Some high order schemes for parabolic Hamilton-Jacobi-Bellman equations

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We consider second order Hamilton-Jacobi-Bellman (HJB) equations:

\[
\begin{align*}
    v_t + \sup_{a \in \Lambda} \mathcal{L}^a(t, x, v, D_x v, D_x^2 v) &= 0 & x \in \mathbb{R}^d, t \in (0, T) \\
    v(0, x) &= \psi(x) & x \in \mathbb{R}^d,
\end{align*}
\]

where \( \mathcal{L}^a : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{S}^{d \times d} \to \mathbb{R} \) takes the form

\[
\mathcal{L}^a(t, x, r, p, Q) = \left\{ -b(t, x, a) \cdot p - \frac{1}{2} Tr[\sigma \sigma^T(t, x, a) Q] - f(t, x, a) r + \ell(t, x, a) \right\}.
\]

- \( \Lambda \subset \mathbb{R}^m \) (set of control values): compact set;
- \( T > 0 \): terminal time;
- **Existence and uniqueness** of a viscosity solution of the HJB equation.
Lions (’83), Fleming-Soner (’93), Yong-Zhou (’99), et al.: 
\( v \) is the value function of a stochastic optimal control problem

\[
\nu(t, x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[ \int_t^T e^{F(t, s, x, \alpha)} \ell(s, X_{t,x}^{\alpha}(s), \alpha(s)) \, ds + e^{F(t, T, x, \alpha)} \psi(X_{t,x}(T)) \right]
\]

where \( F(t, \cdot, x, \alpha) = \int_t^T f(s, X_{t,x}^{\alpha}(s), \alpha(s)) \, ds \) and \( X_{t,x}(\cdot) \) solves

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\frac{dX(s)}{ds} = b(s, X(s), \alpha(s)) \, ds + \sigma(s, X(s), \alpha(s)) \, dW(s) \\
X(t) = x.
\end{array}
\right.
\end{align*}
\]
Example 1: Mean-variance asset allocation

Wealth equation ($\alpha$ control: proportion of wealth invested in risky asset)

$$dX(s) = [(r + \alpha \xi \sigma)X(s) + c] \, ds + \alpha \sigma X(s) \, dW(s),$$

with contribution rate $c$, $\xi = (b - r)\sigma^{-1}$. Mean-variance asset allocation problem:

$$\sup_{\alpha \in \mathcal{A}} \left( \mathbb{E}[X^\alpha_{t,x}(T)] - \lambda \text{Var}[X^\alpha_{t,x}(T)] \right).$$

This problem can be embedded in the following (Zhou and Li ('00)):

$$v(t, x) = \inf_{\alpha \in \mathcal{A}} \mathbb{E} \left[ (X^\alpha_{t,x}(T) - \gamma)^2 \right],$$

associated with the following HJB equation

$$\begin{cases} v_t + \sup_{a \in \Lambda} \left\{ -\frac{1}{2} (\sigma ax)^2 v_{xx} - (c + x(r + a\sigma \xi))v_x \right\} = 0, \\
 v(0, x) = (x - \frac{\gamma}{2})^2. \end{cases}$$
Example 2: Uncertain volatility model

An underlying asset is assumed to follow
\[ dX(s) = rX(s)\,dt + \sigma_s X(s)\,dW(s), \]
where we only assume \( \sigma_s \in [\sigma_{\text{min}}, \sigma_{\text{max}}] \), an “uncertain” volatility.

The lowest arbitrage-free price is given by
\[ v(t, x) = \inf_{\sigma} \mathbb{E} \left[ e^{r(T-t)} \psi(X_{t,x}(T)) \right] \]
associated to the following HJB equation
\[
\left\{ \begin{array}{c}
 v_t + \sup_{\sigma \in \{\sigma_{\text{min}}, \sigma_{\text{max}}\}} \left( -\frac{\sigma^2}{2} x^2 v_{xx} \right) - r x v_x + rv = 0, \\
 v(0, x) = \psi(x),
\end{array} \right.
\]
where \( \psi \) is a European option payoff, and similar for the upper bound.
Consider $d = 1$.

