_b})\otimes T^\vee_{B,b}=H^1(X_b,T_{X_b})\otimes T^\vee_{B,b}. \tag{1.3}$$

where $b \in B^0$.

All the extensions ξ_b can be encoded in a unique object thanks to the notion of relative extension sheaf $\mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)$. Recall that the relative extension sheaf $\mathcal{E}xt^p_f$ is by definition the p-th derived functor of $f_*\mathcal{H}om$, hence $\mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)$ is a sheaf on the base B isomorphic to $\mathcal{E}xt^1_f(\Omega^1_{X/B}, \mathcal{O}_X) \otimes \omega_B$; see cf. [33]. Now by applying the functor $f_*\mathcal{H}om$ to Sequence (1.1) there is a morphism

$$\mathcal{O}_B \to \mathcal{E}xt^1_f(\Omega^1_{X/B}, \mathcal{O}_X) \otimes \omega_B.$$

We call *Global Kodaira–Spencer map of the family* f the image of $1 \in H^0(B, \mathcal{O}_B)$. It can then be seen as a sheaf morphism

$$\rho(\xi)\colon T_B \to \mathcal{E}xt^1_f(\Omega^1_{X/B}, \mathcal{O}_X) \tag{1.4}$$

whose restriction to the general $b \in B$ gives back the usual Kodaira–Spencer map.

It remains defined an homomorphism

$$\rho\colon \operatorname{Ext}^1(\Omega^1_{X/B}, f^*\omega_B) \to H^0(B, \mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)) \tag{1.5}$$

which is surjective and it is also an isomorphism when the general fiber X_b is of general type; see Section 4 for all the details on this construction.

1.3. Globally supported deformation

We refer the reader to Section 2 for the notion of Massey triviality; in particular see Definition 2.3. Here we recall that the condition of Massey triviality is a basic tool to study both the vector bundle K_∂ of holomorphic 1-forms on the fibers X_b which are locally liftable to X, and the local system $\mathbb{D}^1$ of holomorphic 1-forms on the fibers which are liftable to *closed* holomorphic forms of X. See [22, 31, 14, 15].

Now consider $L \leq \Gamma(A, \mathbb{D}^1)$, $\dim L \geq n$, and recall that we denote by $\mathcal{D}$ the horizontal divisor associated to L and with D_b the restriction of $\mathcal{D}$ to the general fiber X_b.

We define the following sheaf on $A \subseteq B$

$$\mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}), f^*\omega_B) := \mathcal{E}xt^1_f(\Omega^1_{X/B_{|f^{-1}(A)}}(-\mathcal{D}), f^*\omega_{B|A})$$

and we can finally recall the definition of supported class.

Defintion 1.1. We say that $\rho(\xi)$ is supported on $\mathcal{D}$ if

$$\rho(\xi)_{|A} \in \operatorname{Ker} H^0(A, \mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)) \to H^0(A, \mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}), f^*\omega_B)). \tag{1.6}$$

The following result is a full generalization of [23, Theorem 1.5.1] and [30, Theorem A].

Theorem [C]. *Let $L \leq \Gamma(A, \mathbb{D}^1)$ be a vector space of $\dim L \geq n$. Assume that L is Massey trivial and that it generically generates $\Omega^1_{X_b}$ on the general fiber. Then $\rho(\xi)$ is supported on $\mathcal{D}_{|f^{-1}(A')}$, where $A' \subset A$ is an open dense subset.*

Viceversa, assume that $\rho(\xi)$ is supported on $\mathcal{D}$. If $f_*\mathcal{O}_X(\mathcal{D})$ is a line bundle then the vector space L is Massey trivial.

See: Theorem 4.13 and 4.17. We point out that actually Theorem [A] is a consequence of this result.

In Section 5 we construct the theory, parallel to the above one, in the case of volume forms on the fibers X_b, that is we consider $n-1$-forms instead of 1-forms. In particular, we prove Theorem 5.4 which is an analogue of Theorem [A] and shows conditions on volume forms which guarantee the existence of a variety of general type Y as in Theorem [A]. In this section we also give some bound on the geometric genus of Y in the case of a relatively minimal fibered threefold $f\colon X \to B$ under some hypotheses on the canonical map $\phi_{|K_{X_b}|}$ of the general fiber X_b following [25].

1.4. Other results

Finally we point out that in Section 2 we revise the theory of Massey products according a new perspective and this lead us to show, in Section 3, a relative version of our old theorem on adjoint quadrics [30, Theorem B]. Indeed we think that Theorem 3.4 has its own interest as a criterion for Massey triviality and as a tool to show finiteness results on certain monodromy groups, see Corollary 3.6.

2. Massey products and local systems

In this section we briefly recall and discuss the main constructions of [31], in particular we give the rigorous definition of the vector bundle K_∂, the local system $\mathbb{D}^1$ and the notion of Massey triviality mentioned in the Introduction.

2.1. Local systems of certain liftable holomorphic forms

Let X be a smooth complex compact n-dimensional variety and B a smooth complex curve. From the Introduction we recall that we consider semistable fibrations $f\colon X \to B$ where $X_b = f^{-1}(b)$ denotes the fiber over a point $b \in B$. All the fibers X_b are either smooth or reduced and normal crossing divisors. Let B_0 be the locus of singular values of f and $B^0 = B \setminus B_0$ the open set of regular values. Consider the exact sequence

$$0 \to f^*\omega_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0 \tag{2.1}$$

defining the sheaf of relative differentials $\Omega^1_{X/B}$. It is not difficult to see that, under our hypothesis on f, the sheaf $\Omega^1_{X/B}$ is torsion free but not locally free in general. In the following we will denote by $\Omega^p_{X/B}$ the wedge product of $\Omega^1_{X/B}$, that is $\Omega^p_{X/B} = \bigwedge^p \Omega^1_{X/B}$, and by $\omega_{X/B}$ the relative dualizing sheaf of f.

Taking the pushforward of Sequence (2.1) we obtain the long exact sequence on B

$$0 \to \omega_B \to f_*\Omega^1_X \to f_*\Omega^1_{X/B} \to R^1 f_*\mathcal{O}_X \otimes \omega_B \to \cdots \tag{2.2}$$

and we call K_∂ the cokernel in the exact sequence

$$0 \to \omega_B \to f_*\Omega^1_X \to K_\partial \to 0. \tag{2.3}$$

Intuitively we can think of K_∂ as the vector bundle of holomorphic 1-forms on the fibers of f which are locally liftable to the variety X. A key property of K_∂ is given in the following Lemma, see [22, Lemma 3.5] or [31, Lemma 2.2].

Lemma 2.1. *If $f\colon X \to B$ is a semistable fibration, the exact sequence*

$$0 \to \omega_B \to f_*\Omega^1_X \to K_\partial \to 0 \tag{2.4}$$

splits.

This means that the above intuitive idea is not only true locally around the fibers, but the liftability holds on every open subset of B. For a more complete study of K_∂ see [22, 14, 15] for the case $n = 2$, [31] for the general case.

If in Sequence (2.3) we consider, instead of Ω^1_X, the sheaf $\Omega^1_{X,d}$ of de Rham closed differential forms, we obtain the exact sequence

$$0 \to \omega_B \to f_*\Omega^1_{X,d} \to \mathbb{D}^1 \to 0. \tag{2.5}$$

It turns out that $\mathbb{D}^1$ is a local system on the curve B as shown in [22] for 1-dimensional fibers and in [31] for any dimension. Note that $\mathbb{D}^1$ is a subsheaf of K_∂ and we can interpret $\mathbb{D}^1$ as the local system of holomorphic 1-forms on the fibers of f which are liftable to *closed* holomorphic forms of the variety X. Finally note that, by Lemma 2.1, also the exact Sequence (2.5) splits.

2.2. Massey products

Massey products, originally called adjoint forms, have been introduced in [10] and [23]. They have been useful for the study of infinitesimal deformations and also for the study of the monodromy of the above mentioned local systems.

We now recall their construction. This presentation is slightly different but equivalent to the one in [31], and it will be more convenient for the applications contained in this paper.

According to Lemma 2.1, Sequence (2.3) splits; from now on for simplicity we choose and fix one of these splittings. The following wedge product sequence also splits

$$0 \longrightarrow \bigwedge^{n-1} K_\partial \otimes \omega_B \longrightarrow \bigwedge^{n} f_*\Omega^1_X \longrightarrow \bigwedge^{n} K_\partial \longrightarrow 0 \tag{2.6}$$

and we take the composition of this splitting with the natural wedge map and obtain the morphism

$$\lambda\colon \bigwedge^n K_\partial \to \bigwedge^n f_*\Omega^1_X \to f_* \bigwedge^n \Omega^1_X = f_*\omega_X. \tag{2.7}$$

Now consider n sections $\eta_1, \dots, \eta_n \in \Gamma(A, K_\partial)$ on an open subset $A \subseteq B$; call $s_1, \dots, s_n \in \Gamma(A, f_*\Omega^1_X)$ liftings of $\eta_1, \dots, \eta_n$ according to the above chosen splitting.

Defintion 2.2. We call ω_i, $i = 1, \dots, n$, the wedge $s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_n \in \Gamma(A, f_*\Omega^{n-1}_X)$ and $\mathcal{W}$ the submodule of $f_*\omega_X$ generated by $\langle \omega_i \rangle \otimes \omega_B$.

Defintion 2.3. The *Massey product or adjoint image* of $\eta_1, \dots, \eta_n$ is the section $\omega \in \Gamma(A, f_*\omega_X)$ given by $\omega = \lambda(\eta_1 \wedge \cdots \wedge \eta_n)$. We say that the sections $\eta_1, \dots, \eta_n$ are *Massey trivial* if their Massey product is contained in the submodule $\mathcal{W}$.

Remark 2.4. The Massey product is given explicitly by $s_1 \wedge \cdots \wedge s_n$ and being Massey trivial means that locally

$$s_1 \wedge \cdots \wedge s_n = \sum_i \omega_i \otimes \sigma_i$$

where the ω_i are as in Definition 2.2 and σ_i are local sections of ω_B.

As a section of $f_*\omega_X$, the Massey product certainly depends on the choice of the splitting mentioned above. On the other hand, the condition of being Massey trivial does not; see [31]. In Proposition 2.7 we will show that if the sections $\eta_1, \dots, \eta_n$ are Massey trivial, there is a very convenient choice for this splitting.

In the literature mentioned at the beginning, the construction of Massey products is done pointwise, that is for a fixed regular value $b \in B$ and working on the fiber X_b and on an infinitesimal neighbourhood of this fiber. It is not difficult to see that all the pointwise defined Massey products can be glued together and this agrees exactly with Definition 2.3 on suitable open subsets $A \subset B$.

