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Some results for the Asymptotics and the Strong Minimum Principle for solutions to some nonlinear parabolic equations

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Abstract. *We extend some of the results in [7] on strong minimum principle and asymptotics of positive viscosity solutions to a class of doubly nonlinear parabolic equations,*

$$H(Du, D^2u) - f(u)u_t = 0, \ \ k \geq 1, \quad in\ \Omega \times [0, T),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $0 < T \leq \infty$. The spatial operator H is homogeneous of degree k.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, and $\overline{\Omega}$ be its closure. For $0 < T \leq \infty$, define $\Omega_T = \Omega \times (0, T)$. If $T = \infty$, we write $\Omega_\infty = \Omega \times (0, \infty)$. Let $P_T = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times [0, T))$, and $P_\infty = P_T$ with $T = \infty$, denote the parabolic boundaries of Ω_T and Ω_∞ respectively. Let $u = u(x, t) \ : \Omega_T \to [0, \infty)$. For $k \geq 1$, set

$$\Gamma_k[u] := H(Du, D^2u) - f(u)u_t, \ \ k \geq 1, \tag{1.1}$$

where H is an operator that is elliptic, homogeneoeus of degree k, and satisfies conditions described later in this section. The function f is a non-decreasing C^1 function. In this work, H could be degenerate, and fully nonlinear, see below.

We introduce notation for the work. The letters x, y and z denote points in $\mathbb{R}^n$, and o is the the origin. Let S^n be the set of all $n \times n$ real symmetric matrices, I is the identity matrix and O is the zero $n \times n$ matrix. The letters e and σ often stand for unit vectors in $\mathbb{R}^n$. Also, $B_\rho(x)$ is the $\mathbb{R}^n$ ball centered at $x \in \mathbb{R}^n$ with radius ρ.

In [7], we studied non-negative viscosity solutions of the parabolic equation

$$H(Du, D^2u) - u^{k-1}u_t = 0, \ \text{ in } \Omega_T, \ \ k \geq 1.$$

We showed that if $k = 1$, the Strong Maximum Principle and the Hopf boundary Principle are true for a large class of operators H. However, if $k > 1$, these results could fail to hold. Included in this work was also a discussion of long time asymptotics for the equations. Our effort in the current work is to extend some of the results in [7] to (1.1), for $k > 1$.

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We state the problem as follows:

$$\Gamma_k[u] = 0, \;\text{ in } \Omega_T,\; u \geq 0, \text{ and } u = h \text{ on } P_T, \tag{1.2}$$

where Γ_k is as in (1.1) and $h = h(x,t) \in C(P_T)$. We allow $T = \infty$ in what follows.

The function $h = h(x,t)$, for $(x,t) \in P_T$, includes the initial and side conditions, and is as given below:

$$h(x,t) = \begin{cases} h(x,0) & \forall x \in \overline{\Omega},\; t = 0, \\ h(x,t) & \forall (x,t) \in \partial\Omega \times [0,T). \end{cases} \tag{1.3}$$

Let $y \in \partial\Omega$. The function h is continuous at $(y,0)$, if

(i) $(x,t) \in \partial\Omega \times (0,T)$ and $(x,t) \to (y,0)$, then $\lim_{(x,t)\to(y,0)} h(x,t) = h(y,0)$,

(ii) $x \in \overline{\Omega}$ and $x \to y$, then $\lim_{(x,0)\to(y,0)} h(x,0) = h(y,0)$.

By $h \in C(P_T)$, we mean that the above holds on $\partial\Omega \times \{0\}$, and h is continuous elsewhere.

We assume throughout that

$$0 < \inf_{P_T} h(x,t) \leq \sup_{P_T} h(x,t) < \infty. \tag{1.4}$$

We list the conditions satisfied by H, these hold throughout the work.

Condition A (Monotonicity): Assume that $H\colon \mathbb{R}^n \times S^n \to \mathbb{R}$ is continuous, and $H(\wp, O) = 0$, for any $\wp \in \mathbb{R}^n$. For any $X,\ Y \in S^n$ with $X \leq Y$,

$$H(\wp, X) \leq H(\wp, Y), \;\; \forall \wp \in \mathbb{R}^n.$$

Condition B (Homogeneity): There is a constant $k_1 \geq 0$ such that $\forall (\wp, X) \in \mathbb{R}^n \times S^n$,

$$H(\theta\wp, X) = |\theta|^{k_1} H(\wp, X), \;\; \forall\, \theta \in \mathbb{R}, \;\text{ and }\; H(\wp, \theta X) = \theta H(\wp, X), \;\; \forall\, \theta > 0.$$

We do not assume that H is odd in X. Also, if $k_1 = 0$ then $H(\wp, X) = H(X)$.

Set $k = k_1 + 1$, using Condition B, for $\theta > 0$,

$$H(\theta\wp, \theta X) = \theta^k H(\wp, X), \;\; \forall\, (\wp, X) \in \mathbb{R}^n \times S^n. \tag{1.5}$$

For the next condition, let $\lambda \in \mathbb{R}$, and $e \in \mathbb{R}^n$ be a unit vector. Define

$$\begin{aligned} m(\lambda) &= \min\left(\min_{|e|=1} H\left(e, I - \lambda e \otimes e\right),\; -\max_{|e|=1} H(e, \lambda e \otimes e - I) \right), \;\text{ and} \\ M(\lambda) &= \max\left(\max_{|e|=1} H\left(e, I - \lambda e \otimes e\right),\; -\min_{|e|=1} H(e, \lambda e \otimes e - I) \right). \end{aligned} \tag{1.6}$$

Note that $m(\lambda) \le M(\lambda)$, and, both are non-increasing functions of λ. Moreover, if $\lambda \le 1$ then $m(\lambda) \ge 0$, since $I - \lambda e \otimes e \ge 0$. However, if $\lambda > 1$ then no definite statement can be made about $I - \lambda e \otimes e$. Condition C addresses this issue.

Condition C (Coercivity): We require that H satisfy

$$\text{C(i)} \quad m(\lambda) > 0,\ \forall \lambda < 1, \quad \text{and} \quad \text{C(ii)} \quad M(\lambda) < 0,\ \forall \lambda \ge \lambda_1, \tag{1.7}$$

for some $\lambda_1 \ge 1$.

Observe that if $\lambda = 0$ then C(i) implies that

$$\text{(i)} \quad H(e, I) \ge m(0) > 0, \quad \text{and} \quad H(e, -I) \le -m(0) < 0. \tag{1.8}$$

The motivation for studying equation (1.2) arises from [9, Chap. II]. As an example, consider the parabolic equation

$$(*) \quad \operatorname{div}(|Du|^{p-2}Du) + |Du|^p = u_t,\ p > 1.$$

Using $v = e^u$ in $(*)$, we obtain the well-known doubly nonlinear parabolic equation

$$(**) \quad \operatorname{div}(|Dv|^{p-2}Dv) = v^{p-2}v_t.$$

See Section 2 for more details.

The operator $H(Du, D^2u) := \operatorname{div}(|Du|^{p-2}Du)$ is quasilinear, $k = p - 1$, and odd in the second derivatives. It is easy to see that Conditions A and B are satisfied, if $p \ge 2$. Also,

$$H(e, I - \lambda e \otimes e) = (n + p - 2) - (p - 1)\lambda.$$

If $n \ge 2$, Condition C is satisfied. Thus, our results would hold for $(*)$, for $p \ge 2$.

Further examples of operators H that satisfy Conditions A, B and C include, the pseudo p-Laplacian ($p \ge 2$), the infinity-Laplacian and the Pucci operators, see [5, Section 3] for a detailed discussion. For related works, see [1, 2, 3, 4, 5, 10, 13].

In the first part of the current work, we discuss cases where the Strong Minimum Principle and the Hopf Boundary Principle may not hold. It turns out that if $k > 1$, these may fail regardless of whether f is a constant function or a non-constant function. In the former (f constant), the sign of the solution u plays no role. In the latter (f non-constant), we consider $u \ge 0$, and a distinction between the cases $\inf_{\Omega_T} u > 0$ and $\inf_{\Omega_T} u = 0$ needs to be made. This appears in Theorem 1.2. For $k = 1$ and f, a positive constant, both the Strong Minimum Principle and the Hopf Boundary Principle are true, even when H is fully nonlinear, see [7]. However, if $k > 1$ and $f(u) = u^{k-1}$, then these were shown to fail.

The second set of results extends the large time asymptotic behaviour of positive solutions, shown in [7], to (1.2). It turns out that the results shown therein continue to hold.

In this work, sub-solution, super-solutions and solutions are in the viscosity sense. We provide a definition below.

Definition 1.1 (Viscosity Solution)**.** Let $U \subset \mathbb{R}^{n+1}$ be a domain. By $usc(lsc)(U)$, we mean the set of all upper semi-continuous (lower semi-continuous) functions defined on the set U.

Our work studies viscosity solution of

$$\Gamma_k[u] \equiv H(Du, D^2u) - f(u)u_t = 0, \quad \text{in } \Omega_T \quad \text{and} \quad u = h \quad \text{on } P_T. \tag{1.9}$$

A function $u \in usc(\Omega_T)$, $u \geq 0$, is said to be a viscosity sub-solution of the differential equation in (1.9) in Ω_T (or solves $\Gamma_k[u] \geq 0$ in Ω_T), if, for any ψ, C^2 in x and C^1 in t, such that $u - \psi$ has a maximum at some point $(y,t) \in \Omega_T$, we have

$$H(D\psi, D^2\psi)(y,t) - f(u(y,t))\psi_t(y,t) \geq 0.$$

We say u is a sub-solution of the problem in (1.9), if $u \in usc(\Omega_T \cup P_T)$, $\Gamma_k[u] \geq 0$ in Ω_T, and $u \leq h$ on P_T.

Similarly, $u \in lsc(\Omega_T)$, $u \geq 0$, is said to be a viscosity super-solution of the differential equation in (1.9) in Ω_T (or solves $\Gamma_k[u] \leq 0$, in Ω_T), if, for any ψ, C^2 in x and C^1 in t, such that $u - \psi$ has a minimum at some $(y,t) \in \Omega_T$, we have

$$H(D\psi, D^2\psi)(y,t) - f(u(y,t))\psi_t(y,t) \leq 0.$$

We say u is a super-solution of the problem in (1.9), if $u \in lsc(\Omega_T \cup P_T)$, $u > 0$, $\Gamma_k[u] \leq 0$ in Ω_T, and $u \geq h$ on P_T.

