Metric limits of Calabi-Yau manifolds

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Complex elliptic curves

\[ y^2 = x^3 + ax + b, \quad 27b^2 + 4a^3 \neq 0 \]

\[ x, y, z, a, b \in \mathbb{C} \]

Projective compactification in \( \mathbb{P}^2 = (\mathbb{C}^3 \backslash \{(0, 0, 0)\})/\mathbb{C}^* \ni [x : y : z] \)

\[ y^2z = x^3 + axz^2 + bz^3 \]

This equation describes a torus
This elliptic curve is isomorphic to $\mathbb{C}/\Lambda$

\[
\Lambda = \text{Span}_\mathbb{Z}(1, \tau)
\]

\[
\tau = \tau(a, b) \in \mathbb{C}, \quad \text{Im}\tau > 0
\]

Topologically, it is a compact orientable (real) 2-dimensional surface $\Sigma$ with $\chi(\Sigma) = 0$

It inherits a \emph{flat} Riemannian metric from any Euclidean metric in $\mathbb{C}$, \emph{not} the one pictured above

It also inherits a never-vanishing holomorphic 1-form, induced by $dz$ in $\mathbb{C}$
Complex Manifolds

$X^n$ covered by atlas $U_\alpha \subset \mathbb{C}^n$, transition maps are holomorphic

- Riemann surfaces

- Complex tori $\mathbb{C}^n/\Lambda \cong (S^1)^{2n}$, $\Lambda \cong \mathbb{Z}^{2n}$

- Projective spaces $\mathbb{P}^n = (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$

- Smooth projective varieties $X = \{P_1 = \cdots = P_m = 0\} \subset \mathbb{P}^N$

Ex: $\{z_0^d + z_1^d + \cdots + z_N^d = 0\} \subset \mathbb{P}^N$
Kähler Metrics

$X^n$ complex manifold. A Kähler metric is a smooth positive definite closed real $(1,1)$ form

$$\omega = i \sum_{j,k=1}^{n} g_{jk}(z) dz_j \wedge d\bar{z}_k, \quad (g_{jk}) > 0, \quad d\omega = 0$$

We say the manifold $X$ is Kähler if it admits a Kähler metric (Kähler 32)

Riemann surfaces, complex tori are all Kähler.

On $\mathbb{P}^n = (\mathbb{C}^{n+1}\setminus\{0\})/\mathbb{C}^*$ the Fubini-Study metric is induced by

$$\omega_{FS} = \frac{i}{2\pi} \partial \bar{\partial} \log(|z_0|^2 + \cdots + |z_n|^2) \text{ on } \mathbb{C}^{n+1}\setminus\{0\}, \quad d = \partial + \bar{\partial}$$

The Kähler property is inherited by restriction, hence all smooth projective varieties are Kähler
The Kähler Cone

$X$ compact Kähler, de Rham cohomology

$$H^2(X, \mathbb{R}) = \left\{ \alpha \in \Lambda^2 X \mid d\alpha = 0 \right\}$$

$$d(\Lambda^1 X)$$

$H^{1,1}(X, \mathbb{R}) \subset H^2(X, \mathbb{R})$ classes of type $(1, 1)$

Kähler cone

$$C_X = \{ [\omega] \in H^{1,1}(X, \mathbb{R}) \mid \omega \text{ Kähler metric on } X \}$$

Open convex cone in $H^{1,1}(X, \mathbb{R})$

$L \to X$ holomorphic line bundle, $h$ Hermitian metric on $L$

$$R_h = -\frac{i}{2\pi} \partial \overline{\partial} \log h, \quad dR_h = 0, \quad c_1(L) := [R_h] \in H^{1,1}(X, \mathbb{R}) \text{ first Chern class}$$
The ample cone

$L \to X$ holomorphic line bundle $\leadsto c_1(L) \in H^{1,1}(X, \mathbb{R})$ first Chern class

The classes one obtains in this way are the Néron-Severi group

$$NS(X) = H^2(X, \mathbb{Z})_{\text{free}} \cap H^{1,1}(X, \mathbb{R}), \quad NS(X, \mathbb{R}) = NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

Ample cone $C_X \cap NS(X, \mathbb{R})$

Integral points $C_X \cap NS(X)$ are first Chern classes of ample line bundles

These exist precisely when $X$ admits a projective embedding (Kodaira 54)

Classes in $\partial C_X \cap NS(X)$ are first Chern classes of nef line bundles
Ricci curvature