Of course since $\mathbb{D}^1$ is a subsheaf of K_∂, it makes sense to construct Massey products starting from sections of $\mathbb{D}^1$, i.e. consider sections $\eta_i \in \Gamma(A, \mathbb{D}^1)$. One of the key points in [22] and [31] is exactly to consider this setting.

To conclude this section we recall the notion of strictness and its relation with Massey triviality. Let $A \subseteq B$ be an open subset and $W \leq \Gamma(A, K_\partial)$ a vector subspace of dimension at least n.

Defintion 2.5. We say that W is Massey trivial if any n-uple of linearly independent sections in W is Massey trivial (according to Definition 2.3).

Following [5, Definition 2.1 and 2.2], we have

Defintion 2.6. We say that W is strict if the morphism

$$\bigwedge^{n-1} W \otimes \omega_{B_{|A}} \to f_*\omega_{X_{|A}}$$

is an injection of vector bundles.

The following proposition shows how Massey triviality and strictness give a preferred choice of liftings as we anticipated in Remark 2.4.

Proposition 2.7. *Let $W \leq \Gamma(B, K_\partial)$ be a strict subspace of global sections of K_∂ and let $A \subseteq B$ be an open contractible subset. If the sections of W are Massey trivial when restricted to A then there exist a unique lifting $\widetilde{W} \leq \Gamma(B, f_*\Omega^1_X)$ such that*

$$\bigwedge^n \widetilde{W} \to \Gamma(B, f_*\omega_X)$$

is zero. If furthermore $W \leq \Gamma(B, \mathbb{D}^1)$ then $\widetilde{W} \leq \Gamma(B, f_*\Omega^1_{X,d})$.

For the proof see [31, Proposition 4.10]. As seen in Remark 2.4, if the sections $\eta_i \in W$ are Massey trivial, for any choice of liftings s_i we have a relation of the form

$$s_1 \wedge \cdots \wedge s_n = \sum_i \omega_i \otimes \sigma_i.$$

This proposition tells us that actually there is a preferred choice of liftings $\tilde{s}_i$ such that

$$\tilde{s}_1 \wedge \cdots \wedge \tilde{s}_n = 0.$$

It also tells us that local Massey triviality implies global Massey triviality.

Remark 2.8. We stress that the strictness condition is essential to prove Proposition 2.7 if $\dim W > n$.

3. Relative Adjoint quadrics

As a natural continuation of [31], in this section we study the generalization of the notion of adjoint quadrics, introduced in [30]. As we will see, the presence or absence of certain quadratic relations is strictly related to the notion of Massey triviality.

In the following, consider as before an open subset $A \subseteq B$ and $\{\eta_1, \ldots, \eta_n\}$ a basis of an n-dimensional vector space $W \leq \Gamma(A, \mathbb{D}^1)$. Choosing a splitting of

$$0 \longrightarrow \omega_B \longrightarrow f_*\Omega^1_{X,d} \longrightarrow \mathbb{D}^1 \longrightarrow 0 \tag{3.1}$$

and $s_1, \ldots, s_n \in \Gamma(A, f_*\Omega^1_{X,d})$ liftings of $\eta_1, \ldots, \eta_n$ accordingly, we denote by ω the Massey product of the η_i. With our choice of liftings, ω is explicitly given by $s_1 \wedge \cdots \wedge s_n$. Also recall that by definition $\omega_i := s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_n$. We have the following definition

Defintion 3.1. A *relative adjoint quadric* is a local quadratic relation of sections of $\Gamma(A, f_*\omega_X \otimes f_*\omega_X)$ of the form

$$\omega^2 = \sum (\omega_i \wedge \sigma_i) \cdot \rho_i$$

where σ_i are local sections of ω_B, ρ_i of $f_*\omega_X$ and ω, ω_i are as above.

To study the role of these relations we construct a commutative diagram as follows. Take the exact sequence

$$0 \to f^*\omega_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0$$

and the surjective morphism from its top wedge product

$$\Omega^{n-1}_X \to \Omega^{n-1}_{X/B}.$$

We take the tensor product with $f^*\omega_B$ followed by the direct image f_*. Denoting by K the kernel of the resulting morphism, we have a long exact sequence of sheaves on B

$$0 \to K \to f_*\Omega^{n-1}_X \otimes \omega_B \to f_*\Omega^{n-1}_{X/B} \otimes \omega_B \to \cdots \tag{3.2}$$

It is well known that we have a map $f_*\Omega^{n-1}_{X/B} \to f_*\omega_{X/B}$ (which is an isomorphism on B^0). From this map we actually obtain $f_*\Omega^{n-1}_{X/B} \otimes \omega_B \to f_*\omega_{X/B} \otimes \omega_B = f_*\omega_X$ hence we can add the diagonal morphism

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & f_*\Omega^{n-1}_X \otimes \omega_B & \longrightarrow & f_*\Omega^{n-1}_{X/B} \otimes \omega_B & \longrightarrow & \cdots \\ & & & & & & & \searrow & \\ & & & & & & & & f_*\omega_X \end{array} \tag{3.3}$$

We complete the diagram on A with the following second row and appropriate morphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & f_*\Omega^{n-1}_X \otimes \omega_B & \longrightarrow & f_*\Omega^{n-1}_{X/B} \otimes \omega_B & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \phi_\omega} & & \downarrow & \searrow & \\ & & & & & & & & f_*\omega_X \\ & & & & & & & \swarrow{\scriptstyle \cdot\omega} & \\ 0 & \longrightarrow & K' & \longrightarrow & \bigwedge^{n-1} \mathbb{D}^1 \otimes \omega_B \otimes f_*\omega_X & \xrightarrow{\nu\otimes id} & f_*\omega_X \otimes f_*\omega_X & \longrightarrow & \cdots \end{array} \tag{3.4}$$

The maps above are defined as follows.

Firstly $\cdot\omega$ is just the multiplication by the Massey product ω, sending a section τ of $f_*\omega_X$ to $\tau \cdot \omega$.

The map $\nu\colon \bigwedge^{n-1} \mathbb{D}^1 \otimes \omega_B \to f_*\omega_X$ is given by taking $n-1$ sections of $\mathbb{D}^1$, call them $\mu_1, \dots, \mu_{n-1}$, liftings of these sections, $t_1, \dots, t_{n-1}$, according to our fixed splitting of Sequence (3.1) and defining $\nu(\mu_1 \wedge \cdots \wedge \mu_{n-1} \otimes \sigma) = t_1 \wedge \cdots \wedge t_{n-1} \wedge \sigma$ for σ in ω_B. In particular note that

$$\nu(\eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_n \otimes \sigma) = \omega_i \wedge \sigma. \tag{3.5}$$

Finally ϕ_ω is given by the aforementioned liftings $s_1, \dots, s_n$ as follows. Locally, given a section s of $f_*\Omega^{n-1}_X$, the image $\phi_\omega(s \otimes \sigma)$ is

$$\phi_\omega(s \otimes \sigma) = \sum_i (-1)^i \eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_n \otimes \sigma \otimes s \wedge s_i. \tag{3.6}$$

It is easy to see by commutativity that ϕ_ω restricts to a map between the kernels of the two sequences $\psi\colon K \to K'$.

Defintion 3.2. We say that the Massey product $\omega \in \Gamma(A, f_*\omega_X)$ is locally liftable if it is in the image of the sheaf morphism

$$f_*\Omega^{n-1}_X \otimes \omega_B \to f_*\omega_X$$

of Diagram 3.4.

Remark 3.3. We recall that, by a famous result of Fujita, see [11] and [12], the direct image $f_*\omega_{X/B}$ is a sum

$$f_*\omega_{X/B} \cong \mathcal{U} \oplus \mathcal{A} \tag{3.7}$$

where $\mathcal{U}$ is a unitary flat vector bundle and $\mathcal{A}$ is ample. The local system associated to $\mathcal{U}$ is usally denoted by $\mathbb{U}$. By taking the tensor product with ω_B, we also get a direct sum decomposition for $f_*\omega_X$.

Now consider the sheaf $\Omega^{n-1}_{X,d}$ of de Rham closed $n-1$-forms. The Massey products in the image of $f_*\Omega^{n-1}_{X,d} \otimes \omega_B \to f_*\omega_X$ are in particular locally liftable and furthermore they are elements $\mathbb{U} \otimes \omega_B$. This means that this theory is well suited to approach the natural question of what happens when the Massey product of sections $\eta_i \in \Gamma(A, \mathbb{D}^1)$ ends up in $\Gamma(A, \mathbb{U} \otimes \omega_B)$, that is in the part of $f_*\omega_X$ given by the local system of the Fujita decomposition.

See Section 5 of this paper for more details on the local systems of relative $n-1$-forms.

The generalization of [30, Theorem 2.1.2] is:

Theorem 3.4. *Let $f\colon X \to B$ be a semistable fibration. Assume that there exist $\{\eta_1, \dots, \eta_n\}$, basis of an n-dimensional vector space $W \leq \Gamma(A, \mathbb{D}^1)$, such that their Massey product $\omega \in \Gamma(A, f_*\omega_X)$ is locally liftable and furthermore there are no relative adjoint quadrics, then W is Massey trivial.*

Proof. Since the Massey product ω is locally liftable, we call $\tilde{\omega}_\alpha$ the local lifting of $\omega_{|A_\alpha}$ in $f_*\Omega^{n-1}_X \otimes \omega_B$, $A = \bigcup A_\alpha$ an open covering. The difference between two such liftings is in K, hence by the commutativity of the first square of Diagram (3.4), we have that $(\nu \otimes id)(\phi_\omega(\tilde{\omega}_\alpha))$ glue together to a section of $\Gamma(A, f_*\omega_X \otimes f_*\omega_X)$ which we will denote, by abuse of notation, $(\nu \otimes id)(\phi_\omega(\tilde{\omega}))$.

Consider now the commutative square

$$\begin{array}{ccc} f_*\Omega^{n-1}_X \otimes \omega_B & \longrightarrow & f_*\omega_X \\ \big\downarrow{\scriptstyle \phi_\omega} & & \big\downarrow{\scriptstyle \cdot\omega} \\ \bigwedge^{n-1} \mathbb{D}^1 \otimes \omega_B \otimes f_*\omega_X & \xrightarrow{\nu\otimes id} & f_*\omega_X \otimes f_*\omega_X \end{array} \tag{3.8}$$

coming from Diagram (3.4). We have that $\omega^2 = (\nu \otimes id)(\phi_\omega(\tilde{\omega}))$. Now note that by definition $\phi_\omega(\tilde{\omega}_\alpha)$ is a sum containing the wedges $\eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_n$ as we have seen in (3.6). Now applying ν all these wedges $\eta_1 \wedge \cdots \wedge \widehat{\eta_i} \wedge \cdots \wedge \eta_n$ produces the sections ω_i as seen in (3.5).