A function $u \in C(\Omega_T)$ is a solution of $\Gamma_k[u] = 0$ in Ω_T, if it is both a sub-solution and a super-solution. Similarly, $u \in C(\Omega_T \cup P_T)$ is a solution of the problem in (1.9), if it is both a sub-solution and a super-solution of (1.9). The above definitions can be extended to the case $T = \infty$.

In the rest of the work, operator H will be assumed to satisfy Conditions A, B and C, unless otherwise mentioned. Additionally, f satisfies conditions which are discussed in greater detail in Section 2, see Comment I and Note II. These are needed for a version of the comparison principle to hold, see Section 2. Also, the results stated here hold if $f > 0$ is a constant function. In this case, there are no sign restrictions on u. However, we do not state this explicitly in the theorems, our focus being mainly on the case where f is a non-constant function.

We assume throughout that $k > 1$.

We now state the main results. Theorem 1.2 addresses the Strong Minimum Principle. We place no restrictions on $\partial\Omega$.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be any bounded domain and $T > 0$. Suppose that $f\colon [0,\infty) \to [0,\infty)$ is C^1, non-decreasing, and $f^{1/(k-1)}$ is concave. Let $u \in lsc(\Omega_T)$, $u \geq 0$, be a super-solution, i.e.,*

$$\Gamma_k[u] \equiv H(Du, D^2u) - f(u)u_t \leq 0 \quad \text{in } \Omega_T.$$

Set $m = \inf_{\Omega_T} u$. The following hold:

(a) Let $m > 0$. If for some $(p, \tau) \in \Omega_T$, $u(p, \tau) > m$ then there is a $\rho > 0$ such that $u > m$ in the cylinder $B_\rho(p) \times [\tau, T)$. As a consequence, if $u(p, \tau) = m$ then $u(p, s) = m$ for all $0 < s < \tau$.

(b) Suppose that $u \geq 0$. If $m = 0$ and $(p, \tau) \in \Omega_T$ is such that $u(p, \tau) = 0$. Assume that $u \in C(\Omega_T)$. Then there is a sequence of points $\{(x_\ell, t_\ell)\}_{\ell=1}^{\infty} \subset \Omega_T$, such that $t_\ell < \tau$, $u(x_\ell, t_\ell) = 0$ and $(x_\ell, t_\ell) \to (p, \tau)$.

A proof appears in Section 3. Parts (a) and (b) cannot be improved, thus showing that the Hopf Boundary Principle and the Strong Minimum Principle do not hold, in general.

The next two results address large time asymptotic behaviour. See [1, 4, 10]. Here, f is as in Theorem 1.2. For Theorem 1.3, we place no restrictions on $\partial\Omega$. However, $\partial\Omega$ satisfies a uniform outer ball condition in Theorem 1.4.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, and $h \in C(P_\infty)$, $h > 0$, satisfy (1.3) and (1.4).*

(a) Let $u \in lsc(\Omega_\infty \cup P_\infty)$, $u > 0$, be a super-solution to (1.2), i.e., $\Gamma_k[u] \leq 0$. Assume that $u = h$ on $\partial\Omega \times [T, \infty)$, for some $T > 0$.

Let $h_{\inf} = \lim_{t\to\infty} \left(\inf_{\partial\Omega\times[t,\infty)} h\right)$. If $h_{\inf}$ exists then

$$\lim_{t\to\infty} \left(\inf_{\overline{\Omega}\times[t,\infty)} u \right) = h_{\inf}.$$

(b) Let $u \in usc(\Omega_\infty \times P_\infty)$, $u > 0$, be a sub-solution to (1.2), i.e., $\Gamma_k[u] \geq 0$. Assume that $u = h$ on $\partial\Omega \times [T, \infty)$, for some $T > 0$.

Let $h_{\sup} = \lim_{t\to\infty} \left(\sup_{\partial\Omega\times[t,\infty)} h\right)$. If $h_{\sup}$ exists then

$$\lim_{t\to\infty} \left(\sup_{\overline{\Omega}\times[t,\infty)} u \right) = h_{\sup}.$$

The next result addresses the case where $h \equiv$ constant. See [7] for the case $k = 1$.

Theorem 1.4. *Let Ω be a bounded domain that satisfies a uniform outer ball condition. Suppose that, for some $\nu \in \mathbb{R}$, $h = \nu$, on $\partial\Omega \times [T, \infty)$ for some $T \geq 0$.*

Assume that $\nu > 0$. Suppose that the sub(super)-solution u satisfies $u = \nu$ on $\partial\Omega \times [T, \infty)$. The following holds for any $\alpha < 1/(k-1)$.

(a) If $u > 0$ is a subsolution then $\lim_{t\to\infty} t^\alpha \left(\sup_{\Omega\times[t,\infty)} u - \nu \right) = 0.$

(b) If $u > 0$ is a supersolution then $\lim_{t\to\infty} t^\alpha \left(\nu - \inf_{\Omega\times[t,\infty)} u \right) = 0.$

See Section 4 for the proofs of Theorems 1.3 and 1.4. In this work, we do not address existence issues for the parabolic problems (1.2). Instead, we direct the reader to [5, Theorems 1.2 and 1.3] for such issues, see also [2]. A somewhat more refined version of Condition C is used in [5]. Under conditions on $M(\lambda)$ (we require $\partial\Omega$ to be smooth in some cases) existence of a positive solution is shown in Ω_T, for any $T > 0$. However, if u is allowed to vanish somewhere in Ω_T, then these results may not apply. See [5] for more details.

The proofs, in the current work, follow [7] closely. To make it self-contained, we have included the details.

2. Preliminaries

We present some elementary calculations that will be useful in the work. Included here is a version of a comparison principle for parabolic equations in (1.2).

2.1. Radial functions

Let $z \in \mathbb{R}^n$ and $r = |x - z|$. Suppose that $v(x) = v(r),\ r \geq 0$, is C^2 in $r > 0$. Set $e = (x - z)/r$, in $r > 0$. Then for $x \neq z$,

$$H(Dv, D^2v) = H\left(v'(r)e, \frac{v'(r)}{r}(I - e \otimes e) + v''(r)e \otimes e\right), \tag{2.1}$$

where I is the $n \times n$ identity matrix. If $v(r) = r^\alpha,\ \alpha > 0$, then

$$H(Dv, D^2v) = \alpha^k r^{\alpha k - (k+1)} H(e, I + (\alpha - 2)e \otimes e).$$

2.2. Change of variable formula

See Lemma 2.3 in [5] for a more general statement. Let $f\colon [0, \infty) \to [0, \infty)$ be a C^1 non-decreasing function that satisfies $f(s) = 0$ if and only if $s = 0$.

Suppose that $u \in usc(lsc)(\Omega_T),\ u > 0$, satisfies

$$H(Du, D^2u) - f(u)u_t \geq (\leq) 0 \quad \text{in } \Omega.$$

For $k > 1$, let $F_k(s)$ be a primitive

$$F_k(s) = \int^s \frac{d\theta}{f(\theta)^{1/(k-1)}}, \quad s > 0. \tag{2.2}$$

Note that F_k is a C^2 function, and is increasing and concave. Define $w = F_k(u)$; thus, if $u \in usc(lsc)(\Omega_T)$ then $w \in usc(lsc)(\Omega_T)$, and

$$H\left(Dw, D^2w + \left[\frac{1}{F_k'(u)}\right]' Dw \otimes Dw\right) - w_t \geq (\leq) 0 \quad \text{in } \Omega_T.$$

Here, $[1/F_k'(u)]' = [f(u)^{1/(k-1)}]'$. The above is in the sense of viscosity. For deriving a comparison principle, we require that $[f(u)^{1/(k-1)}]'$ is non-increasing in w, i.e, non-increasing in u. Recall that w is increasing if and only if u is increasing.

See also [2]. A formal derivation appears in Appendix A.1.

Comment I: The aim of the change in variable is to derive a comparison principle for Γ_k. It will be seen that under some conditions, the function $w = F_k(u)$ satisfies a comparison principle. Thus a version for u holds, see Theorem 2.3 below. Since F_k is an increasing continuous function, $F_k(u) \in usc(lsc)(\Omega_T \cup P_T)$, if $u \in usc(lsc)(\Omega_T \cup P_T)$.

For a version of the comparison principle to hold, we require that $f(s)^{1/(k-1)}$ be a concave function. If $f(s) = s^\alpha$, $\alpha \in \mathbb{R}$, this leads to the requirement

$$0 \leq \alpha \leq k - 1.$$

Observe also that $J(s) \equiv [f(s)^{1/(k-1)}]'$ could be unbounded near $s = 0$. If $f(s) = s^\alpha$ and $\alpha < k - 1$, $J(s)$ is unbounded near $s = 0$. However, if $\alpha = k - 1$, $J(s) = 1$.

Finally, if $\lim_{s \to 0^+} F_k(1) - F_k(s) < \infty$, we define

$$F_k(0) \equiv \lim_{s \to 0^+} F_k(s) > -\infty.$$

If, instead, $\lim_{s \to 0^+} F_k(1) - F_k(s) = \infty$, we define

$$\lim_{s \to 0^+} F_k(s) = -\infty.$$

Note II: In the rest of the work, we assume that (i) $f'(u) \geq 0$, and (ii) $f(u)^{1/(k-1)}$ is concave in u, see (2.2). If f is a constant function, the requirement that $u \geq 0$ may be dropped, as $F_k(u) = u$.

2.3. Parabolic Comparisons

We discuss a version of the comparison principle used in this work. Note that $\Omega \subset \mathbb{R}^n$ is a bounded domain and $0 < T < \infty$. However, many of these continue to hold for $T = \infty$, by letting $T \to \infty$.

We begin with a well known result about sub-solutions that we state without proof.

Lemma 2.1. *Suppose that H satisfies Condition A. For $i = 1, 2$, let $u_i \in usc(\Omega_T \cup P_T)$, such that $u_i \geq 0$ solve $\Gamma_k[u_i] \geq 0$ in Ω_T. Then the function $u = \max\{u_1, u_2\}$ solves*

$$\Gamma_k[u] \geq 0 \quad \text{in } \Omega_T.$$

An analogous statement holds for super-solutions with max *replaced by* min.

Suppose that $F\colon \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^n \times S^n \to \mathbb{R}$ is continuous and satisfies

$$F(t, \nu, \wp, X) \leq F(t, \nu, \wp, Y), \tag{2.3}$$

for any $(t, \nu, \wp) \in (0, T) \times \mathbb{R} \times \mathbb{R}^n$, with $X \leq Y$,

Lemma 2.2 (Comparison principle)**.** *Let F be as in (2.3), and $E\colon \mathbb{R}\to[0,\infty)$ be continuous and non-increasing. Suppose that $u\in usc(\Omega_T\cup P_T)$ and $v\in lsc(\Omega_T\cup P_T)$ satisfy*

$$F(t,Du,D^2u+E(u)Du\otimes Du)-g(t)u_t\geq 0,$$
$$\textit{and}\quad F(t,Dv,D^2v+E(v)Dv\otimes Dv)-g(t)v_t\leq 0,$$

in Ω_T. If $\sup_{P_T} v<\infty$ and $u\leq v$ on P_T then $u\leq v$ in Ω_T.