$(X^n, \omega)$ Kähler manifold. Volume form

$$\omega^n = n! \det(g_{j\bar{k}}) idz_1 \wedge d\bar{z}_1 \cdots idz_n \wedge d\bar{z}_n$$

Ricci curvature $R_{j\bar{k}} = -\frac{\partial}{\partial z_j} \frac{\partial}{\partial \bar{z}_k} \log \det(g_{p\bar{q}})$

Ricci form

$$\text{Ric}(\omega) = i \sum_{j,k=1}^{n} R_{j\bar{k}}(z) dz_j \wedge d\bar{z}_k, \quad d\text{Ric}(\omega) = 0$$

$$\text{Ric}(\omega) - \text{Ric}(\tilde{\omega}) = i\bar{\partial} \partial \log \frac{\tilde{\omega}^n}{\omega^n}$$

$$[\text{Ric}(\omega)] = 2\pi c_1(X) \in H^{1,1}(X, \mathbb{R})$$
Calabi-Yau Manifolds

$X$ compact Kähler manifold with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ is Calabi-Yau

\[ X = \mathbb{C}^n/\Lambda, \ \Lambda \cong \mathbb{Z}^{2n}, \ \text{complex } n\text{-tori} \]

$K3$ surfaces are a simply connected Calabi-Yau surfaces ($n = 2$)
Ex: $x^4 + y^4 = z^4 + w^4$ in $\mathbb{P}^3$

Enriques surfaces are double quotients of $K3$

\[ X \text{ a smooth complex hypersurface in } \mathbb{P}^{n+1} \text{ of degree } n+2, \ X = \{P = 0\}, \ \deg P = n+2 \]
Ex: $z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 = 0$ in $\mathbb{P}^4$
Classification of Calabi-Yau Manifolds?

Calabi-Yau manifolds form one of the basic building blocks for the birational classifications of all complex projective varieties. Up to finite cover, they have $\Omega \neq 0$ holomorphic $n$-form

$X$ Calabi-Yau of dimension 1 is an elliptic curve

(Kodaira 64): $X$ Calabi-Yau of dimension 2 is isomorphic to

- A complex 2-torus
- A finite quotient of a 2-torus
  
  Ex: $T = \mathbb{C}^2/(\mathbb{Z} \oplus i\mathbb{Z})^2$, $\iota: (z_1, z_2) \mapsto (z_1 + \frac{1}{2}, -z_2)$, $X = T/\iota$
- A K3 surface
- An Enriques surface

In particular, Calabi-Yau surfaces have only finitely many homotopy types

Not known to be true in dimensions 3 or higher! A large number of examples is known (e.g. complete intersections in products of (weighted) projective spaces)
Theorem (Yau 76, Conjectured by Calabi 54)

Let $X$ be a compact Kähler manifold. Then $X$ admits Ricci-flat Kähler metrics iff $X$ is Calabi-Yau.

More precisely, if $X$ is Calabi-Yau and $[\omega] \in \mathcal{C}_X$, there is a unique Kähler metric $\tilde{\omega}$ with $\text{Ric}(\tilde{\omega}) = 0$ and $[\tilde{\omega}] = [\omega]$ in $H^{1,1}(X, \mathbb{R})$.

The metric $\tilde{\omega}$ is obtained by solving a nonlinear PDE on $X$ of complex Monge-Ampère type

$$\det \left( g_{j\bar{k}} + \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k} \right) = F \det(g_{j\bar{k}}),$$

$$\tilde{g}_{j\bar{k}} = g_{j\bar{k}} + \frac{\partial^2 \varphi}{\partial z_j \partial \bar{z}_k}.$$  

Solution only known explicitly on tori!
$X^n$ Calabi-Yau manifold. We wish to study degenerations of Ricci-flat Kähler metrics. Two basic ways to do this:

- Degenerate the Kähler class, fixing the complex structure
- Degenerate the complex structure, fixing the Kähler class

In each situation we will have a family of Ricci-flat Kähler metrics $\omega_t$ which is degenerating as $t \to 0$ and would like to understand their limiting behavior.
First Setup

$(X^n, \omega)$ unit-volume Ricci-flat Calabi-Yau manifold. Given $[\alpha] \in \partial C_X$, $[\alpha] + t[\omega] \in C_X$, $t > 0$

By Yau’s Theorem there is a unique Ricci-flat Kähler metric $\omega_t \in [\alpha] + t[\omega]$, which degenerates as $t \to 0$

$$\omega_t = \alpha + t\omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$$

$$(\alpha + t\omega + \sqrt{-1} \partial \bar{\partial} \varphi_t)^n = c_t \omega^n$$

$$c_t = \text{Vol}(X, \omega_t) \to \int_X \alpha^n$$

**Main Question:** Understand the behavior of $\omega_t$ as $t \to 0$. 
Second Setup

\((\mathcal{X}^{n+1}, \omega_{\mathcal{X}})\) Kähler manifold with \(\pi : \mathcal{X} \to \Delta \subset \mathbb{C}\) surjective proper holomorphic map with connected fibers, with \(\pi\) submersion over \(\Delta^*\) and \(X_t = \pi^{-1}(t)\) Calabi-Yau for \(t \in \Delta^*\).