We deduce that $\nu \otimes id(\phi_\omega(\tilde{\omega}))$ is locally of the form $\sum \omega_i \wedge \sigma_i \cdot \rho_i$ where σ_i are local sections of ω_B and ρ_i of $f_*\omega_X$. Now assume by contradiction that ω is not Massey trivial, then the relation $\omega^2 = \nu \otimes id(\phi_\omega(\tilde{\omega}))$ is a true quadratic relation (and not just the square of a linear relation) and gives a relative adjoint quadric. By our hypothesis these do not exist hence the contradiction and ω is Massey trivial. □

The first application of Theorem 3.4 comes from [31, Theorem B] and gives information on the monodromy associated to local systems generated by Massey trivial vector spaces. From now on we call L a vector subspace $L \leq \Gamma(A, \mathbb{D}^1)$ and $\mathbb{L}$ the local system generated by L, i.e. the stalk of $\mathbb{L}$ is $\sum_{g \in G} g \cdot L$ where G is the monodromy group acting non-trivially on $\mathbb{D}^1$.

Defintion 3.5. If L is Massey trivial, we will say that $\mathbb{L}$ is Massey trivial generated.

See [22, Definition 5.5]. Consider the action of the fundamental group $\pi_1(B, b)$ on the stalk of $\mathbb{L}$ and call $H_{\mathbb{L}}$ the subgroup of $\pi_1(B, b)$ acting trivially on $\mathbb{L}$ and $G_{\mathbb{L}} = \pi_1(B, b)/H_{\mathbb{L}}$ the associated monodromy group.

Corollary 3.6. *Let L be a strict vector space such that every Massey product of sections of L is locally liftable and assume that there are no relative adjoint quadrics. Then L is Massey trivial and the local system $\mathbb{L}$ is Massey trivial generated. In particular $\mathbb{L}$ has finite monodromy.*

Proof. Take n linearly independent sections of L and consider the associated Massey product. The Massey triviality follows from the previous theorem, hence L is a Massey trivial vector space. The local system $\mathbb{L}$ generated under the monodromy action is then Massey trivial generated by definition. Local systems generated by a strict and Massey trivial vector space have finite monodromy by [31, Theorem B]. □

For applications of this result see [31].

4. Global supported deformations

We recall that originally, see [28, 29, 30], Massey products have been used as a tool for the study of infinitesimal deformations. Here we generalize this setting in the case of semistable families $f\colon X \to B$, see also [26], before giving another consequence of Theorem 3.4.

4.1. The Global Kodaira–Spencer map

Consider again the exact sequence

$$0 \to f^*\omega_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0. \tag{4.1}$$

The restriction of Sequence (4.1) on a smooth fiber X_b is the sequence

$$0 \to \mathcal{O}_{X_b} \otimes T^\vee_{B,b} \to \Omega^1_{X|X_b} \to \Omega^1_{X_b} \to 0 \tag{4.2}$$

which is associated to an element

$$\xi_b \in H^1(X_b, T_{X_b}) \otimes T^\vee_{B,b} = \mathrm{Ext}^1(\Omega^1_{X_b}, \mathcal{O}_{X_b}) \otimes T^\vee_{B,b}. \tag{4.3}$$

Since $H^1(X_b, T_{X_b})$ is the space of first order deformations of X_b, the class ξ_b naturally corresponds to the deformation of the fiber X_b induced by the family $f\colon X \to B$. The key to encode all the extensions ξ_b in a unique object is the notion of relative extension sheaf. We have learned this tool from [33].

Defintion 4.1. Given a morphism of schemes $f\colon X \to Y$, the relative extension sheaf $\mathcal{E}xt^p_f$ is the p-th derived functor of $f_*\mathcal{H}om$.

For all the properties of the relative extension sheaves we refer to [4, Chapter 1]. Here we only recall the following:

Theorem 4.2. *The sheaves $\mathcal{E}xt^p_f$ satisfy:*

1. *If f is projective and $\mathcal{F}, \mathcal{G}$ are coherent $\mathcal{O}_X$-modules, then $\mathcal{E}xt^p_f(\mathcal{F}, \mathcal{G})$ is a coherent $\mathcal{O}_X$-module.*
2. *$\mathcal{E}xt^p_f(\mathcal{F}, \mathcal{G})$ is the sheaf associated to the presheaf*
$$U \mapsto \mathrm{Ext}^p(\mathcal{F}|_{f^{-1}(U)}, \mathcal{G}|_{f^{-1}(U)}).$$
In particular it holds that
$$\mathcal{E}xt^p_f(\mathcal{F}, \mathcal{G})_{|U} \cong \mathcal{E}xt^p_f(\mathcal{F}_{|f^{-1}(U)}, \mathcal{G}_{|f^{-1}(U)}).$$
3. *$\mathcal{E}xt^p_f(\mathcal{O}_X, \mathcal{G}) = R^p f_* \mathcal{G}$.*
4. *If $\mathcal{L}$ and $\mathcal{N}$ are locally free sheaves of finite rank on X and Y, respectively, then*
$$\mathcal{E}xt^p_f(\mathcal{F} \otimes \mathcal{L}, - \otimes f^*\mathcal{N}) \cong \mathcal{E}xt^p_f(\mathcal{F}, - \otimes \mathcal{L}^\vee \otimes f^*\mathcal{N}) \cong$$
$$\cong \mathcal{E}xt^p_f(\mathcal{F}, - \otimes \mathcal{L}^\vee) \otimes \mathcal{N}.$$
5. *For any $\mathcal{O}_X$-modules $\mathcal{F}, \mathcal{G}$ there is a spectral sequence, called* local to global spectral sequence,
$$E_2^{p,q} = R^p f_* \mathcal{E}xt^q(\mathcal{F}, \mathcal{G}) \Longrightarrow \mathcal{E}xt^{p+q}_f(\mathcal{F}, \mathcal{G})$$
where $\mathcal{E}xt^q$ is the usual extension sheaf on X, that is the derived functor of $\mathcal{H}om$.

6. Under the same hypotheses of (5), we also have the spectral sequence

$$E_2^{p,q} = H^p(B, \mathcal{E}xt_f^q(\mathcal{F}, \mathcal{G})) \Longrightarrow \mathrm{Ext}^{p+q}(\mathcal{F}, \mathcal{G}).$$

The spectral sequences in (5) and (6) can both be seen as a consequence of a result of Grothendieck that computes the derived functor of the composition of two functors F and G knowing the derived functors of F and G separately, cf. [19, Theorem 12.10]. In (5) we take $F = f_*$ and $G = \mathcal{H}om$ and in (6) $F = \Gamma$ and $G = f_*\mathcal{H}om$.

Now if we apply the functor $f_*\mathcal{H}om(-, f^*\omega_B)$ to the exact Sequence (4.1) we obtain, from the resulting long exact sequence, the morphism

$$f_*\mathcal{H}om(f^*\omega_B, f^*\omega_B) \to \mathcal{E}xt_f^1(\Omega^1_{X/B}, f^*\omega_B)$$

which translates, by the properties mentioned in Theorem 4.2, into

$$\mathcal{O}_B \to \mathcal{E}xt_f^1(\Omega^1_{X/B}, \mathcal{O}_X) \otimes \omega_B.$$

Defintion 4.3. The image of $1 \in H^0(B, \mathcal{O}_B)$ is a morphism

$$T_B \to \mathcal{E}xt_f^1(\Omega^1_{X/B}, \mathcal{O}_X) \tag{4.4}$$

which is called the Global Kodaira–Spencer map.

In this paper, we will mainly consider the extension sheaf

$$\mathcal{E}xt_f^1(\Omega^1_{X/B}, f^*\omega_B) = \mathcal{E}xt_f^1(\Omega^1_{X/B}, \mathcal{O}_X) \otimes \omega_B.$$

The following lemma shows how this sheaf behaves on a suitable Zariski open set $B' \subset B$ and justifies the name Kodaira–Spencer for the morphism in Definition (4.3).

Lemma 4.4. *There is an injection*

$$R^1 f_*\mathcal{H}om(\Omega^1_{X/B}, f^*\omega_B) \hookrightarrow \mathcal{E}xt_f^1(\Omega^1_{X/B}, f^*\omega_B)$$

which is an isomorphism over an open dense subset of B. In particular, for general $b \in B$ we have the isomorphism

$$\mathcal{E}xt_f^1(\Omega^1_{X/B}, f^*\omega_B) \otimes \mathbb{C}(b) \cong H^1(X_b, T_{X_b}) \otimes T^\vee_{B,b} \cong \mathrm{Ext}^1(\Omega^1_{X_b}, \mathcal{O}_{X_b}) \otimes T^\vee_{B,b}.$$

Proof. The five term exact sequence associated to the local to global spectral sequence recalled in Theorem 4.2 Point (5)

$$\begin{aligned} 0 \to R^1 f_*\mathcal{H}om(\Omega^1_{X/B}, f^*\omega_B) &\to \mathcal{E}xt_f^1(\Omega^1_{X/B}, f^*\omega_B) \to f_*\mathcal{E}xt^1(\Omega^1_{X/B}, f^*\omega_B) \\ &\to R^2 f_*\mathcal{H}om(\Omega^1_{X/B}, f^*\omega_B) \to \mathcal{E}xt_f^2(\Omega^1_{X/B}, f^*\omega_B) \end{aligned}$$

gives the desired injection. Note that on $X^0 = f^{-1}(B^0)$, $\Omega^1_{X/B}$ is locally free, hence $\mathcal{E}xt^1(\Omega^1_{X/B}, f^*\omega_B)$ is zero and this injection is an isomorphism on B^0:

$$\mathcal{E}xt_f^1(\Omega^1_{X/B}, f^*\omega_B)_{|B^0} \cong R^1 f_*\mathcal{H}om(\Omega^1_{X/B}, f^*\omega_B) \cong R^1 f_*(T_{X/B}) \otimes \omega_B. \tag{4.5}$$

The last statement is the Proper base change theorem [16, Theorem 12.11]. $\square$

We note that specializing the Global Kodaira–Spencer

$$T_B \to \mathcal{E}xt^1_f(\Omega^1_{X/B}, \mathcal{O}_X) \tag{4.6}$$

in $b \in B'$ we get the well known Kodaira–Spencer map at the point b

$$T_{B,b} \to H^1(X_b, T_{X_b}) \cong \operatorname{Ext}^1(\Omega^1_{X_b}, \mathcal{O}_{X_b}). \tag{4.7}$$

By a famous general result, the Global Kodaira–Spencer morphism is zero on an open subset of B if and only if the family is locally trivial on this set. In particular all the fibers are isomorphic and the map (4.7) is zero in every point. Conversely it is not true that if (4.7) is zero in every point, then the Global Kodaira–Spencer is also zero. This holds however when the family is regular, i.e. the dimension of the complex vector space $H^1(X_b, T_{X_b})$ is the same for all points in the set. See for example [17, Section 4].