See [5, Lemma 4.1, Section 4]. See [8], for a more general result. We apply the above (see Comment I and Note II) to obtain the comparison principle in Theorem 2.3.

We introduce additional notation for the following theorem. Let $\delta>0$ be small. Define

$$(*)\qquad \Omega^\delta=\{x\in\Omega\ :\ \mathrm{dist}(x,\mathbb{R}^n\setminus\Omega)\geq\delta\}.$$

Let $\theta>0$, be small. By $P(\delta,T-\theta)$ we denote the parabolic boundary of $\Omega^\delta\times(\delta,T-\theta)$. Note that $P(0,T-\theta)=P_{T-\theta}$.

Theorem 2.3 (Comparison principle)**.** *Let H satisfy Conditions A and B, and f satisfy Note II, in Subsection 2.2. Suppose that $u\in usc(\Omega_T\cup P_T)$, and $v\in lsc(\Omega_T\cup P_T)$, satisfy*

$$\Gamma_k[u]\geq 0,\ \ \textit{and}\ \Gamma_k[v]\leq 0,\ \ \textit{in}\ \Omega_T,$$

where $\Gamma_k[w]=H(Dw,D^2w)-f(w)w_t$.

(i) Let $f(u)=c>0$. Then $u-v\leq\sup_{P_T}(u-v)$. This holds without any sign restrictions on u and v. Moreover, $u\leq\sup_{P_T}u$, and $v\geq\inf_{P_T}v$.

In what follows, suppose that $f(u)$ is a non-constant function. Let F_k be the function defined in (2.2). Assume in parts (ii) and (iii) that $v>0$ in Ω_T. The following hold.

(ii) Assume that $u\geq\nu>0$, for some $\nu>0$, and $v>0$ on P_T. Then $F_k(u)-F_k(v)\leq\sup_{P_T}(F_k(u)-F_k(v))$. If $u\leq v$ on P_T then $u\leq v$. Also, $u\leq\sup_{P_T}u$, and $v\geq\inf_{P_T}v$.

(iii) Suppose that $u\geq 0$ on $\Omega_T\cup P_T$. We address two cases.

(iii-a) If $v>0$ on P_T, then $F_k(u)-F_k(v)\leq\sup_{P_T}(F_k(u)-F_k(v))$. The remaining conclusions in Part (ii) hold also. Moreover, if $u=0$ on P_T then $u\equiv 0$ in Ω_T.

(iii-b) Suppose that $v=0$ somewhere on P_T.

- *If $\lim_{s\to0^+}F_k(s)>-\infty$, then $F_k(u)-F_k(v)\leq\sup_{P_T}(F_k(u)-F_k(v))$.*

Thus, if $u\leq v$ on P_T, $u\leq v$ in Ω_T. As a result, $u\leq\sup_{P_T}u$, and $v\geq\inf_{P_T}v$.

- *If $\lim_{s\to0^+}F_k(s)=-\infty$, then*

$$F_k(u)-F_k(v)\leq\lim_{\delta\to0}\Big[\sup_{P(\delta,T-\delta)}(F_k(u)-F_k(v))\Big].$$

Moreover, $u\leq\sup_{P_T}u$, and $v\geq\inf_{P_T}v$. In particular, if $u=0$ on P_T, $u=0$ in Ω_T.

Proofs of parts (i) and (ii): The conclusion in (i) follows from Lemma 2.2, see also [8]. To show the maximum principles, take v =constant in one case, and u =constant in the other.

For part (ii), see [5, Theorem 4.3, Section 4]. We apply Lemma 2.2 to the transformed functions $F_k(u)$ and $F_k(v)$, see Comment I and Note II in Subsection 2.2. Since F_k is increasing, and $F_k(s) > -\infty$, for $s > 0$, the conclusion $u \le v$ in Ω_T follows, if $u \le v$ on P_T. The maximum principles follow as in part (i) since $F_k(u) \le \sup_{P_T} F_k(u) \le F_k(\sup_{P_T} u)$. The second inequality follows as $u(x,t) \le \sup_{P_T} u$, $\forall\,(x,t) \in P_T$.

Proof of part (iii): We begin by showing that the claim holds if $v > 0$ on $\Omega_T \cup P_T$.

Proof of (iii-a) Assume that $u \ge 0$, and $v > 0$ in $\Omega_T \cup P_T$. For a fixed, small $\varepsilon > 0$, set $u_\varepsilon = \max\{u, \varepsilon\}$. By Lemma 2.1, u_ε is a sub-solution, since $w = \varepsilon$ is a sub-solution. Assume that $u > 0$ somewhere in Ω.

(a1): Suppose that $F_k(0) \equiv \lim_{s\to 0^+} F_k(s) > -\infty$. Thus, $F_k\colon [0,\infty) \to [F_k(0),\infty)$ is right continuous at 0. Recall from Comment I that $F_k(u) \in usc(\Omega_T \cup P_T)$, and $F_k(v) \in lsc(\Omega_T \cup P_T)$.

If $(x,t) \in \Omega_T$, then $(x,t) \in \Omega_{T-\theta}$, for some $\theta > 0$, small. Set

$$M_\varepsilon \equiv \sup_{\{0\le u\le\varepsilon\}\cap P_{T-\theta}} [F_k(\varepsilon) - F_k(v)].$$

Since $F_k(u) \le F_k(u_\varepsilon)$, applying part (ii) of the theorem, for any $\varepsilon > 0$,

$$\begin{aligned} &F_k(u(x,t)) - F_k(v(x,t)) \le F_k(u_\varepsilon(x,t)) - F_k(v(x,t)) \\ &\quad\le \sup_{P_{T-\theta}} (F_k(u_\varepsilon) - F_k(v)) = \max\left\{ M_\varepsilon, \sup_{\{u>\varepsilon\}\cap P_{T-\theta}} [F_k(u) - F_k(v)] \right\} \\ &\quad\le \max\left\{ M_\varepsilon, \sup_{\overline{P_{T-\theta}}} [F_k(u) - F_k(v)] \right\}. \end{aligned} \tag{2.4}$$

Choose $\eta > 0$, small. For every $\varepsilon > 0$ ($\varepsilon \to 0$), let $(x_\varepsilon, t_\varepsilon) \in \{0 \le u \le \varepsilon\} \cap P_{T-\theta}$ such that $M_\varepsilon \le F_k(\varepsilon) - F_k(v(x_\varepsilon, t_\varepsilon)) + \eta$. Since M_ε is decreasing, $0 \le u(x_\varepsilon, t_\varepsilon) \le \varepsilon$, and $F_k(\varepsilon) - F_k(v) \in usc(\Omega_T \cup P_T)$,

$$\begin{aligned} \lim_{\varepsilon\to 0} M_\varepsilon &\le \limsup_{\varepsilon\to 0} [F_k(\varepsilon) - F_k(v(x_\varepsilon, t_\varepsilon))] + \eta \le F_k(0) - F_k(v(x_0, t_0)) + \eta, \\ &\le F_k(u(x_0,t_0)) - F_k(v(x_0,t_0)) + \eta \le \sup_{\overline{P_{T-\theta}}} [F_k(u) - F_k(v)] + \eta. \end{aligned}$$

for some $(x_0, t_0) \in \overline{\mathrm{P}_{\mathrm{T}-\theta}}$. Since the above holds for any η, (2.4) implies that

$$F_k(u(x,t)) - F_k(v(x,t)) \le \sup_{P_T} F_k(u) - F_k(v).$$

We show the last part. Assume that $u = 0$ on P_T and $u(\bar{x}, \bar{t}) > 0$ at some $(\bar{x}, \bar{t}) \in \Omega_T$. Set $v = 1$, then, for any $\varepsilon > 0$,

$$\begin{aligned} (F_k(u) - F_k(1))\,(\bar{x}, \bar{t}) &\le (F_k(u_\varepsilon) - F_k(1))\,(\bar{x}, \bar{t}) \\ &\le \sup_{P_T} (F_k(u_\varepsilon) - F_k(1)) = F_k(\varepsilon) - F_k(1). \end{aligned}$$

Hence, $F_k(u(\bar{x},\bar{t})) \le F_k(\varepsilon)$. Taking $\varepsilon < u(\bar{x},\bar{t})$, we get a contradiction. Thus, $u = 0$ in Ω_T.

(a2): Suppose that $\lim_{s\to 0^+} F_k(s) = -\infty$. If $u = 0$ on P_T, using the argument in **(a1)**, we get $u = 0$ in Ω_T. We assume that $u > 0$ somewhere in Ω_T, hence, $u > 0$ somewhere on P_T. Note that the functions $u_\varepsilon = \varepsilon$ and $v = 1$ are positive solutions of $\Gamma_k = 0$.

We observe that (2.4) (see **(a1)**) continues to hold, i.e.,

$$F_k(u) - F_k(v) \le \max\left\{ \sup_{\{0\le u\le\varepsilon\}\cap P_T} [F_k(\varepsilon) - F_k(v)], \sup_{\{u>0\}\cap P_T} [F_k(u) - F_k(v)] \right\}, \quad \forall\varepsilon > 0.$$

Since $\inf_{P_T} v > -\infty$, and $\lim_{\varepsilon\to 0} F_k(\varepsilon) = -\infty$, one can choose ε, small, so that

$$\sup_{\{0\le u\le\varepsilon\}\cap P_T} [F_k(\varepsilon) - F_k(v)] \le F_k(\varepsilon) - F_k(\inf_{P_T} v) \le \sup_{\{u>0\}\cap P_T} [F_k(u) - F_k(v)].$$

It follows from above that

$$F_k(u) - F_k(v) \le \sup_{\{u>0\}\cap P_T} [F_k(u) - F_k(v)] \le \sup_{P_T} [F_k(u) - F_k(v)].$$

Proof of (iii-b). We now consider the case $v \ge 0$ on P_T. Recall that $F_k(v) \in lsc(\Omega_T)$.

Let $\delta > 0$ and $\theta > 0$ be small, and Ω^δ, and $P(\delta, T-\theta)$ be as defined above (see $(*)$). Let $(x,t) \in \Omega_T$. There are $\delta > 0$ and $\theta > 0$ such that $(x,t) \in \Omega^\delta \times (\delta, T-\theta)$. Note that

$$v > 0 \text{ on } \overline{\Omega^\delta \times (\delta, T-\theta)}.$$

(b1) Let $F_k(0) > -\infty$. It follows that $F_k(u) \in usc(\Omega_T \cup P_T)$.