For \(t \in \Delta^*\) let \(\omega_t \in [\omega_{\mathcal{X}}|_{X_t}]\) be the unique Ricci-flat Kähler metric on \(X_t\).

Example include families of projective Calabi-Yaus:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\pi} & \mathbb{P}^N \times \Delta \\
\downarrow & & \downarrow \\
\Delta & \xrightarrow{\pi} & \Delta
\end{array}
\]

Main Question: Understand the behavior of \(\omega_t\) as \(t \to 0\).
Diameter bound

Let us consider the first setup: $X^n$ Calabi-Yau, $[\alpha] \in \partial C_X$, $\omega_t \in [\alpha] + t[\omega]$ Ricci-flat Kähler metrics, $t > 0$.

What geometric information do we have about $\omega_t$?

Volumes of submanifolds: if $V \subset X$ closed $k$-dimensional complex submanifold (or irreducible closed analytic subvariety), then

$$\text{Vol}(V, \omega_t) = \int_V \omega^k_t = \int_V (\alpha + t\omega)^k \to \int_V \alpha^k \geq 0,$$

So $\text{Vol}(V, \omega_t) \to 0 \iff \int_V \alpha^k = 0$.

**Theorem (T., Zhang 07)**

*There is a constant $C$ so that for all $t > 0$*

$$\text{diam}(X, \omega_t) \leq C.$$
Gromov-Hausdorff convergence

\((X, d_X), (Y, d_Y)\) compact metric spaces. We say that their Gromov-Hausdorff distance is at most \(\varepsilon > 0\) if \(\exists F : X \to Y, G : Y \to X\)

- \(|d_X(x_1, x_2) - d_Y(F(x_1), F(x_2))| \leq \varepsilon, \ \forall x_1, x_2 \in X\)
- \(|d_Y(y_1, y_2) - d_X(G(y_1), G(y_2))| \leq \varepsilon, \ \forall y_1, y_2 \in Y\)
- \(d_X(x, G(F(x))) \leq \varepsilon, \ \forall x \in X\)
- \(d_Y(y, F(G(y))) \leq \varepsilon, \ \forall y \in Y\)

**Theorem (Gromov 81)**

The space of compact \(n\)-dimensional Riemannian manifolds with a uniform lower bound on the Ricci curvature and a uniform upper bound on the diameter is precompact in the Gromov-Hausdorff topology.

Therefore \((X, \omega_t)\) subconverge to some compact metric space \((Z, d)\). Is it unique? Can we identify it?
Noncollapsing Case

**Theorem (T. 07, Rong-Zhang 10, Collins-T. 15)**

\((X^n, \omega)\) Calabi-Yau, \([\alpha] \in \partial C_X\) with \(\int_X \alpha^n > 0\). Then

\[ E = \bigcup_{V \subset X, \int_V \alpha^{\dim V} = 0} V \]

is a nonempty proper closed analytic subvariety of \(X\), and as \(t \to 0\) the Ricci-flat metrics \(\omega_t \in [\alpha] + t[\omega]\) converge in \(C^\infty_{\text{loc}}(X \setminus E)\) to an incomplete Ricci-flat metric \(\omega_0\), and \((X, \omega_t)\) converges in Gromov-Hausdorff to \((Z, d) = \text{the metric completion of } (X \setminus E, \omega_0)\)

- Kobayashi-Todorov 87, LeBrun-Singer 94: Kummer K3 surfaces
- Song 14: if \([\alpha] \in NS(X)\) then there is \(f : X \to Y\) birational, \(Y = \text{singular Calabi-Yau variety, } [\alpha] = [f^*\omega_Y]\), and \(Z\) homeomorphic to \(Y\)
- Höring 18: if \(n = 3\), there is \(f : X \to Y\) bimeromorphic, \(Y = \text{singular Calabi-Yau variety, } [\alpha] = [f^*\omega_Y]\)
Kummer $K3$ surfaces