Remark 4.5. To our knowledge $B^0 = B'$ if the fibration is regular. In general the relation between B' and B^0 seems to be not fully clarified,

Lemma 4.6. *We have a surjective morphism*

$$\rho\colon \operatorname{Ext}^1(\Omega^1_{X/B}, f^*\omega_B) \to H^0(B, \mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)) \tag{4.8}$$

which is also an isomorphism if the general fiber of $f\colon X \to B$ is of general type. Calling $\xi \in \operatorname{Ext}^1(\Omega^1_{X/B}, f^*\omega_B)$ the element corresponding to Sequence (4.1), ρ maps ξ to the Global Kodaira–Spencer map $\rho(\xi)$ which associates to $b \in B'$ the element $\xi_b \in H^1(X_b, T_{X_b}) \otimes T^\vee_{B,b}$ as defined in (4.3).

Proof. From the spectral sequence in Theorem 4.2 Point (6), we get the beginning of the associated five terms exact sequence:

$$\begin{aligned} 0 \to H^1(B, f_*\mathcal{H}om(\Omega^1_{X/B}, f^*\omega_B)) &\to \operatorname{Ext}^1(\Omega^1_{X/B}, f^*\omega_B) \\ &\xrightarrow{\rho} H^0(B, \mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)) \to H^2(B, f_*\mathcal{H}om(\Omega^1_{X/B}, f^*\omega_B)) \to \cdots \end{aligned}$$

The fourth term is zero because B is a curve, hence ρ is surjective.

Now note in the first term of this sequence that

$$f_*\mathcal{H}om(\Omega^1_{X/B}, f^*\omega_B) = f_*\mathcal{H}om(\Omega^1_{X/B}, \mathcal{O}_X) \otimes \omega_B = f_*T_{X/B} \otimes \omega_B.$$

Since $f_*T_{X/B}$ is torsion free, it is a line bundle on B and if the general fiber of f is of general type then $f_*T_{X/B} = 0$ and we get the desired isomorphism.

For the last statement, ρ maps ξ to a global section of $\mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)$ which associates to the general $b \in B'$ the element $\xi_b \in H^1(X_b, T_{X_b}) \otimes T^\vee_{B,b}$ as defined in (4.3); see [18, Lemma 2.1]. □

Remark 4.7. If the general fiber is of general type, the map ρ is actually surjective even if $\dim B > 1$.

4.2. Global Kodaira–Spencer supported on a horizontal divisor

Let $L \leq \Gamma(A, \mathbb{D}^1)$ be a l-dimensional vector space of sections of the local system $\mathbb{D}^1$ and choose η_i, $i = 1, \ldots, l$, forming a basis for L. Denote by s_i the liftings of these sections via the splitting of (3.1) fixed above. From L we define the following divisors in $f^{-1}(A)$.

Defintion 4.8. Let $\mathcal{D}^A$ be the divisor in $f^{-1}(A)$ given by the common zeroes of the sections $s_{i_1} \wedge \cdots \wedge s_{i_{n-1}} \wedge \sigma$ where the s_{i_j} run among the liftings above and σ over the local sections of ω_B on A.

Denote by $\mathcal{D}^A_{Hor}$ the divisor obtained by the horizontal components of $\mathcal{D}^A$ and with D_b the restriction $\mathcal{D}^A_{Hor}$ to the general fiber X_b. We call $\mathcal{D}^A_{Hor}$ the horizontal divisor associated to L.

Note that D_b is the fixed part of the sections $\eta_{i_1} \wedge \cdots \wedge \eta_{i_{n-1}}$ where the η_i run among the elements of the basis of L.

Remark 4.9. First note that $\mathcal{D}^A$ and $\mathcal{D}^A_{Hor}$ do not depend on the choice of the splitting of (3.1) fixed above. In fact a different choice gives new liftings $\tilde{s}_i$, with $s_i - \tilde{s}_i \in \Gamma(A, \omega_B)$.

Furthermore consider a Massey product $\omega = s_{i_1} \wedge \cdots \wedge s_{i_n}$ of sections of L. By local computation it is clear that ω vanishes on $\mathcal{D}^A_{Hor}$.

We can define the following sheaf on A:

$$\mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}^A_{Hor}), f^*\omega_B) := \mathcal{E}xt^1_f(\Omega^1_{X/B|f^{-1}(A)}(-\mathcal{D}^A_{Hor}), f^*\omega_{B|A}).$$

Alternatively recall that by $\mathbb{L}$ we denote the local system generated by L, $H_{\mathbb{L}}$ the subgroup of $\pi_1(B, b)$ acting trivially on $\mathbb{L}$ and $G_{\mathbb{L}} = \pi_1(B, b)/H_{\mathbb{L}}$ the monodromy group. Let $\widetilde{B} \to B$ the covering classified by the subgroup $H_{\mathbb{L}}$ and $\tilde{f} \colon \widetilde{X} \to \widetilde{B}$ the associated pullback fibration. The inverse image of the local system $\mathbb{L}$ on $\widetilde{B}$ is trivial, in particular the sections η_i are global and their liftings s_i are global closed 1-forms on $\widetilde{X}$. This means that $\mathcal{D}^A$ and $\mathcal{D}^A_{Hor}$ define global divisors $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}_{Hor}$ on $\widetilde{X}$. Hence $\mathcal{E}xt^1_f(\Omega^1_{\widetilde{X}/\widetilde{B}}(-\widetilde{\mathcal{D}}_{Hor}), f^*\omega_{\widetilde{B}})$ is defined on the whole base $\widetilde{B}$.

Remark 4.10. When $\mathbb{L}$ is Massey trivial generated and strict, by [31, Theorem B] the monodromy of $\mathbb{L}$ is finite hence the covering $\widetilde{B} \to B$ is also finite and $\tilde{f} \colon \widetilde{X} \to \widetilde{B}$ is a fibration of compact varieties. So, under these hypotheses, it is not restrictive to assume that everything is globally defined, since this is true up to a finite covering which does not impact the local deformation data of the fibers. Note that $\rho(\xi)$ is a global section of $\mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)$ which defines a global section $\widetilde{\rho(\xi)}$ of $\mathcal{E}xt^1_{\tilde{f}}(\Omega^1_{\widetilde{X}/\widetilde{B}}, \tilde{f}^*\omega_{\widetilde{B}})$; for example by Theorem 4.2 Point (2).

Finally we note that the relative $\mathcal{E}xt$ functors are contravariant in the first component and we obtain a sheaf morphism (on A)

$$\mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B) \to \mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}^A_{Hor}), f^*\omega_B).$$

Remark 4.11. By the same arguments seen in Lemma 4.4, we have that

$$\mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}^A_{Hor}), f^*\omega_B) \otimes \mathbb{C}(b) \cong \operatorname{Ext}^1(\Omega^1_{X_b}(-D_b), \mathcal{O}_{X_b}) \otimes T^\vee_{B,b}$$

for general $b \in A$.

We recall that $\xi_b \in H^1(X_b, T_{X_b})$ is *supported* on a divisor E_b in X_b if

$$\xi_b \in \operatorname{Ker} H^1(X_b, T_{X_b}) \to H^1(X_b, T_{X_b}(E_b)). \tag{4.9}$$

See [30]. The new concept of global supported deformation is Definition 1.1, that we recall:

Defintion 4.12. We say that $\rho(\xi)$ is supported on a horizontal divisor $\mathcal{E}$ in $f^{-1}(A)$ if

$$\rho(\xi)_{|A} \in \operatorname{Ker} H^0(A, \mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)) \to H^0(A, \mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{E}), f^*\omega_B)). \tag{4.10}$$

By what we have seen so far, if $\rho(\xi)$ is supported on $\mathcal{D}^A_{Hor}$ then ξ_b is supported on D_b for the general $b \in B$. The viceversa does not hold, since the sheaf $\mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}^A_{Hor}), f^*\omega_B)$ in general has a torsion part.

Note also that if $\rho(\xi)$ is supported on $\mathcal{D}^A_{Hor}$, we have that in the following diagram of torsion free sheaves on $f^{-1}(A)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\omega_B & \longrightarrow & \mathcal{E} & \longrightarrow & \Omega^1_{X/B}(-\mathcal{D}^A_{Hor}) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & f^*\omega_B & \longrightarrow & \Omega^1_X & \longrightarrow & \Omega^1_{X/B} & \longrightarrow & 0 \end{array} \tag{4.11}$$

the top row splits when restricted to the general fiber. Of course this does not mean that the top row itself splits.

4.3. Global supported deformations and Massey triviality

In this subsection we Prove Theorem [C] from the Introduction. As a first step, in light of the Adjoint theorem [30, Theorem A] we have the following result

Theorem 4.13. *Let $L \leq \Gamma(A, \mathbb{D}^1)$ be a vector space of $\dim L \geq n$. Assume that L is Massey trivial and that it generically generates $\Omega^1_{X_b}$ on the general fiber. Then $\rho(\xi)$ is supported on $\mathcal{D}^{A'}_{Hor}$, where $A' \subset A$ is an open dense subset. Furthermore if $\mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}^A_{Hor}), f^*\omega_B)$ is torsion free, then $\rho(\xi)$ is supported on $\mathcal{D}^A_{Hor}$.*

Proof. Choose generic $\eta_{i_1}, \dots, \eta_{i_n}$ linearly independent elements of L. They are Massey trivial by hypothesis hence by the Adjoint Theorem [30, Theorem A] we have that on a smooth fiber X_b the infinitesimal deformation ξ_b is supported on a divisor $D_b^{i_1,\dots,i_n}$, defined as the fixed part of the n sections $\eta_{i_1} \wedge \dots \wedge \widehat{\eta_{i_j}} \wedge \dots \wedge \eta_{i_n}$.

By [23, Proposition 3.1.6], if L generically generates $\Omega^1_{X_b}$, it turns out that actually $D_b^{i_1,\dots,i_n}$ does not depend on the choice of the η_i and it is exactly the divisor D_b.

We have proved that ξ_b is supported on D_b which of course is the restriction of $\mathcal{D}^A_{Hor}$ on the fiber X_b. The thesis follows easily by the discussion following Definition 4.12. □

Remark 4.14. One could also add the strictness hypothesis and state the theorem globally over $\widetilde{B}$.

Corollary 4.15. *Let $f\colon X \to B$ be a family such that the general fiber X_b is a variety of general type with $p_g(X_b) = \dim L = n$. If L is strict and $\Omega^1_{X_b}$ is generated by the elements of L, then f is isotrivial on an appropriate dense open set of the base.*

Proof. For such an X_b it is not difficult to see that L is Massey trivial and $D_b = 0$, see for example [30, Corollary 2.2.2]. The idea is that if we take a basis $\eta_1, \dots, \eta_n$ of L, $\bigwedge^{n-1} L \cong H^0(X_b, \omega_{X_b})$ and this implies that the η_i are necessarily Massey trivial. We also have $D_b = 0$ since L generates $\Omega^1_{X_b}$.