As shown in part **(a1)** above (using u_ε),

$$F_k(u(x,t)) - F_k(v(x,t)) \le \sup_{P(\delta,T-\theta)} (F_k(u) - F_k(v)).$$

.

We now show that

$$F_k(u(x,t)) - F_k(v(x,t)) \le \sup_{\overline{P_{T-\theta}}} F_k(u) - F_k(v).$$

Let $\eta > 0$ be small; fix θ. For each $\delta > 0$, let $(x_\delta, t_\delta) \in P(\delta, T-\theta)$ (i.e., on the parabolic boundary of $\Omega^\delta \times (\delta, T-\theta)$) be such that

$$\begin{aligned} F_k(u(x,t)) - F_k(v(x,t)) &\le \sup_{P(\delta,T-\theta)} F_k(u) - F_k(v) \\ &\le F_k(u(x_\delta,t_\delta)) - F_k(v(x_\delta,t_\delta)) + \eta. \end{aligned}$$

Since, as $\delta \to 0$, $\Omega^\delta \times (0, T-\theta) \nearrow \Omega_{T-\theta}$, there is a sub-sequence (x_δ, t_δ) and a $(y, s) \in \partial\Omega \times [0, T-\theta]$ such that $(x_\delta, t_\delta) \to (y, s)$. Thus,

$$\begin{aligned}(F_k(u) - F_k(v))(x,t) &\le \lim_{\delta\to 0} \Big[\sup_{P(\delta, T-\theta)} (F_k(u) - F_k(v)) \Big] \\ &\le \limsup_{\delta \to 0} \left[F_k(u(x_\delta, t_\delta)) - F_k(v(x_\delta, t_\delta))\right] + \eta \\ &\le F_k(u(y,s)) - F_k(v(y,s)) + \eta \\ &\le \sup_{P_T} (F_k(u) - F_k(v)) + \eta.\end{aligned}$$

The above follows as $F_k(u) - F_k(v)$ is upper semi-continuous. The claim follows.

(b2) Suppose that $\lim_{s\to 0^+} F_k(s) = -\infty$. The assumption $v > 0$ in Ω_T continues to hold.

Thus, $F_k(u) \in usc((\Omega_T \cup P_T) \cap \{u > 0\})$, and $F_k(v) \in lsc(\Omega_T \cup (P_T \cap \{v > 0\}))$.

Arguing as in **(a1)** and **(b1)**, we get that $F_k(u) - F_k(v) \le \sup_{P(\delta, T-\delta)} F_k(u) - F_k(v)$, if $\delta > 0$ is small enough. Hence,

$$F_k(u) - F_k(v) \le \lim_{\delta \to 0} \left[\sup_{P(\delta, T-\delta)} F_k(u) - F_k(v) \right].$$

Firstly, by arguing as in **(a1)**, we can show that if $u = 0$ on P_T then $u = 0$ in Ω_T. Suppose that $\sup_{P_T} u > 0$. Choose $0 < \varepsilon < \sup_{P_T} u$. By **(a2)** (take $v = 1$),

$$F_k(u) \le F_k(u_\varepsilon) \le \sup_{P_T} F_k(u_\varepsilon) \le F_k(\sup_{P_T} u).$$

This concludes the proof. □

Corollary 2.4. *In Theorem 2.3, take $f(s) = s^q$, where $0 \le q \le k-1$. Let $u \in usc(\Omega_T \cup P_T)$ and $v \in lsc(\Omega_T \cup P_T)$ solve*

$$H(Du, D^2u) - u^q u_t \ge 0 \quad \text{and} \quad H(Dv, D^2v) - v^q v_t \le 0 \ \text{ in } \Omega_T.$$

(i) For $0 \le q < k-1$, define

$$\alpha = \frac{k-1}{k-1-q}.$$

Suppose that $u \ge 0$ and $v > 0$ in Ω_T. If $v \ge 0$ on P_T, then

$$u^{1/\alpha} - v^{1/\alpha} \le \sup_{P_T} \left(u^{1/\alpha} - v^{1/\alpha} \right).$$

(ii) Let $q = k-1$. Assume that $u > 0$ and $v > 0$ in $\Omega_T \cup P_T$. Then $\log u - \log v \le \sup_{P_T} (\log u - \log v)$. Clearly, the following quotient type comparison result holds in Ω_T:

$$u/v \le \sup_{P_T} (u/v).$$

The above quotient continues to hold in case $u \ge 0$ and $v > 0$ in $\Omega_T \cup P_T$. If $v > 0$ in Ω_T and $v \ge 0$ on P_T, then

$$u/v \le \lim_{\delta \to 0} \left[\sup_{P(\delta, T-\delta)} u/v \right].$$

Proof. Parts (i) and (ii) follow from Theorem 2.3(iii). For $0 \le q < k-1$,

$$F_k(s) = \int^s f(s)^{-1/(k-1)} ds = \left(1 - \frac{q}{k-1}\right)^{-1} s^{1-q/(k-1)} = \alpha s^{1/\alpha}.$$

Note that $w = F_k(u) = \alpha u^{1/\alpha}$. Moreover, from Comment I and Note II,

$$H\left(Dw, D^2 w + \frac{\alpha - 1}{w} Dw \otimes Dw\right) - w_t \ge (\le) 0.$$

Clearly, $\lim_{s \to 0^+} F_k(s) > -\infty$.

If $q = k-1$, then $F_k(s) = \log s$, $w = \log u$, and

$$H(Dw, D^2 w + Dw \otimes Dw) - w_t \ge (\le) 0.$$

Here, $\lim_{s \to 0^+} F_k(s) = -\infty$. $\square$

Corollary 2.5. *Let $\bar{u} \in usc(\Omega_T \cup P_T)$ and $\bar{v} \in lsc(\Omega_T \cup P_T)$, $\bar{v} > -\infty$. Assume that $\inf_{\Omega_T} \bar{u} > -\infty$ with, possibly, $\inf_{\Omega_T \cup P_T} \bar{u} = -\infty$. If*

$$H(D\bar{u}, D^2\bar{u} + D\bar{u} \otimes \bar{u}) - \bar{u}_t \le 0 \quad and \quad H(D\bar{v}, D^2\bar{v} + D\bar{v} \otimes \bar{v}) - \bar{v}_t \ge 0, \quad in\ \Omega_T,$$

then, $\bar{u} - \bar{v} \le \max_{P_T}(\bar{u} - \bar{v})$.

Proof. For $\varepsilon \in \mathbb{R}$, $\bar{u}_\varepsilon = \max\{\bar{u}, \varepsilon\}$ is a sub-solution. Apply Subsection 2.2 and Lemma 2.2. $\square$

3. Proof of Theorem 1.2: Strong Minimum Principle

In this section, we show that the Strong Minimum Principle and the Hopf Boundary Principle for a non-negative super-solution u may fail, if $k > 1$ and $m \equiv \inf_{\Omega_T} u > 0$. This conclusion holds regardless of f is constant or increasing (see Comment I and Note II at the end of Subsection 2.2). The case $k > 1$ differs from $k = 1$, even when $f \equiv$constant.

However, things are not clear in the case $m = 0$ (f increasing, $k > 1$), and we provide a partial result. One of the difficulties seems to be that the comparison principle (see Theorem 2.3) becomes unclear at places where both the sub-solution and the super-solution vanish.

We consider super-solutions $u > 0$ of doubly nonlinear equations of the type:

$$H(Du, D^2 u) - f(u) u_t \le 0,$$

where $f\colon [0, \infty) \to [0, \infty)$, is non-decreasing, and $f(s) = 0$ if and only if $s = 0$.

3.1. Case $m > 0$:

Before presenting the proof of the the theorem, we discuss an example that shows the failure of the Strong Minimum Principle and the Hopf Boundary Principle.

Example: We construct a super-solution ξ, in an appropriate cylinder Ω_T, such that for some $p \in \Omega$, and some $T > 0$, $\xi(p,t) = m \equiv \inf_{\Omega_T} \xi$, for $0 < t \le T$. However, $\xi > m$ in the rest of Ω_T. Actually, our construction produces a super-solution in $\mathbb{R}^n \times (0,T)$, for any fixed $T > 0$.

Take $p = o$, and any $T > 0$. Set $r = |x - o| = |x|$ and $\phi(r) = r^{(k+1)/(k-1)}$. Using (2.1) (see Subsection 2.1) and (1.7) i.e, Condition C(i),

$$H(D\phi, D^2\phi) = cr^{(k+1)/(k-1)} H\left(e, I - \frac{k-3}{k-1} e \otimes e\right) \le c\phi(r)L, \tag{3.1}$$

for some constants $0 < c = c(k) < \infty$, and $0 < L = L(k) < \infty$.

For any $R > 0$, we take $\Omega_T = B_R(o) \times [0,T)$. Define

$$\xi(x,t) = m + \phi(r)\eta(t), \text{ where } \eta(t) = \left(\frac{1}{E(2T-t)}\right)^{1/(k-1)} \text{ and } E = \frac{c(k-1)L}{f(m)}.$$

Note that $f(m) > 0$ as $m > 0$, and

$$\eta'(t) = E\eta^k/(k-1) > 0.$$

Using (3.1), we get in, $0 < r < R$

$$\begin{aligned}\Gamma_k[\xi] &= H(D\xi, D^2\xi) - f(\xi)\xi_t \le c\phi\eta^k L - f(m + \phi\eta)\phi\eta' \\ &= c\phi\eta^k L - \frac{Ef(m+\phi\eta)\eta^k}{k-1} \le \phi\eta^k \left[cL - \frac{Ef(m)}{(k-1)}\right] \le 0.\end{aligned}$$

We verify below that ξ is a super-solution in Ω_T, i.e., also at (o,t). But first, we make the following observations. Clearly,

$$\xi(o,t) = m, \quad 0 < t \le T, \quad \text{and} \quad \xi(x,t) > m, \quad x \ne o.$$

This shows that u does not attain its minimum value anywhere except along (o,t), $0 < t < T$.

Next, let ∇ be the $\mathbb{R}^{n+1}$ gradient. Then $\nabla\xi(o,t) = 0$, $0 < t < T$. Let $z \ne o$ and $\rho = |z|$. Let $U = B_\rho(z) \times [0,T]$, and $r = |x|$, as defined above. Thus, $\xi > m$ is a super-solution in U and $\xi(o,t) = m$, $0 < t < T$. This is a t-segment on the parabolic boundary of U. Since $\nabla\xi(o,t) = 0$, the Hopf Boundary Principle fails.