$T = \mathbb{C}^2 / \Lambda$ a complex 2-torus, $\iota : T \to T, \iota(z_1, z_2) = (-z_1, -z_2)$

$Y = T / \iota$ has 16 $\mathbb{Z}/2$-quotient singularities (orbifold)

$f : X \to Y$ blowup of 16 singular points, each one replaced by a $\mathbb{P}^1$

$X$ is a Kummer $K3$ surface. If $\omega_Y$ is a flat Euclidean (orbifold) metric $[f^* \omega_Y] = [\alpha] \in \partial \mathcal{C}_X$

$\int_X \alpha^2 > 0, \quad \int_C \alpha = 0 \iff C =$one of the 16 $\mathbb{P}^1$'s
Collapsing on tori

\[ X = \mathbb{C}^n / \Lambda \text{ torus} \]

Then \( \omega \) Ricci-flat Kähler \( \Leftrightarrow \) \( \omega \) flat \( \Leftrightarrow \) a positive definite \( n \times n \) Hermitian matrix

While \( [\alpha] \in \partial C_X \Leftrightarrow \) a semipositive definite \( n \times n \) Hermitian matrix with nontrivial kernel \( K \subset \mathbb{C}^n \)

Therefore here we are always collapsing.

If \( K \) is \( \mathbb{Q} \)-defined modulo \( \Lambda \) then
\[
[\alpha] = f^* [\omega_Y] \quad \text{where} \quad Y = \mathbb{C}^m / \Lambda', \quad m < n \text{ and } f : X \to Y \text{ fibration. The metrics } \omega_t \text{ shrink the fibers of } f.
\]

If \( K \) not \( \mathbb{Q} \)-defined modulo \( \Lambda \) then \( K \) defines a foliation with non-closed leaves.
Collapsing fibrations

Many examples of classes $[\alpha] \in \partial C_X$ with $\int_X \alpha^n = 0$ come from fibrations as follows.

$f : X \to Y$ surjective holomorphic map with connected fibers onto a compact Kähler manifold $Y^m, 0 < m < n$, and $[\alpha] = f^*[\omega_Y]$

$D \subset Y$ critical values of $f$, $S = f^{-1}(D) \subset X$,

$f : X \setminus S \to Y \setminus D$ proper holomorphic submersion

Fibers $X_y = f^{-1}(y)$ with $y \in Y \setminus D$ are Calabi-Yau $n - m$ folds, pairwise diffeomorphic

Examples: products, elliptic fibrations $X = K3 \to Y = \mathbb{P}^1$, holomorphic Lagrangian fibrations of hyperkähler manifolds,
How to find such fibrations?

**Conjecture (Bogomolov, Hassett-Tschinkel, Huybrechts 99)**

\[ X^n \text{ projective Calabi-Yau manifold, } 0 \neq [\alpha] \in \partial C_X \cap NS(X) \text{ with } \int_X \alpha^n = 0, \text{ so } [\alpha] = c_1(L) \text{ for } L \to X \text{ a nef line bundle. Then there is a fibration } f : X \to Y \text{ as above with } [\alpha] = f^*[\omega_Y]. \]

Classic result for $K3$ surfaces, where sections of $L^k$ define an elliptic fibration.

Fischer-Grauert 64: for such a fibration, the fibers $X_y$ are all pairwise isomorphic for all $y \in U \subset Y \setminus D$ open, if and only if $f$ is a holomorphic fiber bundle over $U$.

**Theorem (T.-Zhang 13)**

\[ X^n \text{ Calabi-Yau manifold and } f : X \to Y \text{ holomorphic submersion with connected fibers onto a compact Kähler manifold } Y. \text{ Then } f \text{ is a holomorphic fiber bundle.} \]

So for all “nontrivial” fibrations, $S \neq \emptyset$
Collapsing away from the singular fibers

**Theorem**

Let $X^n$ be a Calabi-Yau manifold, $[\alpha] = f^* [\omega_Y]$ for a fibration $f : X \to Y$ as before. Let $\omega_t \in [\alpha] + t[\omega_X]$ be Ricci-flat Kähler metrics. Then there is $\omega_0$ Kähler on $Y \setminus D$, $\text{Ric}(\omega_0) = \omega_{\text{WP}} \geq 0$, such that $\omega_t \to f^* \omega_0$ in