Hence by Theorem 4.13 we have that $\rho(\xi)$ is supported on an empty divisor, that is $\rho(\xi)$ is trivial.

This means that the fibration is isotrivial on an appropriate open set of the base. □

Remark 4.16. We stress that Corollary 4.15 is applicable to one dimensional families where $p_g(X_b) = q(X_b) = \dim X_b + 1$.

Finally we prove a viceversa of Theorem 4.13. These two results together are Theorem [C].

Theorem 4.17. *Let $L \leq \Gamma(A, \mathbb{D}^1)$ be a vector space of $\dim L \geq n$. Assume that $\rho(\xi)$ is supported on $\mathcal{D}^A_{Hor}$, the horizontal divisor associated to L. If $f_*\mathcal{O}_X(\mathcal{D}^A_{Hor})$ is a line bundle then the vector space L is Massey trivial.*

Proof. We want to prove that L is Massey trivial, that is every choice of n linearly independent sections in L is Massey trivial. We fix such a choice $\eta_1, \dots, \eta_n$ and call $W = \langle \eta_1, \dots, \eta_n \rangle \leq L$. We start by considering the exact sequence

$$0 \to f^*\omega_B \to \Omega^1_X \to \Omega^1_{X/B} \to 0 \tag{4.12}$$

and applying the functor $f_*\mathcal{H}om(\cdot, f^*\omega_B)$ to obtain the long exact sequence

$$0 \to f_*T_{X/B} \otimes \omega_B \to f_*T_X \otimes \omega_B \to \mathcal{O}_B \to \mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B) \to \cdots \tag{4.13}$$

Recall that the image of $1 \in H^0(B, \mathcal{O}_B)$ is $\rho(\xi) \in H^0(B, \mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B))$.

We ask the reader to accept the following easier and more compact notation for the rest of this proof: $X := f^{-1}(A)$ and $B := A$, that is we restrict everything locally on A.

Sequence (4.12) together with its tensor by $\mathcal{O}_X(-\mathcal{D}^A_{Hor})$ fits into the commutative diagram

$$\begin{CD}
0 @>>> f^*\omega_B @>>> \Omega^1_X @>>> \Omega^1_{X/B} @>>> 0\\
@. @| @AAA @AAA @.\\
0 @>>> f^*\omega_B @>>> \mathcal{E} @>>> \Omega^1_{X/B}(-\mathcal{D}^A_{Hor}) @>>> 0\\
@. @AAA @AAA @| @.\\
0 @>>> f^*\omega_B(-\mathcal{D}^A_{Hor}) @>>> \Omega^1_X(-\mathcal{D}^A_{Hor}) @>>> \Omega^1_{X/B}(-\mathcal{D}^A_{Hor}) @>>> 0
\end{CD}$$

Applying the functor $f_*\mathcal{H}om(\cdot, f^*\omega_B)$ we obtain

$$\begin{CD}
f_*T_X\otimes\omega_B @>>> \mathcal{O}_B @>>> \mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)\\
@VVV @| @VVV\\
f_*\mathcal{E}^\vee\otimes\omega_B @>>> \mathcal{O}_B @>>> \mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}^A_{Hor}), f^*\omega_B)\\
@VVV @VVV @|\\
f_*T_X(\mathcal{D}^A_{Hor})\otimes\omega_B @>>> f_*\mathcal{O}_X(\mathcal{D}^A_{Hor}) @>>> \mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}^A_{Hor}), f^*\omega_B)
\end{CD}$$

Thanks to this diagram we can interpret our hypothesis that $\rho(\xi)$ is supported on $\mathcal{D}^A_{Hor}$ as follows.

As pointed out above, the identity element $1\in H^0(B,\mathcal{O}_B)$ in the first row is mapped to $\rho(\xi)$ in $\mathcal{E}xt^1_f(\Omega^1_{X/B}, f^*\omega_B)$. By hypothesis $\rho(\xi)$ goes to zero in $\mathcal{E}xt^1_f(\Omega^1_{X/B}(-\mathcal{D}^A_{Hor}), f^*\omega_B)$, hence the identity element in the second row is in the image of the morphism $f_*\mathcal{E}^\vee\otimes\omega_B\to\mathcal{O}_B$. This means that if we take a point $b\in B$, we can, locally around b, find a lifting of the identity in $f_*\mathcal{E}^\vee\otimes\omega_B$. We denote by θ_b the image of this local lifting in $f_*T_X(\mathcal{D}^A_{Hor})\otimes\omega_B$. Denote by $\bigwedge^{n-1}W$ the vector space with basis the sections $\omega_i = s_1\wedge\cdots\wedge\widehat{s_i}\wedge\cdots\wedge s_n$ as in Definition 2.2 and consider the following commutative square

$$\begin{CD}
f_*T_X(\mathcal{D}^A_{Hor})\otimes\omega_B @>\alpha>> f_*\mathcal{O}_X(\mathcal{D}^A_{Hor})\\
@V{\alpha'}VV @VV{\beta}V\\
\bigwedge^{n-1}W\otimes f_*\mathcal{O}_X(\mathcal{D}^A_{Hor})\otimes\omega_B @>{\beta'}>> f_*\omega_X
\end{CD} \tag{4.14}$$

The horizontal arrow α is the same as in the above diagram, and the horizontal arrow β' is given by the fact that the ω_i are elements of $f_*\Omega^{n-1}_X$ and furthermore $\mathcal{D}^A_{Hor}$ is a divisor of common zeroes of $\omega_i\wedge\sigma$ for arbitrary σ in ω_B, that is we can see $\omega_i\wedge\sigma\in\bigwedge^{n-1}W\otimes\omega_B$ as an element of $f_*\omega_X(-\mathcal{D}^A_{Hor})$.

The vertical arrow α' is given by taking a section θ of $f_*T_X(\mathcal{D}^A_{Hor}) \otimes \omega_B$ and sending it to

$$\theta \mapsto \sum_i (-1)^i \theta(s_i) \otimes \omega_i$$

where $\theta(s_i)$ indicates the contraction, since s_i is in $f_*\Omega^1_X$.

The vertical arrow β is given by the Massey product ω since we recall that ω vanishes on $\mathcal{D}^A_{Hor}$, see Remark 4.9.

With these definitions, it is not difficult to see that the square commutes, hence $\beta\alpha(\theta_b) = \beta'\alpha'(\theta_b)$. On one side $\beta\alpha(\theta_b) = \beta(1) = \omega$ since θ_b is a lifting of the identity.

On the other side, note that we are working locally around the general point b. The germ of the section θ_b can be decomposed as a sum of elements of the form $v_b \otimes \sigma_b$ with $v_b \in H^0(X_b, T_{X_b}(D_b))$ and $\sigma_b \in \omega_{B,b}$. Its image via α' is then a sum of sections $v_b(s_i) \otimes \sigma_b \otimes \omega_i$ where now $v_b(s_i) \in H^0(X_b, \mathcal{O}_{X_b}(D_b))$. Since by hypothesis $f_*\mathcal{O}_X(\mathcal{D}^A_{Hor})$ is a line bundle, we have that $h^0(X_b, \mathcal{O}_{X_b}(D_b)) = 1$. This implies that the poles of $v_b(s_i)$ are exactly the zeroes of $\omega_i \otimes \sigma_b$, hence the image via β' is exactly an element in the submodule generated by $\langle \omega_i \rangle \otimes \omega_B$.

Hence by the commutativity we conclude that the Massey product ω is in the submodule generated by $\langle \omega_i \rangle \otimes \omega_B$, that is it is Massey trivial by Definition 2.3. □

4.4. Global supported deformations and morphisms to product varieties

In this subsection we prove Theorem [A]. The main ingredients are Theorem [C] and a Generalized Castelnuovo–de Franchis theorem, see [5, Theorem 1.14] and [24, Prop II.1]. See also [31, Theorem 5.6] for the following refined version.

Theorem 4.18. *Let Z be an n-dimensional compact Kähler manifold and $w_i \in H^0(Z, \Omega^1_Z)$, $i = 1, \dots, l$, linearly independent 1-forms such that $w_{j_1} \wedge \cdots \wedge w_{j_{k+1}} = 0$ for every $j_1, \dots, j_{k+1}$ and that no collection of k linearly independent forms in the span of $w_1, \dots, w_{j_{k+1}}$ wedges to zero. Then there exists a holomorphic map $f \colon Z \to Y$ over a normal variety Y of dimension $\dim Y = k$ and such that $w_i \in f^*H^0(Y, \Omega^1_Y)$. Furthermore Y is of general type.*

We are now ready to prove Theorem [A] which follows from the following corollary of Theorem 4.17, and a nice interpretation of Theorem 4.18.

Corollary 4.19. *Assume that there exist a strict subspace $L \leq \Gamma(A, \mathbb{D}^1)$, $\dim L \geq n$, such that $f_*\mathcal{O}_X(\mathcal{D}^A_{Hor})$ is a line bundle and $\rho(\xi)$ is supported on $\mathcal{D}^A_{Hor}$, the horizontal divisor associated to L. Then there exist a surjective morphism*

$$h_A \colon f^{-1}(A) \to Y$$

onto a normal $n-1$-dimensional variety Y of general type.

Furthermore, up to a finite étale covering $\widetilde{B} \to B$, the associated base change $\widetilde{X}$ also has a surjective morphism $h \colon \widetilde{X} \to Y$ onto Y.

Proof. By Theorem 4.17, we know that L is Massey trivial. In particular by Proposition 2.7, there exists a unique lifting $\widetilde{L}$ such that the wedge

$$\bigwedge^n \widetilde{L} \to \Gamma(A, f_*\omega_X)$$

is zero. We recall that the sections in $\widetilde{L}$ can be seen as 1-forms in $\Omega^1_{X,d}$.

Since their wedge is zero, we can then apply Theorem 4.18 and this give a morphism with connected fibers $h_A\colon f^{-1}(A) \to Y$ onto a normal $n-1$ dimensional variety Y. Note that even if $f^{-1}(A)$ is not compact, Theorem 4.18 can still be applied because the sections in $\widetilde{L}$ are *closed* by Proposition 2.7. This is actually enough to ensure that the arguments of Theorem 4.18 applies. In particular see [31, Remark 5.7].

For $\widetilde{X}$ the proof is similar and relies on the fact that Proposition 2.7 basically allows to pass from a local to a global condition. More precisely, by [31, Theorem B] the monodromy group $G_{\mathbb{L}}$ is finite, hence the associated covering $\widetilde{B}$ is also finite. Furthermore the sections of L give global sections on $\widetilde{B}$. So by Proposition 2.7 applied on $\widetilde{B}$, the elements of $\widetilde{L}$ can be seen as global closed 1-forms on $\widetilde{X}$ such that

$$\bigwedge^n \widetilde{L} \to \Gamma(\widetilde{B}, \tilde{f}_*\omega_{\widetilde{X}})$$

is zero. This is a global condition deriving from the local condition of Massey triviality.