We now show that ξ is a super-solution in Ω_T by showing that it is a super-solution at (o,t). Let ζ, C^2 in x and C^1 in t, be such that $\xi - \zeta$ has a minimum at (o,s) for some $0 < s < T$. Then $\xi(x,t) - \xi(o,s) \ge \zeta(x,t) - \zeta(o,s)$. Since ξ is C^1 in both x and t, we get that $D\xi(o,s) = D\zeta(o,s) = 0$ and $\xi_t(o,s) = \zeta_t(o,s) = 0$. Since $k > 1$, we get, by applying Condition B ($k_1 > 0$) that

$$H(D\zeta(o,s), D^2\zeta(o,s)) - \xi(o,s)^{k-1}\zeta_t(o,s) = 0.$$

This finishes the proof. □

Proof of Theorem 1.2 Part (a). We now show that if $u > 0$ satisfies $\Gamma_k[u] \le 0$, in Ω_T, and $u(p,\tau) > m$, for some $(p,\tau) \in \Omega_T$, then there is a cylinder $C \equiv B_\rho(p) \times [\tau, T)$, for some $\rho > 0$, such that $u(x,t) > m$ in C. As a result, if $u(p,\tau) = m$ then $u(p,t) = m$, for all $0 < t < \tau$. As the above example shows, this result cannot be improved.

Suppose that $u(p,\tau) > m$. Since u is lower semicontinuous, there are $\bar{\varepsilon} > 0$ and $0 < \rho < 1$ such that

$$u(x,\tau) \ge m + \bar{\varepsilon}, \quad \text{in } B_\rho(p).$$

We construct a sub-solution. Set $\delta = T - \tau$, $r = |x - p|$ and $C = B_\rho(p) \times [\tau, T)$. In C, we define ψ, C^2 function in both x and t, as follows. For $0 < \varepsilon \le \bar{\varepsilon}$, to be chosen,

$$\psi(x,t) = m + \varepsilon\phi(r)^2\eta(t), \quad \text{where} \quad \phi(r) = \rho^2 - r^2, \quad \text{and} \quad \eta(t) = \frac{T - t + \delta}{2\delta}. \tag{3.2}$$

Using (2.1) and Condition B, we get

$$\begin{aligned} H(D\psi, D^2\psi) &= (\varepsilon\eta)^k H\left(-4r\phi e, -4\phi(I - e\otimes e) + (-4\phi + 8r^2)e\otimes e\right) \\ &= (4\varepsilon\phi\eta)^k r^{k-1} H\left(e, \frac{2r^2}{\phi} e\otimes e - I\right). \end{aligned}$$

Hence,

$$\begin{aligned} \Gamma_k[\psi] &= H(D\psi, D^2\psi) - f(\psi)\psi_t \\ &= (4\varepsilon\eta\phi)^k r^{k-1} H\left(e, \frac{2r^2}{\phi} e\otimes e - I\right) + \frac{\varepsilon f(\psi)\phi^2}{2\delta}. \end{aligned} \tag{3.3}$$

We divide the interval $[0,\rho)$ into two sub-intervals: $[0,\sigma]$ and $[\sigma,\rho)$, where σ is such that

$$\forall\, r \in [\sigma,\rho), \quad \frac{2r^2}{\phi(r)} = \frac{2}{(\rho/r)^2 - 1} \ge \frac{2}{(\rho/\sigma)^2 - 1} \ge \lambda_1,$$
$$\text{where} \quad \sigma = \rho\nu, \quad \text{and} \quad \nu \equiv \sqrt{\frac{\lambda_1}{\lambda_1 + 2}},$$

where λ_1 is defined in (1.7) C(ii). See also (1.6).

Thus, $H(e, 2r^2/\phi(r) - I) \ge 0$, in $[\sigma,\rho)$. By (3.3), $\Gamma_k[\psi] \ge 0$, in $[\sigma,\rho) \times (\tau, T)$.

Next, we consider $[0,\sigma]$. We estimate

$$H(e, 2r^2/\phi(r) - I) \ge \min_{|e|=1} H(e, -I) \ge -|M| > -\infty.$$

Observe that $1/2 \le \eta \le 1$, $(1-\nu^2)\rho^2 = \phi(\sigma) \le \phi(r) \le \rho^2$, and $m \le \psi \le m + \varepsilon$.

Applying these in (3.3), we see

$$\begin{aligned}\Gamma_k[\psi] &\geq -(4\varepsilon\eta\phi)^k r^{k-1}|M| + \frac{\varepsilon f(\psi)\phi^2(r)}{2\delta} \\ &\geq \frac{\varepsilon f(m)\phi(\sigma)^2}{2\delta} - (4\varepsilon)^k \phi(0)^k r^{k-1}|M| \\ &\geq \varepsilon\left[\frac{f(m)(1-\nu^2)^2\rho^4}{2\delta} - 4^k\varepsilon^{k-1}\rho^{3k-1}|M|\right] \\ &= \varepsilon\rho^4\left[\frac{f(m)(1-\nu^2)^2}{2\delta} - 4^k\varepsilon^{k-1}\rho^{3k-5}|M|\right].\end{aligned} \tag{3.4}$$

If $\varepsilon > 0$ is small enough then ψ is a sub-solution in $B_\rho(p) \times [\tau, T)$.

Next, we observe that $u \geq \psi = m$, on $\partial B_\rho(p) \times [\tau, T)$, and $u(x,\tau) \geq m + \varepsilon \geq \psi(x,\tau)$, for $x \in B_\rho(p)$. By using the comparison principle Theorem 2.3, $\psi \leq u$ in C. Thus, for any $(x,t) \in C$,

$$u(x,t) \geq \psi(x,t) = m + \varepsilon(\rho^2 - |x-p|^2)^2\left(\frac{T-t+\delta}{2\delta}\right) > m.$$

The claim holds. □

3.2. Case $m = 0$:

We assume that $u \in C(\Omega_T)$.

Proof of Theorem 1.2 Part (b). We show that the zeros of u are not isolated. Assume to the contrary. Suppose that $u(p,\tau) = 0$, and there is a cylinder $C \equiv B_\rho(p) \times (\tau - \delta, \tau) \subset \Omega_T$, for some $\rho > 0$ and $\delta > 0$, such that $u > 0$ in $\overline{C} \setminus \{(p,\tau)\}$.

Let P be the parabolic boundary of C. Since $u > 0$ on P, there is a $\mu > 0$ such that $u \geq \mu$ on P. Recall the calculations done in the proof of Part (a), (3.1) and (3.3). Define in C,

$$\psi(x,t) = \frac{\mu}{2} + \varepsilon(\rho^2 - r^2)^2\left(\frac{\tau - t}{2\delta}\right), \quad r = |x-p|,$$

where $0 < \varepsilon \leq \min\{\mu/(2\rho^4),\ \bar{\varepsilon}\}$. As shown above, if ε small enough, ψ is a sub-solution in C, see (3.4). Moreover, $\psi \leq \nu \leq u$ on P. Hence, by Theorem 2.3, $u \geq \psi$ in C. In particular, $u(p,t) \geq \mu/2$, $\tau - \delta \leq t < \tau$. Since u is continuous, $u(p,\tau) \geq \mu/2 > 0$, a contradiction. The claim holds. □

4. Proof of Theorem 1.3: Asymptotics

The proof extends Theorem 1.2 in [7] to a somewhat more general equation. We recall a few items, and introduce two auxiliary functions before presenting the proof. In this section, all the sub-solutions and super-solutions are positive.

We recall that $\Omega_\infty = \Omega \times (0,\infty)$ and $P_\infty = (\overline{\Omega} \times \{0\}) \cup (\partial\Omega \times (0,\infty))$. For $t > 0$, set

$$\mathcal{Q}_t = \overline{\Omega} \times [t,\infty) \quad \text{and} \quad \mathcal{S}_t = \partial\Omega \times [t,\infty).$$

Let $T > 0$ be as in the statement of the theorem. We assume that $u = h$ on $\mathcal{S}_T$. Set

$$m = \min_{\mathcal{S}_T} h > 0 \quad \text{and} \quad M = \sup_{\mathcal{S}_T} h < \infty.$$

By a sub(super)-solution u of (1.9), we mean $u \in usc(lsc)(\overline{\Omega}_\infty)$, $u \le (\ge) h$ on P_∞, and $\Gamma_k[u] \ge (\le) 0$. Thus, Theorem 2.3 implies that

$$\begin{aligned} &\text{If } u > 0 \text{ is a sub-solution then } u \le \max\{\sup_{\overline{\Omega}} u(x,T),\ M\} \quad \text{in } \mathcal{Q}_T, \\ &\text{If } u > 0 \text{ is a super-solution then } u \ge \min\{\inf_{\overline{\Omega}} u(x,T),\ m\} \quad \text{in } \mathcal{Q}_T. \end{aligned} \tag{4.1}$$

To see (4.1), apply the comparison principle in the cylinder $\Omega \times (T,s)$, for $s > T$, and then let $s \to \infty$.

We make a remark about f that will be useful in the sequel.

Remark: We note a property of $f^{1/(k-1)}$ that follows from concavity. Since $f^{1/(k-1)}(s)/s$ is decreasing,

$$\sup_{s\ge\theta} \frac{f(s)}{s^{k-1}} = \frac{f(\theta)}{\theta^{k-1}} \equiv \mathcal{F}(\theta) < \infty. \tag{4.2}$$

If f is non-constant, we assume that $f(0) = 0$.

We introduce notation and quantities that are needed for constructing the auxiliary functions. In what follows, D, E, F and a are positive constants, where a depends on E. We choose D, E and F in the proof of the theorem.