- $\mathcal{C}^\alpha_{\text{loc}}(X \setminus S)$, $0 < \alpha < 1$
- $\mathcal{C}^\infty_{\text{loc}}(X \setminus S)$ if $X_y$ tori
- $\mathcal{C}^\infty_{\text{loc}}(X \setminus S)$ if $X_y$ all pairwise isomorphic, $y \in Y \setminus D$

- $\mathcal{C}^\infty_{\text{loc}}(X \setminus S)$ is conjectured to hold in general (in progress)
- $\omega_{\text{WP}} \equiv 0 \iff$ all $X_y, y \in Y \setminus D$ are isomorphic
- Proved by Gross-Wilson 00 for elliptic $K3$ with 24 singular fibers
- Earlier weaker results by T. 09, T.-Weinkove-Yang 14
Adiabatic limit

Theorem (T.-Zhang 14)

For every \( x \in X \setminus S, y = f(x) \), the pointed rescaled spaces \((X, \frac{\omega_t}{t}, x)\) converge smoothly modulo diffeomorphisms to the product cylinder

\[
\left( \mathbb{C}^m \times X_y, \omega_{\mathbb{C}^m} \oplus \omega_y \right),
\]

where \( \omega_y \) is the Ricci-flat Kähler metric on \( X_y \) in \([\omega_X|_{X_y}]\).

Li 18: similar result also on singular fibers, for certain Calabi-Yau 3-folds fibered by \( K3 \)
Conjecture

In above theorem, \((X, \omega_t) \rightarrow (Z, d)\) in Gromov-Hausdorff, where \((Z, d) = \text{metric completion of } (Y \setminus D, \omega_0)\), and \(Z_{\text{sing}} = Z \setminus (Y \setminus D)\) satisfies

\[
\text{codim}_{\mathbb{R}} (Z_{\text{sing}}) \geq 2,
\]

and \(Z\) is homeomorphic to \(Y\).

Theorem

This is true when

- (Gross-T.-Zhang 13) \(\dim Y = 1\) (covers all elliptic K3 surfaces)
- (T.-Zhang 17) \(X\) projective hyperkähler
- (Gross-T.-Zhang 19) when \(X\) projective and \(D\) a divisor with simple normal crossings

Song-Tian-Z.Zhang 19 proved the Gromov-Hausdorff and homeomorphism parts in general.
Singularities of the limit

Theorem (Gross-T.-Zhang 19)

Assume $X$ projective and $D = \cup_i D_i$ a divisor with simple normal crossings. Then on $Y \setminus D$ the collapsed limit metric $\omega_0$ is quasi-isometric to a conical metric

$$\omega_{\text{cone}} = \sum_{i=1}^{k} \frac{\sqrt{-1}dz_i \wedge d\bar{z}_i}{|z_i|^{2(1-\alpha_i)}} + \sum_{i=k+1}^{m} \sqrt{-1}dz_i \wedge d\bar{z}_i,$$

$(\alpha_i \in 2\pi\mathbb{Q} \cap (0, 2\pi])$ up to logarithmic errors.

Hodge theory is used to find the precise asymptotic behavior of $\omega_0^m = f^*\omega^n_X$ near $D$, which looks like $\omega_{\text{cone}}^m$ up to logarithmic errors. Then estimates for solution of Monge-Ampère equations with conical singularities have to be extended to this case.
Collapsing Non-Fibration Case

What if $[\alpha]$ does not come from a fibration on $X$?

$(X, \omega)$ Calabi-Yau, $[\alpha] \in \partial C_X$ with $\int_X \alpha^n = 0$ and $\omega_t \in [\alpha] + t[\omega]$, $t > 0$ Ricci-flat metrics. Do we have smooth convergence of $\omega_t$ away from an analytic subvariety?

Theorem (Filip-T. 17, 18)

No! There are examples where $X = K3$ and this fails. Here $[\alpha]$ contains no smooth semipositive representative.

These come from holomorphic dynamics on $K3$ surfaces (Cantat, McMullen 99). They construct examples of $T : X \to X$ chaotic automorphisms (positive topological entropy $h$), where $\omega_t \to \eta$ weakly as currents, $\eta =$closed positive current in $[\alpha]$, $T^*\eta = e^{h} \eta$.