Hence on $\widetilde{X}$ we can work globally and we get the morphism h again by Theorem 4.18. □

Remark 4.20. Note that h in general does not descend to a morphism $X \to Y$ due to the monodromy of the local system $\mathbb{L}$. Nevertheless, for every $U \subseteq B$ open subset trivializing the local system $\mathbb{L}$, h gives a surjective morphism $h_U\colon f^{-1}(U) \to Y$.

The following Corollary is Theorem [A].

Corollary 4.21. *Assume that there exists L as above such that $f_*\mathcal{O}_X(\mathcal{D}^A_{Hor})$ is a line bundle and $\rho(\xi)$ is supported on $\mathcal{D}^A_{Hor}$. Then there exists a generically finite surjective morphism $\widetilde{X} \to \widetilde{B} \times Y$ where Y is of general type.*

Proof. Note that the map h from the previous corollary is surjective when restricted to the general fiber X_b thanks to the strictness hypothesis. Hence the map $\tilde{f} \times h\colon \widetilde{X} \to \widetilde{B} \times Y$ is generically finite. □

Remark 4.22. Consider the morphism $h\colon \widetilde{X} \to Y$ and X_b a general fiber of $\tilde{f}$ over $b \in \widetilde{B}$. If the ramification of $h|_{X_b}$, denoted by R_b, is the restriction to X_b of a divisor $\mathcal{R}$ on $\widetilde{X}$ contained in the critical locus of h, then the deformation ξ_b is trivial.

In fact in this case the pullback $h^*\omega_Y$ is not only contained in $\Omega^{n-1}_{\widetilde{X}}$ but also in $\Omega^{n-1}_{\widetilde{X}}(-\mathcal{R})$ and on the fiber X_b we have the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^{n-2}_{X_b}(-R_b) & \longrightarrow & \Omega^{n-1}_{\widetilde{X}}|_{X_b}(-R_b) & \longrightarrow & \omega_{X_b}(-R_b) & \longrightarrow & 0 \\
 & & & & \uparrow & \nearrow & & & \\
 & & & & (h^*\omega_Y|_{X_b})^{\vee\vee} & & & &
\end{array}
\tag{4.15}
$$

The diagonal arrow is an isomorphism. This gives the splitting of the exact sequence in (4.15) which we note is associated to ξ_b after tensoring by R_b.

5. The case of volume forms on the fibers

In this section we study the case of $n-1$ forms on the fibers. The previous results, as Theorem 4.17 and its corollaries, work only in the case of highly irregular general fibers, that is $h^0(X_b, \Omega^1_{X_b}) \geq n$. In this section we show similar results which work with top forms on the fibers instead of 1-forms.

Recall again that, by a famous result of Fujita, see [11] and [12, 7, 6], the direct image $f_*\omega_{X/B}$ is a sum

$$f_*\omega_{X/B} \cong \mathcal{U} \oplus \mathcal{A} \tag{5.1}$$

where $\mathcal{U}$ is a unitary flat vector bundle and $\mathcal{A}$ is ample. Hence $\mathcal{U}$ is associated to a local system of relative $(n-1)$-forms.

In this section we restrict ourselves to a contractible subset $A \subset B^0$. The local system associated to $\mathcal{U}$ is given by the holomorphic $(n-1)$-forms on the fibers which can be locally lifted to de Rham closed holomorphic forms on X. That is, denoting this local system by $\mathbb{D}^{n-1}$ in analogy with $\mathbb{D}^1$, we have the exact sequence

$$0 \to \omega_A \otimes f_*\Omega^{n-2}_{f^{-1}(A)/A} \to f_*\Omega^{n-1}_{f^{-1}(A),d} \to \mathbb{D}^{n-1} \to 0 \tag{5.2}$$

where we use the compact notation $\omega_A := \omega_{B_{|A}}$. The main reason for working locally is that we do not know if in general this sequence splits globally on B, contrary to the case of 1-forms of (3.1).

Note that there is a map given by taking the wedge exact sequence

$$\mathrm{Ext}^1(\Omega^1_{f^{-1}(A)/A}, f^*\omega_A) \to \mathrm{Ext}^1(\Omega^p_{f^{-1}(A)/A}, f^*\omega_A \otimes \Omega^{p-1}_{f^{-1}(A)/A}),$$

hence by Theorem 4.2 Point (2), we have a morphism of sheaves

$$\mathcal{E}xt^1_f(\Omega^1_{f^{-1}(A)/A}, f^*\omega_A) \to \mathcal{E}xt^1_f(\Omega^p_{f^{-1}(A)/A}, f^*\omega_A \otimes \Omega^{p-1}_{f^{-1}(A)/A}).$$

Similarly we also have a morphism

$$\mathcal{E}xt^1_f(\Omega^1_{f^{-1}(A)/A}(-\mathcal{E}), f^*\omega_A) \to \mathcal{E}xt^1_f((\Omega^p_{f^{-1}(A)/A}(-\mathcal{E}), f^*\omega_A \otimes \Omega^{p-1}_{f^{-1}(A)/A})$$

where $\mathcal{E}$ is a horizontal divisor.

Remark 5.1. It is easy to see that these are actually isomorphisms when $p = n-1$, in particular we note that $\rho(\xi)$ is supported on a horizontal divisor $\mathcal{E}$ (according to Definition 4.12) if and only if its image in $H^0(A, \mathcal{E}xt^1_f(\Omega^{n-1}_{f^{-1}(A)/A}(-\mathcal{E}), f^*\omega_A \otimes \Omega^{n-2}_{f^{-1}(A)/A}))$ is zero.

The results of this section will use the analogue of the Castelnuovo–de Franchis theorem 4.18 in the case of $n-1$-forms, that is [32, Theorem 2.3]. For the reader benefit and for final use we recall the set up.

Let Z be a smooth variety $w_1, \dots, w_l \in H^0(Z, \Omega^p_Z)$, where $p \leq n-1$ and $l \geq p+1$, be linearly independent p-forms such that $w_i \wedge w_j = 0$ (as an element of $\bigwedge^2 \Omega^p_Z$ and not of Ω^{2p}_Z) for any choice of $i, j = 1, \dots, l$. These forms generate a subsheaf of Ω^p_Z generically of rank 1. Note that the quotients w_i/w_j define a non-trivial global meromorphic function on Z for every $i \neq j$, $i, j = 1, \dots, l$. By taking the differential $d(w_i/w_j)$ we then get global meromorphic 1-forms on Z.

Defintion 5.2. We say that a set of linearly independent p-forms $\{\omega_1, \dots, \omega_l\} \subset H^0(Z, \Omega^p_Z)$, $p \leq n-1$ and $l \geq p+1$, is p-strict if $\omega_i \wedge \omega_j = 0$ for every i, j and there exist p meromorphic differential forms $d(\omega_i/\omega_j)$ that do not wedge to zero.

For this setting, this condition is analogous to the strictness condition considered in Definition 2.6.

We need Theorem 2.3 in [32]:

Theorem 5.3. *Let Z be an n-dimensional smooth projective variety and consider a p-strict subset $\{w_1, \dots, w_l\} \subset H^0(Z, \Omega^p_Z)$. Then there exists a rational dominant map $f\colon Z \dashrightarrow Y$, defined in codimension 2, over a p-dimensional smooth variety Y of general type such that w_i is the pullback of a holomorphic p-form μ_i on Y, that is $f^*\mu_i$ extends to w_i, for $i = 1, \dots, l$.*

Now consider $L \leq \Gamma(A, \mathbb{D}^{n-1})$ a vector space of $n-1$-forms and, in analogy with the case of 1-forms, we will denote by η_i the elements of a basis of L and by $s_i \in \Gamma(A, f_*\Omega^{n-1}_{f^{-1}(A),d})$ a fixed choice of liftings (which exist since we are working locally on A). We will also denote by $\mathcal{D}^{A,(n-1)}_{Hor}$ the horizontal part of the divisor given by the common zeroes of $s_i \wedge \sigma$ where σ is a section of ω_A. This is in analogy with Definition 4.8, but note that the superscript $n-1$ is to remind that this divisor is associated to a vector space L of $n-1$-forms instead of 1-forms as in Definition 4.8.

We can finally prove the following result

Theorem 5.4. *Let $L \leq \Gamma(A, \mathbb{D}^{n-1})$ be a vector space of $\dim L = l \geq n$. Assume that the sections $\{\eta_1, \dots, \eta_l\}$ forming a basis of L are $(n-1)$-strict. If $\rho(\xi)$ is supported on $\mathcal{D}^{A,(n-1)}_{Hor}$ and $f_*\mathcal{H}om(\Omega^{n-1}_{f^{-1}(A)/A}(-\mathcal{D}^{A,(n-1)}_{Hor}), \Omega^{n-2}_{f^{-1}(A)/A})$ is zero then there exists a meromorphic dominant map $f^{-1}(A) \dashrightarrow Y$ over a smooth $(n-1)$-dimensional variety Y of general type.*

Proof. As in Theorem 4.17, we ask the reader to accept the more compact notation $X = f^{-1}(A)$ for reasons of brevity. For the same reason we also denote $\mathcal{D}^{A,(n-1)}_{Hor}$ by $\mathcal{D}^A_{Hor}$.

Consider the exact sequence

$$0 \to f^*\omega_A \otimes \Omega^{n-2}_{X/A} \to \Omega^{n-1}_A \to \Omega^{n-1}_{X/A} \to 0,$$

the $n-1$-th wedge of Sequence 4.12 which fits into the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f^*\omega_A \otimes \Omega^{n-2}_{X/A} & \longrightarrow & \Omega^{n-1}_X & \longrightarrow & \Omega^{n-1}_{X/A} & \longrightarrow & 0 \\ & & \Vert & & \uparrow & & \uparrow & & \\ 0 & \longrightarrow & f^*\omega_A \otimes \Omega^{n-2}_{X/A} & \longrightarrow & \mathcal{G} & \longrightarrow & \Omega^{n-1}_{X/A}(-\mathcal{D}^A_{Hor}) & \longrightarrow & 0 \end{array} \tag{5.3}$$

We apply the functor $f_*\mathcal{H}om(\cdot, f^*\omega_A \otimes \Omega^{n-2}_{X/A})$ and obtain the diagram

$$\begin{array}{ccccc} f_*\mathcal{H}om(\Omega^{n-1}_X, \Omega^{n-2}_{X/A}) \otimes \omega_A & \longrightarrow & f_*\mathcal{H}om(\Omega^{n-2}_{X/A}, \Omega^{n-2}_{X/A}) & \longrightarrow & \mathcal{E}xt^1_f(\Omega^{n-1}_{X/A}, f^*\omega_A \otimes \Omega^{n-2}_{X/A}) \\ \downarrow & & \Vert & & \downarrow \\ f_*\mathcal{H}om(\mathcal{G}, \Omega^{n-2}_{X/A}) \otimes \omega_A & \xrightarrow{\alpha} & f_*\mathcal{H}om(\Omega^{n-2}_{X/A}, \Omega^{n-2}_{X/A}) & \longrightarrow & \mathcal{E}xt^1_f(\Omega^{n-1}_{X/A}(-\mathcal{D}^A_{Hor}), f^*\omega_A \otimes \Omega^{n-2}_{X/A}) \end{array} \tag{5.4}$$

Note that in the middle sheaf of the top row, that is $f_*\mathcal{H}om(\Omega^{n-2}_{X/A}, \Omega^{n-2}_{X/A})$, we have the identity element that we simply denote by 1. Exactly as in the proof of Theorem 4.17, by the hypothesis on $\rho(\xi)$ and Remark 5.1, the section 1 is in the image of α and can be locally lifted to $f_*\mathcal{H}om(\mathcal{G}, \Omega^{n-2}_{X/A}) \otimes \omega_A$. Actually by our hypothesis on the vanishing of $f_*\mathcal{H}om(\Omega^{n-1}_{X/A}(-\mathcal{D}^A_{Hor}), \Omega^{n-2}_{X/A}) = \ker \alpha$, this lifting is global (on A) and unique. We will denote it by $h \in \Gamma(A, f_*\mathcal{H}om(\mathcal{G}, \Omega^{n-2}_{X/A}) \otimes \omega_A)$.