Let $z \in \mathbb{R}^n \setminus \overline{\Omega}$; set

$$r = |x - z|, \quad R = \sup_{x\in\Omega} |x - z|, \quad \mathcal{R} = \inf_{x\in\Omega} |x - z|, \quad \text{and} \quad \mathcal{D} = \operatorname{diam}(\Omega). \tag{4.3}$$

Clearly, $\mathcal{R} > 0$, $r \ge \mathcal{R} > 0$, if $x \in \Omega$, and

$$R \le \mathcal{R} + \mathcal{D} \quad \text{and} \quad \Omega \subset B_{\mathcal{R}+\mathcal{D}}(z) \setminus B_{\mathcal{R}}(z).$$

Auxiliary Function 1 (Sub-solution): Let z and r be defined as above. For constants D, E, F, and a, we define the function $\xi \in C^2(\Omega_\infty)$ as follows:

$$\xi(x,t) = \alpha(r)\tau(t), \quad \text{where} \quad \alpha(r) = De^{Er^2} \ \text{and} \ \tau(t) = \frac{e^{at}}{e^{at} + F}. \tag{4.4}$$

We record that

$$\begin{aligned} &\alpha'(r) = (2Er)\alpha(r), \quad \alpha''(r) = 2E\alpha(r)\left(1 + 2Er^2\right), \\ &\text{and} \quad \tau'(t) = \tau(t)\left(\frac{aF}{e^{at} + F}\right). \end{aligned}$$

Set $\sigma = (x-z)/|x-z|$; we get

$$D\xi = (2Er)\xi\,\sigma, \quad \text{and} \quad \xi_t = \xi\left(\frac{aF}{e^{at}+F}\right).$$

Using (2.1), we get

$$D^2\xi = \tau\left[\frac{\alpha'}{r}(I-\sigma\otimes\sigma) + \alpha''\sigma\otimes\sigma\right] = 2E\xi\left(I + 2Er^2\sigma\otimes\sigma\right).$$

Using the above observations and Conditions A, B and C, and (4.3), we get

$$\begin{aligned}\Gamma_k[\xi] &= (2E\xi)^k r^{k-1} H(\sigma, I + 2Er^2\sigma\otimes\sigma) - f(\xi)\xi\left(\frac{aF}{e^{at}+F}\right)\\ &\geq \xi^k\left[(2E)^k\mathcal{R}^{k-1}H(\sigma,I) - \frac{af(\xi)}{\xi^{k-1}}\right].\end{aligned}$$

By (4.3) and (4.4), $\xi(x,t) \geq De^{E\mathcal{R}^2}/(1+F)$, in Ω_T. For ξ to be a subsolution, we require that

$$0 < a < \frac{(2E)^k\mathcal{R}^{k-1}\min_{|\sigma|=1}H(\sigma,I)}{\mathcal{F}(\theta)}, \quad \text{where} \quad \theta = De^{E\mathcal{R}^2}/(1+F) > 0, \tag{4.5}$$

and $\mathcal{F}(\theta) = \sup_{s\geq\theta} f(s)/s^{k-1}$. See (1.7) C(i), and (4.2). □

Auxiliary Function 2 (Super-solution): Let z be as before, and recall (4.3). For positive D, E, F, and $a > 0$, we set

$$\zeta(x,t) = \beta(r)\gamma(t), \quad \text{where} \;\; \beta(r) = De^{-Er^2} \quad \text{and} \quad \gamma(t) = 1 + Fe^{-at}. \tag{4.6}$$

We impose a condition on E and a, for ζ to be a super-solution. Rest are chosen in the proof of the theorem. Clearly,

$$\begin{aligned}\beta' = (-2Er)\beta, \;\; \beta'' = 2E\beta(2Er^2-1), \;\; \text{and}\\ \gamma' = -aFe^{-at} = -\gamma\left(\frac{aFe^{-at}}{1+Fe^{-at}}\right).\end{aligned}$$

Letting $\sigma = (x-z)/|x-z|$, we have

$$\begin{aligned}D\zeta = (-2Er)\zeta\,\sigma, \qquad \zeta_t = -\zeta\left(\frac{aFe^{-at}}{1+Fe^{-at}}\right),\\ D^2\zeta = \gamma\left[\frac{\beta'}{r}(I-\sigma\otimes\sigma) + \beta''\sigma\otimes\sigma\right]\\ = 2E\zeta\left(2Er^2\sigma\otimes\sigma - I\right).\end{aligned}$$

Thus,

$$\begin{aligned}\Gamma_k[\zeta] &= H\left(-2E\zeta r\,\sigma,\ \ 2E\zeta\left(2Er^2\sigma\otimes\sigma - I\right)\right) + f(\zeta)\zeta\left(\frac{aFe^{-at}}{1+Fe^{-at}}\right)\\ &= (2E\zeta)^k r^{k-1} H\left(\sigma, 2Er^2\sigma\otimes\sigma - I\right) + f(\zeta)\zeta\left(\frac{aFe^{-at}}{1+Fe^{-at}}\right)\\ &= \zeta^k\left[(2E)^k r^{k-1} H\left(\sigma, 2Er^2\sigma\otimes\sigma - I\right) + \frac{af(\zeta)}{\zeta^{k-1}}\left(\frac{Fe^{-at}}{1+Fe^{-at}}\right)\right]. \qquad (4.7)\end{aligned}$$

By (4.3), $\mathcal{R}\le r\le \mathcal{R}+\mathcal{D}$. We choose E (see (1.7) C(i)) small so that

$$0<\kappa\equiv 2E(\mathcal{R}+\mathcal{D})^2\le 1/2 \quad\text{and}\quad L\equiv\max_{|\sigma|=1} H(\sigma,\kappa\,\sigma\otimes\sigma - I)<0.$$

The latter follows as $H(\sigma,\kappa\sigma\otimes\sigma - I)\le H(\sigma,-I/2)=H(\sigma,-I)/2<0$.

Next, set (see (4.2))

$$\theta\equiv\inf_{(x,t)\in\Omega_\infty}\zeta\ge De^{-E(\mathcal{R}+\mathcal{D})^2}\ge De^{-1},\quad\text{and}\quad \mathcal{F}(\theta)=\sup_{[\theta,\infty)}\frac{f(s)}{s^{k-1}}.$$

Select

$$0<a<\frac{(2E)^k\mathcal{R}^{k-1}\,|L|}{\mathcal{F}(\theta)}. \qquad (4.8)$$

Hence,

$$\begin{aligned}\Gamma_K[\zeta] &\le \zeta^k\left[\frac{af(\zeta)}{\zeta^{k-1}} + (2E)^k r^{k-1} H\left(\sigma,\kappa\,\sigma\otimes\sigma - I\right)\right]\\ &\le \zeta^k\left[a\mathcal{F}(\theta) - (2E)^k\mathcal{R}^{k-1}|L|\right]\le 0,\end{aligned}$$

i.e., ζ is a super-solution in Ω_∞. □

We introduce additional notation for the proof of the theorem. Recall that for $t>0$, $\mathcal{Q}_t=\overline{\Omega}\times[t,\infty)$ and $\mathcal{S}_t=\partial\Omega\times[t,\infty)$. Let $t\ge T$. Define

$$\begin{aligned}&\text{(i) } \mu_{\inf}(t)=\inf_{\overline{\mathcal{Q}}_t} u, \quad \text{(ii) } \mu_{\sup}(t)=\sup_{\overline{\mathcal{Q}}_t} u, \qquad (4.9)\\ &\text{(iii) } \nu_{\inf}(t)=\inf_{\mathcal{S}_t} h, \quad\text{and}\quad \text{(iv) } \nu_{\sup}(t)=\sup_{\mathcal{S}_t} h.\end{aligned}$$

Since $u=h$ on S_T, $\mu_{\inf}(t)\le\nu_{\inf}(t)$, and $\nu_{\sup}(t)\le\mu_{\sup}(t)$. Set

$$\nu_{\sup}=\lim_{t\to\infty}\nu_{\sup}(t)\quad\text{and}\quad \nu_{\inf}=\lim_{t\to\infty}\nu_{\inf}(t). \qquad (4.10)$$

Proof of Part (a) of Theorem 1.3. Let $k>1$, and $t\ge T$. Recall the notation in (4.9), and (4.10). Recall also that $u>0$ is a super-solution, and since (1.4) holds, $\mu_{\inf}(t)<\infty$, $\forall\, t>0$.

Since $\mu_{\inf}(t) \le \nu_{\inf}(t)$, the claim follows if we show that

$$\lim_{t\to\infty} \mu_{\inf}(t) \ge \nu_{\inf}.$$

Also, from (4.1), $u \ge \min\{\min_{\Omega} u(x,T),\ m\} \equiv m_0$. Since $\nu_{\inf} \ge \nu_{\inf}(t) \ge \mu_{\inf}(t) \ge m_0$, if $\nu_{\inf} = m_0$, the claim follows. Assume from here on that $\nu_{\inf} > m_0$.
Let $\varepsilon > 0$ be small, and $T_0 \ge T$, large, so that for $t \ge T_0$ (see (4.10))

$$\nu_{\inf}(t) \ge \nu_{\inf} - \varepsilon > m_0 > 0.$$

Fix $z \in \mathbb{R}^n \setminus \Omega$; let r, $\mathcal{R}$ and $\mathcal{D}$ be as in (4.3). We employ Auxiliary Function 1, see (4.4), and recall condition (4.5):

$$\xi(x,t) = De^{Er^2}\left(\frac{e^{a(t-T_0)}}{e^{a(t-T_0)}+F}\right), \quad \text{where}$$
$$0 < a < \frac{(2E)^k \mathcal{R}^{k-1} \min_{|\sigma|=1} H(\sigma, I)}{\mathcal{F}(\theta)}, \quad \text{and} \quad \theta = \inf_{\Omega_T} \xi \ge m_0/2,$$

see (4.2), (4.8), and (4.12) below.
We select

$$D = m_0, \quad E = \frac{1}{(\mathcal{R}+\mathcal{D})^2}\log\left(\frac{\nu_{\inf}-\varepsilon}{m_0}\right), \quad \text{and} \quad F = \frac{\nu_{\inf}-\varepsilon}{m_0} - 1.$$

Hence,

$$e^{E(\mathcal{R}+\mathcal{D})^2} = 1 + F = \frac{\nu_{\inf}-\varepsilon}{m_0}, \quad \text{and} \tag{4.11}$$
$$\xi(x,t) = m_0(1+F)^{r^2/(\mathcal{R}+\mathcal{D})^2}\left(\frac{e^{a(t-T_0)}}{e^{a(t-T_0)}+F}\right).$$

We may bound ξ as follows. For a lower bound, take $r = \mathcal{R}$ (large), and for an upper bound take $r = \mathcal{R}+\mathcal{D}$, to find that, for $t \ge T_0$,

$$\frac{m_0}{2} \le \left(\frac{m_0}{1+F}\right)(1+F)^{\mathcal{R}^2/(\mathcal{R}+\mathcal{D})^2} \le \xi(x,t) \le m_0(1+F) = \nu_{\inf} - \varepsilon. \tag{4.12}$$

The lower bound for ξ influences the choice of a, see (4.2) and (4.5) (take $\theta = m_0/2$).
We show that $u \ge \xi$ in $\mathcal{Q}_{T_0}$. Using (4.1), (4.11), and that $\mathcal{R} \le r \le \mathcal{R}+\mathcal{D}$,

$$m_0(1+F)^{[\mathcal{R}^2/(\mathcal{R}+\mathcal{D})^2]-1} \le \xi(x,T_0) \le m_0 \le u(x,T_0), \quad \forall\, x \in \Omega.$$