Work of Cantat-Dupont 14 and Filip-T. 18 shows that in these examples $\eta$ cannot be $C^0$ away from any subvariety. Here the Gromov-Hausdorff limit is a point.
Second Setup

Back to the second setup where we degenerate the complex structure

\[ \pi : \mathcal{X} \to \Delta \text{ with fibers } X^n_t = \pi^{-1}(t) \]

Calabi-Yau for \( t \in \Delta^* \)

\[ \omega_t \in [\omega_{\mathcal{X}}|_{X_t}] \text{ Ricci-flat on } X_t \]

**Theorem (T. 14, Takayama 15)**

*We have that \( \text{diam}(X_t, \omega_t) \leq C \) as \( t \to 0 \) if and only if \( X_0 \) is a normal Calabi-Yau variety with canonical singularities (up to applying semistable reduction and relative MMP)*

Let \( \tilde{\omega}_t = \text{diam}(X_t, \omega_t)^{-2} \omega_t \), which are volume collapsing if and only if \( X_0 \) has worse than canonical singularities.
The noncollapsed case

**Theorem**

Suppose that in our family $\pi: \mathcal{X} \to \Delta$ we have $X_0$ normal Calabi-Yau variety with canonical singularities. Then

- (Rong-Zhang 10) $(X_t, \omega_t)$ converge locally smoothly to $(X_0^{\text{reg}}, \omega_0)$ the EGZ metric
- (Rong-Zhang 10) In Gromov-Hausdorff $(X_t, \omega_t) \to (Z, d) = \text{metric completion of } (X_0^{\text{reg}}, \omega_0)$
- (Donaldson-Sun 12) $Z$ is homeomorphic to $X_0$

Example: family of $K3$ surfaces in $\mathbb{P}^3$

$$x^4 + x^2 + y^4 + y^2 + z^4 + z^2 = t$$
The collapsed case

Suppose that in our family \( \pi : \mathcal{X} \to \Delta \) the central fiber \( X_0 \) has worse singularities, so \( \text{Vol}(X_t, \tilde{\omega}_t) \to 0 \). Pretending that \( X_0 \) has normal crossings, construct its dual intersection complex \( Sk(\mathcal{X}) \)

\[
0 < \dim_{\mathbb{R}} Sk(\mathcal{X}) \leq n
\]

When \( \dim_{\mathbb{R}} Sk(\mathcal{X}) = n \) we call \( X_0 \) a large complex structure limit

Example: family of elliptic curves in \( \mathbb{P}^2 \) given by

\[
y^2 = x^3 + x^2 + t, \quad Sk(\mathcal{X}) \cong S^1
\]

Example: family in \( \mathbb{P}^{n+1} \) given by

\[
z_0^{n+2} + \cdots + z_{n+1}^{n+2} + \frac{1}{t}z_0 \cdots z_{n+1} = 0, \quad Sk(\mathcal{X}) \cong S^n
\]
Conjecture (Strominger-Yau-Zaslow 95)

Given a large complex structure limit $\pi : \mathfrak{X} \to \Delta$ of Calabi-Yau $n$-folds, for $0 \neq |t| \ll 1$ the manifold $X_t$ admits a Special Lagrangian $T^n$-fibration (with singular fibers) over a half-dimensional base $B$ with singular fibers lying over a subset $S \subset B$ of codimension 2.

Taking fiberwise dual tori one obtains a dual $T^n$ fibration over $B \setminus S$ which should compactify to the “mirror family” of $\mathfrak{X}$

$B$ is expected to be homeomorphic to $S^n$ when $X_t$ are “strict” Calabi-Yau, and to $\mathbb{P}^n$ when $X_t$ are hyperkähler

Recent progress by Li 19 for Fermat hypersurfaces
Large complex structure limits

**Conjecture (Kontsevich-Soibelman, Gross-Wilson, Todorov 00)**

Given a large complex structure limit $\pi : \mathcal{X} \to \Delta$ of Calabi-Yau $n$-folds, with unit-diameter Ricci-flat metrics $\tilde{\omega}_t$ as before. Then $(X_t, \tilde{\omega}_t) \to (Z, d)$ in Gromov-Hausdorff and there is $Z_0 \subset Z$ open dense with complement of codimension $\geq 2$, so that $(Z_0, d)$ is an $n$-dimensional Riemannian manifold.

$Z_0$ also has an affine structure and the limit metric satisfies extra properties. $Z$ is expected to coincide with the base $B$ of the SYZ fibration. Proved by Gross-Wilson 00 for some large complex structure limits of $K3$.

**Theorem (Gross-T.-Zhang 11, T.-Zhang 17)**

This conjecture holds for some large complex structure limits of projective hyperkähler manifolds (those which after hyperkähler rotation reduce to the first setup).

Using this, Odaka-Oshima 18 proved the conjecture for all large complex structure limits of $K3$. 
Thank You!