Now fix η_1, η_2 two linearly independent sections of $L \leq \Gamma(A, \mathbb{D}^{n-1})$ and s_1, s_2 the associated liftings in $\Omega^{n-1}_{X,d} \subset \Omega^{n-1}_X$. By definition of $\mathcal{D}^A_{Hor}$, the s_i can be lifted to $\mathcal{G}$, we call $\hat{s}_i$ these liftings. It is not difficult to see by a local computation that

$$s_1 \wedge s_2 + h(\hat{s}_1) \wedge h(\hat{s}_2) = s_1 \wedge h(\hat{s}_2) - s_2 \wedge h(\hat{s}_1).$$

Hence by taking $\tilde{s}_i := s_i - h(\hat{s}_i)$ we have that $\tilde{s}_i$ are still liftings of the η_i and furthermore $\tilde{s}_1 \wedge \tilde{s}_2 = 0$. Since h is unique, repeating the same argument, we get that $\tilde{s}_i \wedge \tilde{s}_j = 0$ for any pair of sections η_i, η_j of L. Hence we can apply Theorem 5.3 to get the map $f^{-1}(A) \dashrightarrow Y$. As in Corollary 4.19, even if $f^{-1}(A)$ is not compact, the argument is the same since the forms $\tilde{s}_i$ are closed. □

Regarding the vanishing hypothesis in the statement of the Theorem, note that on the smooth fibers, this vanishing is equivalent to the vanishing of the cohomology group $H^0(X_b, T_{X_b}(D^{n-1}_b))$, where D^{n-1}_b is the restriction of $\mathcal{D}^{A,(n-1)}_{Hor}$ on X_b; this can essentially be seen as a hypothesis on the normal sheaf of D^{n-1}_b. In particular it can often be applied if X_b is of general type and D^{n-1}_b has negative self-intersection.

Remark 5.5. Note that if $p_g(X_b) \geq a \cdot p_g(Y) - b$, with $a > 0$ and $b \in \mathbb{Z}$, the rank r of the local system $\mathbb{L}$ generated by L is

$$r \leq \frac{p_g(X_b) + b}{a}.$$

5.1. Fibered threefolds

We can obtain some bounds for the geometric genus $p_g(Y)$, where Y is as in Theorem 5.4, in the case of a relatively minimal fibered threefold $f\colon X \to B$. These bounds are based on the work of [2] and [25].

We start by recalling some standard definitions. Let $f_*\omega_{X/B} = \mathcal{U} \oplus \mathcal{A}$ the second Fujita decomposition of the direct image of the relative dualizing sheaf and $u_f := \operatorname{rank}\mathcal{U}$ the rank of the unitary flat part, so that $\operatorname{rank}\mathcal{A} = p_g(X_b) - u_f$. Denote also by g the genus of B and by K_f the divisor of $\omega_{X/B}$, we have the following invariants for our fibration

$$K_f^3 := K_X^3 - 2(g-1)K_{X_b}^2,$$

$$\Delta_f := deg f_*\mathcal{O}_X(K_f),$$

$$\chi_f := \chi_{X_b}\chi_B - \chi_X.$$

Hence, for fibered threefolds, it makes sense to define two slopes

$$\lambda_f^1 := \frac{K_f^3}{\chi_f}, \quad \lambda_f^2 := \frac{K_f^3}{\Delta_f}.$$

From now on we assume that $\chi_f > 0$ so that, following [2, Lemma 5.6], we have that $\chi_f \leq \Delta_f$ and hence more importantly $\lambda_f^2 \leq \lambda_f^1$. We refer to [2, Theorem 5.7] for examples where $\chi_f \geq 0$.

We also use the following definition from [25].

Defintion 5.6. Let $|M|$ be a linear system on a surface S. We say that

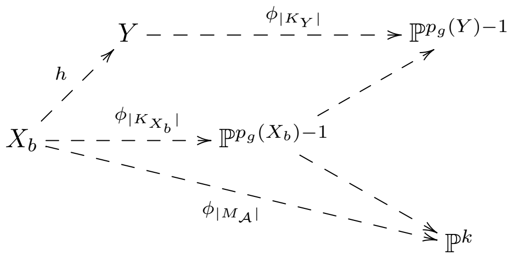
- $|M|$ is g.f.d. if it induces a generically finite map $\phi_{|M|}\colon S \to \mathbb{P}^k$ which is a double cover on the image which is a ruled surface
- $|M|$ is g.f.n.d. if it induces a generically finite map which is not a double cover on a ruled surface
- $|M|$ is a fibration of gonality γ if $\phi_{|M|}\colon S \to \mathbb{P}^k$ is a fibration with general fiber a smooth curve of gonality γ

The key point of the estimates of this section is the assumption that the ample part of the Fujita decomposition $\mathcal{A}$ is semistable. Under this assumption we have that

$$0 \subsetneq \mathcal{A} \subsetneq f_*\omega_{X/B} \tag{5.5}$$

is the Harder–Narasimhan filtration of $f_* \omega_{X/B}$ and we denote by $|M_{\mathcal{A}}|$ the movable part of $\mathcal{A}$ restricted to the fiber X_b.

We have the following diagram which puts together all the relevant pieces for our purposes.



(5.6)

Note that the diagonal maps between the projective spaces are just projections.

The result is the following

Proposition 5.7. *Let $f\colon X \to B$ be a relatively minimal fibered threefold with $\chi_f > 0$ and $g \leq 1$. Assume that the ample part of the Fujita decomposition $\mathcal{A}$ is semistable. Under the hypotheses of Theorem 5.4 we have the following bound on $p_g(Y)$.*

If $\operatorname{rank} \mathcal{A} \geq 2$

- *If $|K_{X_b}|$ and $|M_{\mathcal{A}}|$ are g.f.n.d.*
$$p_g(Y) \leq \frac{63 p_g(X_b) + 20}{66}$$
- *If $|K_{X_b}|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ is g.f.d.*
$$p_g(Y) \leq \frac{65 p_g(X_b) - 4q + 14}{68}$$
- *If $|K_{X_b}|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 5$*
$$p_g(Y) \leq \frac{64 p_g(X_b) + 12}{67}$$
- *If $|K_{X_b}|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 4$*
$$p_g(Y) \leq \frac{65 p_g(X_b) + 11}{68}$$
- *If $|K_{X_b}|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 3$*
$$p_g(Y) \leq \frac{67 p_g(X_b) + 10}{70}$$
- *If $|K_{X_b}|$ is g.f.n.d. and $|M_{\mathcal{A}}|$ defines a fibration of gonality $\gamma \geq 2$*
$$p_g(Y) \leq \frac{67 p_g(X_b) + 9}{70}$$

- *If* $|K_{X_b}|$ *and* $|M_{\mathcal{A}}|$ *are g.f.d.*

$$p_g(Y) \leq \frac{66p_g(X_b) - 6q + 11}{68}$$

- *If* $|K_{X_b}|$ *is g.f.d. and* $|M_{\mathcal{A}}|$ *defines a fibration of gonality* $\gamma \geq 5$

$$p_g(Y) \leq \frac{65p_g(X_b) - 2q + 9}{67}$$

- *If* $|K_{X_b}|$ *is g.f.d. and* $|M_{\mathcal{A}}|$ *defines a fibration of gonality* $\gamma \geq 4$

$$p_g(Y) \leq \frac{66p_g(X_b) - 2q + 8}{68}$$

- *If* $|K_{X_b}|$ *is g.f.d. and* $|M_{\mathcal{A}}|$ *defines a fibration of gonality* $\gamma \geq 3$

$$p_g(Y) \leq \frac{67p_g(X_b) - 2q + 8}{69}$$

- *If* $|K_{X_b}|$ *is g.f.d. and* $|M_{\mathcal{A}}|$ *defines a fibration of gonality* $\gamma \geq 2$

$$p_g(Y) \leq \frac{68p_g(X_b) - 2q + 7}{70}$$

- *If* $|K_{X_b}|$ *defines a fibration of gonality* $\gamma \geq 5$

$$p_g(Y) \leq \frac{62p_g(X_b) + 10}{67}$$

- *If* $|K_{X_b}|$ *defines a fibration of gonality* $\gamma \geq 4$

$$p_g(Y) \leq \frac{16p_g(X_b) + 2}{17}$$

- *If* $|K_{X_b}|$ *defines a fibration of gonality* $\gamma \geq 3$

$$p_g(Y) \leq \frac{22p_g(X_b) + 2}{23}$$

- *If* $|K_{X_b}|$ *defines a fibration of gonality* $\gamma \geq 2$

$$p_g(Y) \leq \frac{34p_g(X_b) + 2}{35}$$

If $\operatorname{rank} \mathcal{A} = 1$

$$p_g(Y) \leq p_g(X_b) - 1$$

Proof. Proposition 4.3.2 in [25] computes a list of all the upper bounds for the rank u_f. Our result follows immediately by noticing that $p_g(Y) \leq u_f$ since all the top forms on Y are de Rham closed and hence their pullback restricted on the fiber is in the local system $\mathbb{D}^{n-1}$ which we recall is the local system associated to the unitary flat vector bundle $\mathcal{U}$, that is $\mathcal{U} = \mathbb{D}^{n-1} \otimes \mathcal{O}_B$; see (5.1).