Use the upper bound in (4.12) to see that, for $x \in \partial\Omega$, $\xi(x,t) \le \nu_{\inf} - \varepsilon \le h(x,t)$, $\forall (x,t) \in \mathcal{S}_{T_0}$. Employing the comparison principle in Theorem 2.3,

$$u \ge \xi \ \text{ in } \mathcal{Q}_{T_0}.$$

Using (4.11), and $r \geq \mathcal{R}$, we have

$$u(x,t) \geq m_0 \left(\frac{\nu_{\inf} - \varepsilon}{m_0}\right)^{\mathcal{R}^2/(\mathcal{R}+\mathcal{D})^2} \left(\frac{e^{a(t-T_0)}}{e^{a(t-T_0)} + F}\right), \quad \forall (x,t) \in \mathcal{Q}_{T_0}.$$

Since the above holds for any $x \in \Omega$, we take the infimum over x to obtain,

$$\mu_{\inf}(t) \geq m_0 \left(\frac{\nu_{\inf} - \varepsilon}{m_0}\right)^{\mathcal{R}^2/(\mathcal{R}+\mathcal{D})^2} \left(\frac{e^{a(t-T_0)}}{e^{a(t-T_0)} + F}\right).$$

Letting $t \to \infty$, and then letting $\mathcal{R} \to \infty$,

$$\lim_{t\to\infty} \mu_{\inf}(t) \geq \nu_{\inf} - \varepsilon.$$

The claim follows since the above is true for any small ε. □

Proof of Part (b). We assume that u is a sub-solution. Recall that $M = \sup_{\mathcal{S}_T} h(x,t)$. Set $M_0 = \max\{u(x,T), \ M\}$. As noted in (4.1), $u(x,t) \leq M_0$ in $\mathcal{Q}_T$. Since $\nu_{\sup} \leq \mu_{\sup}(t) \leq M_0$, if $\nu_{\sup} = M_0$, the statement follows.

Thus, we assume that $\nu_{\sup} < M_0$ and show that $\lim_{t\to\infty} \mu_{\sup}(t) \leq \nu_{\sup}$.

Let $\varepsilon > 0$, small, and $T_0 > T > 0$ be such that

$$\nu_{\sup} \leq \nu_{\sup}(t) \leq \nu_{\sup} + \varepsilon < M_0, \quad \text{for any } t \geq T_0. \tag{4.13}$$

This ensures that $h(x,t) \leq \nu_{\sup} + \varepsilon$ on $\mathcal{S}_{T_0}$.

We employ the super-solution ζ in (4.6): let $z \in \mathbb{R}^n \setminus \Omega$ and $r = |x - z|$. Define

$$\zeta(x,t) = \zeta(r,t) = De^{-Er^2}\left(1 + Fe^{-a(t-T_0)}\right), \quad \forall (x,t) \in \mathcal{Q}_{T_0},$$

where D, E, F and a are positive constants. Recalling (4.3) and (4.8), we choose

$$\begin{aligned} &0 < a < \frac{(2E)^k \mathcal{R}^{k-1}|L|}{\mathcal{F}(\theta)}, \quad \text{where} \quad L = \max_{|\omega|=1} H(\sigma, \kappa\sigma\otimes\sigma - I) < 0, \\ &\kappa \equiv 2E(\mathcal{R}+\mathcal{D})^2 \leq 1/2, \quad \text{and} \quad \theta = De^{-E(\mathcal{R}+\mathcal{D})^2}. \end{aligned} \tag{4.14}$$

Observe that $L \leq \max_{|\sigma|=1} H(\sigma, -I)/2$. Also, a different choice for θ is indicated below.

For a fixed κ, we choose

$$D = e^{\kappa/2}(\nu_{\sup} + \varepsilon), \quad E = \frac{\kappa}{2(\mathcal{R}+\mathcal{D})^2} \quad \text{and} \quad F = \frac{M_0}{\nu_{\sup} + \varepsilon} - 1.$$

Thus, in $\mathcal{Q}_{T_0}$,

$$\zeta(x,t) = (\nu_{\sup} + \varepsilon) \exp\left(\frac{\kappa}{2}\left[1 - \frac{r^2}{(\mathcal{R}+\mathcal{D})^2}\right]\right)\left(1 + Fe^{-a(t-T_0)}\right).$$

Since $\zeta \geq \nu_{\sup} + \varepsilon$, one can choose $\theta = \nu_{\sup} + \varepsilon$, see (4.14).

Recall that $u(x,t) \leq M_0$. Since $\mathcal{R} \leq r \leq \mathcal{R} + \mathcal{D}$, for $x \in \Omega$, by (4.13),

$$\zeta(x, T_0) \geq (\nu_{\sup} + \varepsilon) \left(\frac{M_0}{\nu_{\sup} + \varepsilon} \right) \geq M_0 \geq u(x, T_0).$$

As noted above already, $\zeta \geq \nu_{\sup} + \varepsilon$, and, thus, $\zeta(x,t) \geq h(x,t)$, $\forall (x,t) \in \mathcal{S}_{T_0}$.

Thus, $\zeta \geq u$ on the parabolic boundary of $\mathcal{Q}_{T_0}$, and Theorem 2.3 implies that $\zeta \geq u$ in $\mathcal{Q}_{T_0}$. Observe that for each $x \in \Omega$, $\zeta(x,t)$ is decreasing in t. Thus,

$$\mu_{\sup}(t) \leq \sup_{\mathcal{Q}_t} \zeta \leq (\nu_{\sup} + \varepsilon) \exp\left(\frac{\kappa}{2} \left[1 - \frac{\mathcal{R}^2}{(\mathcal{R} + \mathcal{D})^2} \right] \right) \left(1 + F e^{-a(t - T_0)} \right),$$

for any $t > T_0$.

Let $t \to \infty$ and then let $\mathcal{R} \to \infty$ to obtain that $\lim_{t\to\infty} \mu_{\sup}(t) \leq \nu_{\sup} + \varepsilon$. The claim holds. $\square$

5. Proof of Theorem 1.4

We begin with a useful lemma. See Appendix A.2 for existence, and comparison principles.

Lemma 5.1. *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded domain that satisfies an outer ball condition. Let $k \geq 1$, $\delta \neq 0$ and $\theta \in \mathbb{R}$. Then there is a ψ in $C(\overline{\Omega})$ such that*

$$H(D\psi, D^2\psi) = \delta, \quad \text{in } \Omega, \text{ with } \psi = \theta \text{ on } \partial\Omega.$$

If $\delta > 0$ then $\psi \leq \theta$, and if $\delta < 0$ then $\psi \geq \theta$. Also, $\psi = \theta + |\delta|^{1/k}\eta(x)$, where $H(D\eta, D^2\eta) = \delta/|\delta|$, and $\eta = 0$ on $\partial\Omega$.

Proof of Theorem 1.4 Part (a). Suppose that $\nu > 0$ and $k > 1$. Assume that $u > 0$ is a sub-solution and $u = \nu$ on $\mathcal{S}_T$.

Let $\varepsilon > 0$ be small. By Theorem 1.3, there is a $T_0 \geq T$ such that

$$\nu \leq \sup_{x \in \overline{\Omega}} u(x,t) \leq \nu + \varepsilon, \quad \text{for any } t \geq T_0. \tag{5.1}$$

By Lemma 5.1, there is a function $\psi \geq 1$ in $C(\Omega)$ such that

$$H(D\psi, D^2\psi) = -1 \quad \text{in } \Omega, \text{ and } \psi = 1 \text{ on } \partial\Omega. \tag{5.2}$$

Observe that $\psi \geq 1$ in Ω.

Let $T_1 \geq T_0$, to be determined later. With ψ as in (5.2), set in $\mathcal{Q}_{T_1}$,

$$\phi(x,t) = \nu + \varepsilon\psi(x)\tau(t) \quad \text{in } \mathcal{Q}_{T_1}, \quad \text{where} \quad \tau(t) = \left(\frac{T_1}{t} \right)^{1/(k-1)}.$$

Define $M = \sup_{\overline{\Omega}} \psi$. Clearly,

$$1 \leq \psi \leq M, \quad \nu \leq \phi \leq \nu + \varepsilon M, \quad \text{and} \quad \tau'(t) = \frac{-\tau(t)}{(k-1)t}. \tag{5.3}$$

Using (5.2) and (5.3),

$$\Gamma_k[\phi] = H(D\phi, D^2\phi) - f(\phi)\phi_t = -(\varepsilon\tau)^k + f(\phi)\left(\frac{\varepsilon\tau\psi}{(k-1)t}\right).$$

Since $\tau^{k-1} = T_1/t$, using (5.3),

$$\Gamma_k[\phi] = \varepsilon\tau\left[\frac{\psi f(\phi)}{(k-1)t} - (\varepsilon\tau)^{k-1}\right] \le \frac{\varepsilon\tau}{t}\left[\frac{Mf(\nu+\varepsilon M)}{k-1} - \varepsilon^{k-1}T_1\right].$$

Hence, ϕ is super-solution in $\mathcal{Q}_{T_1}$ if

$$T_1 \ge \max\left\{\frac{Mf(\nu+\varepsilon M)}{(k-1)\varepsilon^{k-1}},\ T_0\right\}.$$

Next, from (5.1) and (5.3),

$$u(x,T_1) \le \nu+\varepsilon \le \phi(x,T_1) \ \text{ and } \ u(x,t) = \nu \le \phi(x,t),\ \forall (x,t)\in \mathcal{S}_{T_1}.$$

By the comparison principle in Theorem 2.3, and (5.1),

$$\nu \le \sup_\Omega u(x,t) \le \sup_\Omega \phi(x,t) \le \nu + \frac{\varepsilon M T_1^{1/(k-1)}}{t^{1/(k-1)}} = \nu + \frac{K}{t^{1/(k-1)}} \quad \text{in } \mathcal{Q}_{T_1},$$

where $K = K(k,\nu,T,M)$. Thus,

$$\lim_{t\to\infty}\left[t^\alpha\left(\sup_\Omega u(x,t) - \nu\right)\right] = 0, \quad \text{for any } \ 0<\alpha<\frac{1}{k-1}.$$

The claim holds. □

Proof of Part (b). We assume that $u>0$ is a super-solution. In Lemma 5.1, let ψ be the solution for $\delta = 1$ and $\theta=-1$. Set $L = \max_{\overline{\Omega}}|\psi|$; thus, $\psi<0$, and

$$1 \le |\psi| \le L.$$

Let $\varepsilon_0>0$, small, such that $\varepsilon_0 L < \nu$. Next, choose T_ε and T_0 as follows.

$$\text{(i)}\quad T_\varepsilon = \frac{f(\nu)L}{(k-1)\varepsilon^{k-1}}, \quad \text{where } \ 0<\varepsilon\le\varepsilon_0, \quad \text{and}$$

$$\text{(ii)}\ \ T_0 \ge T_\varepsilon \text{ such that } \ \forall (x,t)\in \mathcal{Q}_{T_0}, \quad 0<\nu-\varepsilon \le \inf_{\Omega} u(x,t) \le \nu. \tag{5.4}$$

For the second statement, we have used Theorem 1.3.

Next, set

$$\begin{aligned}\phi(x,t) &= \nu + \varepsilon\psi(x)\left(\frac{T_0}{t}\right)^{1/(k-1)}\\ &= \nu - \varepsilon|\psi(x)|\left(\frac{T_0}{t}\right)^{1/(k-1)}, \quad \forall (x,t)\in\mathcal{Q}_{T_0}.\end{aligned}$$

Since $\varepsilon L < \nu$ (see (5.4)) and $\psi \le -1$, we have

$$\phi(x, T_0) \le \nu - \varepsilon, \ \text{ in } \Omega, \ \text{ and } \ 0 < \phi(x,t) \le \nu \ \text{ in } \mathcal{S}_{T_0}. \tag{5.5}$$

Since $H(D\psi, D^2\psi) = 1$, using (5.5), we have that

$$\begin{aligned}\Gamma_k[\phi] &= \varepsilon^k \left(\frac{T_0}{t}\right)^{k/(k-1)} + f(\phi)\left(\frac{\varepsilon\psi}{k-1}\right)\frac{T_0^{1/(k-1)}}{t^{k/(k-1)}} \\ &\ge \frac{\varepsilon T_0^{1/(k-1)}}{t^{k/(k-1)}}\left(\varepsilon^{k-1}T_0 - \frac{f(\nu)L}{k-1}\right) \ge 0.\end{aligned}$$

The last line follows from (5.4).

Since ϕ is sub-solution in $\mathcal{Q}_{T_0}$, by (5.5) and that $u = \nu$ on $\partial\Omega$, $t \ge T_0$, we obtain that $u \ge \phi$ on its parabolic boundary. Using Theorem 2.3,

$$\begin{aligned}u(x,t) \ge \phi(x,t) &= \nu + \varepsilon\psi(x)\left(\frac{T_0}{t}\right)^{1/(k-1)} \\ &\ge \nu - \varepsilon L \left(\frac{T_0}{t}\right)^{1/(k-1)}, \quad \forall (x,t) \in \mathcal{Q}_{T_0}.\end{aligned}$$

Observe that $\inf_\Omega \phi(x,t) \le \inf_\Omega u(x,t) \le \nu$.

If $0 < \alpha < 1/(k-1)$ we have

$$\lim_{t\to\infty}\left[t^\alpha\left(\inf_\Omega u(x,t) - \nu\right)\right] = 0.$$

This proves the claim. $\square$

A. Change of variables and Existence for the elliptic problem

A.1. Change of Variables:

Recall from Subsection 2.2 that

$$F_k(u) = \int^u f(s)^{-1/(k-1)} ds, \ \ u > 0.$$

Setting $w = F_k(u)$, we get

$$\begin{aligned}Du &= f(u)^{1/(k-1)}Dw, \quad u_t = f(u)^{1/(k-1)}w_t \\ D^2u &= f(u)^{1/(k-1)}\left\{D^2w + [f(u)^{1/(k-1)}]'Dw \otimes Dw\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}&H(Du, D^2u) - f(u)u_t \\ &\quad = f(u)^{k/(k-1)}\left[H\left(Dw, D^2w + [f(u)^{1/(k-1)}]'Dw \otimes Dw\right) - w_t\right].\end{aligned}$$

A.2. Existence for the elliptic problem:

The work overlaps with the work in [7]. We begin with a version of the comparison principle that will be used in this section. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. We recall a result proven in [6].

Lemma A.1. *Let $f_i \colon \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, be continuous as in (2.2). Suppose that $u \in usc(\overline{\Omega})$ and $v \in lsc(\overline{\Omega})$ are solutions to*

$$H(Du, D^2u) \geq f_1(x, u(x)) \quad \text{and} \quad H(Dv, D^2v) \leq f_2(x, v(x)), \quad \text{in } \Omega.$$

If $\sup_\Omega(u - v) > \sup_{\partial\Omega}(u - v)$ then there is a point $z \in \Omega$ such that

$$(u - v)(z) = \sup_\Omega(u - v) \text{ and } f_1(z, u(z)) \leq f_2(z, v(z)).$$

Proof. A proof can be worked out as in Theorem 4.1 in [[6]: Section 4]. □

Corollary A.2 (Comparison Principle). *Suppose that s, $t \in \mathbb{R}$ are such that $|s| + |t| > 0$, and $s \leq t$. Let $u \in usc(\overline{\Omega})$ and $v \in lsc(\overline{\Omega})$ satisfy*

$$H(Du, D^2u) \geq t, \quad \text{and} \quad H(Dv, D^2v) \leq s \quad \text{in } \Omega.$$

Then $u - v \leq \sup_{\partial\Omega}(u - v)$.

Proof. Consider $s < t$. By taking $f_1 = t$ and $f_2 = s$, Lemma A.1 implies that $u - v \leq \sup_{\partial\Omega}(u - v)$.

Assume now that $t = s$. We take $\theta > 1$ if $t > 0$, and $0 < \theta < 1$ if $t < 0$. The function $u_\theta = \theta u$ solves $H(Du_\theta, D^2u_\theta) = \theta^k H(Du, D^2u) \geq t\theta^k > s$. Thus, $u_\theta - v \leq \sup_{\partial\Omega}(u_\theta - v)$. The conclusion follows by letting $\theta \to 1$. □

A.3. Existence for Lemma 5.1

Let $\delta > 0$ and $\theta \in \mathbb{R}$. We show now the existence of viscosity solutions to the following problems by using the Perron method.

$$\begin{aligned} &\text{(a) } H(Du, D^2u) = \delta, \text{ in } \Omega, u = \theta \text{ on } \partial\Omega, \text{ and} \\ &\text{(b) } H(Du, D^2u) = -\delta, \text{ in } \Omega, u = \theta \text{ on } \partial\Omega. \end{aligned} \tag{A.1}$$

Corollary A.2 provides the necessary comparison principle. Define

$$d = \operatorname{diam}(\Omega). \tag{A.2}$$

By the outer ball condition, for any $y \in \partial\Omega$, there is a $\rho > 0$ and a $q \in \mathbb{R}^n \setminus \Omega$ such that

$$B_\rho(q) \subset \mathbb{R}^N \setminus \Omega \quad \text{and} \quad y \in \partial\Omega \cap \overline{B}_\rho(q). \tag{A.3}$$

Sub and Super solutions to (A.1)(a): Note that $w(x) = \theta$ is a super-solution of (A.1)(a). Our effort is to construct sub-solutions.

Let $y \in \partial\Omega$. With ρ and q_y as in (A.3), set $r = |x - q|$. Define

$$v_y(x) = \theta + E\left(\frac{1}{r^\alpha} - \frac{1}{\rho^\alpha}\right), \quad \forall x \in \Omega.$$

where $E > 0$ and $\alpha > 0$ are to be determined. Using (2.1), we get, in $r \geq \rho$,

$$\begin{aligned} H(Dv_y, D^2v_y) &= E^k H\left(\frac{-\alpha}{r^{\alpha+1}}e, \frac{-\alpha}{r^{\alpha+2}}(I - e\otimes e) + \frac{\alpha(\alpha+1)}{r^{\alpha+2}}e\otimes e\right) \\ &= \frac{(E\alpha)^k}{r^{\alpha k+k+1}} H\left(e, (\alpha+2)e\otimes e - I\right). \end{aligned} \tag{A.4}$$

Setting $\Lambda = \alpha + 2$, and recalling (1.6) and Condition C(ii) in Section 1 (see (1.7)),

$$\min_{|e|=1} H(e, \Lambda e\otimes e - I) \geq -M(\Lambda) > 0, \quad \text{if } \Lambda > \Lambda_1.$$

Choose $\Lambda > \Lambda_1$ and $\alpha = \Lambda - 2$. Since $\rho \leq r \leq \rho + d$, $\forall x \in \Omega$, (A.4) yields in Ω,

$$H_k[v_y] \geq \frac{(E\alpha)^k |M(\Lambda)|}{(\rho+d)^{k\alpha+k+1}} \geq \delta > 0,$$

if E is chosen large enough. With this choice, we obtain that

$$H(Dv_y, D^2v_y) \geq \delta, \quad v_y(y) = \theta, \quad \text{and} \quad v_y \leq \theta \text{ on } \partial\Omega.$$

For every $y \in \partial\Omega$, the sub-solution v_y attains the boundary value θ at y. The Perron Method leads to a solution $v_y \leq u \leq w = \theta$ of (A.1)(a).

Sub and Super solutions to (A.1)(b): Observe that $v(x) = \theta$ is a sub-solution. Our effort is to construct super-solutions.

Let $y \in \partial\Omega$. With d as in (A.2), and ρ and q as in (A.3), set $r = |x - q|$. Define

$$w_y(x) = \theta + E\left(\frac{1}{\rho^\alpha} - \frac{1}{r^\alpha}\right), \quad \forall x \in \Omega,$$

where $E > 0$ and $\alpha > 0$ are to determined. Using (2.1), and (A.4), we get, in $r > 0$,

$$H(Dw_y, D^2w_y) = \frac{(E\alpha)^k}{r^{\alpha k+k+1}} H(e, I - (\alpha+2)e\otimes e).$$

Set $\Lambda = \alpha + 2$. From (1.6) and Condition C(ii), $\max_{|e|=1} H(e, I - \Lambda e\otimes e) \leq M(\Lambda) < 0$, if $\Lambda > \Lambda_1$. Choose $\alpha > \Lambda - 2$. Since, $\rho \leq r \leq \rho + d$, we see that if $E > 0$ is large enough,

$$H(Dw_y, D^2w_y) = \frac{(E\alpha)^k}{r^{\alpha k+k+1}} H(e, I - (\alpha+2)e\otimes e) \leq \frac{(E\alpha)^k M(\Lambda)}{(\rho+d)^{\alpha k+k+1}} \leq -\delta < 0.$$

Thus, $H(Dw_y, D^2w_y) \leq -\delta$, in Ω, $\bar{w}_y(y) = \theta$, and $w_y \geq \theta$, on $\partial\Omega$. By the Perron method, there is a solution u such that $\theta = v \leq u \leq w_y$. $\square$

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