For the reader's convenience we briefly give an idea on how these bounds are obtained in [25]. Consider the first case of this list, that is $|K_{X_b}|$ and $|M_{\mathcal{A}}|$ are both g.f.n.d. The Harder-Narasimhan filtration of $f_*\omega_{X/B}$ is

$$0 \subsetneq \mathcal{A} \subsetneq f_*\omega_{X/B}$$

with $\mu_1 = \deg f_*\omega_{X/B}/\operatorname{rank}\mathcal{A} = \deg f_*\omega_{X/B}/(p_g(X_b) - u_f)$ and $\mu_2 = 0$ since $\mathcal{U}$ is flat.

The Xiao-Ohno-Konno formula then gives the inequality

$$K_f^3 \geq \mu_1(M_{\mathcal{A}}^2 + M_{\mathcal{A}}K_{X_b} + K_{X_b}^2). \tag{5.7}$$

Thanks to [2, Lemma 5.9], we get the necessary estimates for the quantities appearing in (5.7) and we get

$$K_f^3 \geq \frac{\deg f_*\omega_{X/B}}{p_g(X_b) - u_f}(3(p_g(X_b) - u_f) - 7 + 3(p_g(X_b) - u_f) - 6 + 3p_g(X_b) - 7)$$

that easily gives the lower bound for the slope λ_f^2

$$\lambda_f^2 \geq 9 + \frac{3u_f - 20}{p_g(X_b) - u_f} \tag{5.8}$$

see [25, Theorem 4.2].

The last step consists in using the inequality

$$K_f^3 - 2(g-1)K_{X_b}^2 \leq 72\chi_f, \tag{5.9}$$

see [20]. This inequality for $g \leq 1$ gives

$$\lambda_f^1 \leq 42$$

hence remembering that under our hypothesis $\lambda_f^2 \leq \lambda_f^1$ and putting together with (5.8) we get

$$9 + \frac{3u_f - 20}{p_g(X_b) - u_f} \leq 72. \tag{5.10}$$

Isolating u_f and using $p_g(Y) \leq u_f$ we get the first bound of the list.

The following bounds can be done in a similar way. □

6. A result on the Albanese map

This short final section is a result on the Albanese map which does not directly follow by the previous work but it is in the same spirit. This is Theorem [B].

Theorem 6.1. *Let X be a smooth n-dimensional variety and $\alpha\colon X \to A := Alb(X)$ its Albanese morphism. Assume that $\mathcal{L} := \operatorname{Im}(\alpha^*\Omega_A^{n-1} \to \Omega_X^{n-1})$ is a line bundle on X, then the global sections of $\mathcal{L}$ define a rational map $h\colon X \dashrightarrow Y$ to a variety Y of general type. Furthermore if the sections of $H^0(X,\mathcal{L})$ are $n-1$-strict, we can take h to be a morphism and Y is the Stein factorization of $X \to Z$ where $Z := \alpha(X)$.*

Proof. We begin by showing that $Z := \alpha(X)$ is $(n-1)$-dimensional. For convenience consider α as the composition

$$X \xrightarrow{\alpha'} Z \overset{i}{\hookrightarrow} A.$$

Since $\mathcal{L}$ is not zero, it immediately follows that $\dim Z \geq n-1$. Similarly $\dim Z$ is not n otherwise $\mathcal{L}$ would be of rank n.

Now we define a rational map $h\colon X \dashrightarrow Y$. Indeed the global sections of $\mathcal{L}$ are $n-1$-forms with $\omega_i \wedge \omega_j = 0$ since $\mathcal{L}$ is a line bundle. We can then apply Theorem 5.3 which defines our variety Y. In general we have $\dim Y \leq n-1$.

Finally if the sections of $H^0(X,\mathcal{L})$ are $(n-1)$-strict, $\dim Y = n-1$, again by Theorem 5.3. To show that we have a rational map $Y \dashrightarrow Z$ we note that the kernel of the global sections of $\mathcal{L}$ is a foliation as in [32]. More precisely, any global section of $\mathcal{L}$ defines by contraction a map

$$T_X \to \Omega_X^{n-2}$$

and since the sections are closed, the kernel is closed under Lie bracket and, up to saturation, gives a foliation.

These leaves are contained in the fibers of both $\alpha'\colon X \to Z$ and $h\colon X \dashrightarrow Y$. Since h has connected fibers we have that Y is birational to the variety given by the Stein factorization of α'. In particular it turns out that we can choose h to be a morphism. □

Remark 6.2. If the Albanese of the normalization of Z is A, then $Y = Z$.

The following corollary is in some sense a version of the Volumetric theorem [23, Theorem 1.5.3]

Corollary 6.3. *Under the hypotheses of the Theorem 6.1, if the restrictions of the Albanese map α to the fibers X_b have degree 1, then these fibers are birational to Y.*

Proof. By the previous theorem it follows that $Y \to Z$ is a birational map, hence the fibers X_b are in the same birational class as Y. □

Acknowledgements. The first author has been supported by the Institute for Basic Science (IBS-R003-D1) and (IBS-R032-D1), and by European Union funds, NextGenerationEU. The second author has been supported by the grant DIMA Geometry PRIDZUCC, by PRIN 2017 Prot. 2017JTLHJR "Geometric, algebraic and analytic methods in arithmetics"and by DM737_RIC_COLLAB_ZUCCONI "Birational geometry, Fano manifolds and torus actions" CUP G25F21003390007, European Union funds, NextGenerationEU.

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References

[1] Arbarello, E., Cornalba M.: Su una congettura di Petri. Comment. Math. Helvetici **56**, 1–38 (1981).

[2] Barja, M. Á.: On the slope and geography of fibred surfaces and threefolds. Ph.D. Thesis, http://diposit.ub.edu/dspace/handle/2445/35132 (2000).

[3] Barja, M. Á., González-Alonso, V., Naranjo, J. C.: Xiao's conjecture for general fibred surfaces. J. Reine Angew. Math. (Crelles Journal) **739**, 297–308 (2018).

[4] Birkar C.: Topics in algebraic geometry. https://arxiv.org/abs/1104.5035 (2011).

[5] Catanese, F.: Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations. Invent. Math. **104**, no. 2, 263–289 (1991).

[6] Catanese, F., Dettweiler, M.: The direct image of the relative dualizing sheaf needs not be semiample. C. R. Math. Acad. Sci. Paris **352**, no. 3, 241–244 (2014).

[7] Catanese, F., Dettweiler, M.: Answer to a question by Fujita on Variation of Hodge Structures. Higher Dimensional Algebraic Geometry: In honour of Professor Yujiro Kawamata's sixtieth birthday, Mathematical Society of Japan, Tokyo, Japan, 73–102 (2017).

[8] Cesarano, L., Rizzi, L., Zucconi, F.: On birationally trivial families and Adjoint quadrics. Proc. Edinb. Math. Soc. **65**, 587–617 (2022).

[9] Collino, A., Naranjo, J. C., Pirola, G. P.: The Fano normal function. J. Math. Pures Appl. **98**, no. 3, 346–366 (2012).

[10] Collino, A., Pirola, G. P.: The Griffiths infinitesimal invariant for a curve in its Jacobian. Duke Math. J. **78**, no. 1, 59–88 (1995).

[11] Fujita, T.: On Kähler fiber spaces over curves. J. Math. Soc. Japan **30**, no. 4, 779–794 (1978).

[12] Fujita, T.: The sheaf of relative canonical forms of a Kähler fiber space over a curve. Proc. Japan Acad. Ser. A Math. Sci. **54**, no. 7, 183–184 (1978).

[13] González-Alonso, V.: On deformations of curves supported on rigid divisors. Ann. Mat. Pura Appl. **195**, no. 1, 111–132 (2016).

[14] González-Alonso, V., Stoppino, L., Torelli, S.: On the rank of the flat unitary summand of the Hodge bundle. Trans. Amer. Math. Soc. **372**, 8663–8677 (2019).

[15] González-Alonso, V., Torelli, S.: Families of curves with Higgs field of arbitrarily large kernel. Bull. London Math. Soc. **53**, no. 2, 493–506 (2021).

[16] Hartshorne, R.: Algebraic Geometry. Graduate Texts in Mathematics **52**, Springer-Verlag, New York-Heidelberg, (1977).

[17] Kodaira, K.: Complex Manifolds and Deformation of Complex Structures. A Series of Comprehensive Studies in Math. **283**, Springer-Verlag, New York (1986).

[18] Lange, H.: Universal families of extensions. J. Algebra **83**, 101–112 (1983).

[19] McCleary, J.: A User's Guide to Spectral Sequences. Cambridge Studies in Advanced Mathematics **58**, Cambridge University Press, Cambridge (2001).

[20] Ohno, K.: Some inequality for minimal fibration of surface of general type over curves. J. Math. Soc. Japan **44**, no.4, 643–666 (1992).

[21] Pirola, G. P., Rizzi, C.: Infinitesimal invariant and vector bundles. Nagoya Math. J. **186**, 95–118 (2007).

[22] Pirola, G. P., Torelli, S.: Massey Products and Fujita decompositions on fibrations of curves. Collect. Math. **71**, 39–61 (2020).
[23] Pirola, G. P., Zucconi, F.: Variations of the Albanese morphisms. J. Algebraic Geom. **12**, no. 3, 535–572 (2003).
[24] Ran Z.: On subvarieties of abelian varieties. Invent. Math. **62**, 459–479 (1981).
[25] Riva, E.: Slope inequalities for fibred surfaces and fibred threefolds. Ph.D. Thesis, https://boa.unimib.it/retrieve/handle/10281/374266/584870/phd_unimib_841667.pdf (2021).
[26] Rizzi, L.: Adjoint forms and algebraic families. PhD thesis, https://air.uniud.it/handle/11390/1132961 (2017).
[27] Rizzi, L.: On Massey products and Rational homogeneous varieties. Kyoto J. Math. **64**, no. 2, 421–457 (2024).
[28] Rizzi, L., Zucconi, F.: Generalized adjoint forms on algebraic varieties. Ann. Mat. Pura Appl. **196**, no. 3, 819–836 (2017).
[29] Rizzi, L., Zucconi, F.: On Green's proof of infinitesimal Torelli theorem for hypersurfaces. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. **29**, no. 4, 689–709 (2018).
[30] Rizzi, L., Zucconi, F.: Differential forms and quadrics of the canonical image. Ann. Mat. Pura Appl. **199**, no. 6, 2341–2356 (2020).
[31] Rizzi, L., Zucconi, F.: Fujita decomposition and Massey product for fibered varieties. Nagoya Math. J. **247**, 624–652 (2022).
[32] Rizzi, L., Zucconi, F.: A note on a generalization of the Castelnuovo-de Franchis theorem. Rend. Circ. Mat. Palermo, II. Ser (2024).
[33] Sernesi, E.: General curves on algebraic surfaces. J. Reine Angew. Math. (Crelles Journal) **700**, 209–233 (2015).

Received: 12 December 2023/Accepted: 6 May 2024/Published online: 27 May 2024

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