Let us consider a space and time discretization $h, \tau > 0$:

\[ t_k = k\tau \quad x_i = ih \quad k = 0, \ldots, N \quad i \in \mathbb{I}. \]

\[ u^k_i \sim v(t_k, x_i). \]

Let $u$ be a numerical approximation of $v$ defined by a scheme

\[ \mathcal{I}(t_{k+1}, x_i, u^{k+1}_i, [u])_i = 0 \quad k = 0, \ldots, N - 1, \quad i \in \mathbb{I} \]

\[ u^0_i = \psi(x_i) \quad i \in \mathbb{I} \]

Scheme in “explicit form”:

\[
\begin{aligned}
&u^{k+1}_i = S(t_{k+1}, x_i, [\ldots, u^{k-1}_i, u^k])_i \quad k = 0, \ldots, N - 1, \quad i \in \mathbb{I} \\
u^0_i = \psi(x_i) 
\end{aligned}
\]

If $\mathcal{I}$ one step explicit: \[ \mathcal{I}(t_{k+1}, x_i, u^{k+1}_i, u^k)_i = \frac{(u^{k+1}_i - S(t_{k+1}, x_i, u^k))_i}{\tau} . \]
Key properties of numerical schemes

- **Stability:** \((\tau, h)\) the scheme admits a bounded solution;
- **Consistency:** For any smooth function \(\varphi\), there exists a function \(E\) such that \(E(\tau, h) \to 0\) as \((\tau, h) \to 0\) and
  \[
  \left| \mathcal{S}(t, x, \varphi(t, x), [\varphi]) - \left( \varphi_t + \sup_{a \in \Lambda} \mathcal{L}^a(t, x, \varphi, \varphi_x, \varphi_{xx}) \right) \right| \leq E(\tau, h).
  \]
  \(E(\tau, h)\): consistency error;
  \(E(\tau, h) \leq C(\tau^\rho + h^\rho)\): scheme of order \(\rho\);
- **Monotonicity:** For any bounded functions \(\phi\) and \(\psi\) such that \(\phi \leq \psi\):
  \[
  \mathcal{S}(t, x, r, [\phi]) \geq \mathcal{S}(t, x, r, [\psi]).
  \]

**Theorem (Barles-Souganidis (‘91))**

*If the scheme \(\mathcal{S}\) is stable, consistent and monotone, then its solution \(u\) converges to the unique viscosity solution of the HJB equation.*
Finite difference schemes

For simplicity set $b = \ell = 0$:

$$v_t + \sup_{a \in \Lambda} \left( -\frac{1}{2} \sigma^2(t, x, a) D^2 v - f(t, x, a)v \right) = 0.$$  \hspace{1cm}

Then for $\theta \in [0, 1]$ define a scheme

$$\frac{u_{i}^{k+1} - u_{i}^{k}}{\tau} + \sup_{a \in \Lambda} \left( \theta \left( -\frac{1}{2} \sigma^2(t_{k+1}, x_i, a) D^2 u_{i}^{k+1} + f(t_{k+1}, x_i, a) u_{i}^{k+1} \right) ight.$$

$$\left. + (1 - \theta) \left( -\frac{1}{2} \sigma^2(t_k, x_i, a) D^2 u_{i}^{k} + f(t_n, x_i, a) u_{i}^{k} \right) \right) = 0,$$

where

$$D^2 u_i := \frac{u_{i-1} - 2u_{i} + u_{i+1}}{h^2}.$$  \hspace{1cm}

Typically: $\theta = 1/2$ (CN), $\theta = 0$ (EE), $\theta = 1$ (IE).
The Crank-Nicolson scheme

The IE scheme is: consistent ($\rho = 1$), unconditionally stable and monotone.

The CN scheme is:

- consistent of order $\rho = 2$ in both $\tau$ and $h$;
- unconditionally stable, i.e. for all $\tau$ and $h$;
- monotone only under a CFL condition, a restriction on the timestep $\tau \leq Ch^2$. 
Forsyth et al. (‘01): butterfly options with UV

- Without CFL condition the **CN** scheme may converge to a wrong solution;
- Rannacher timestepping (fully implicit scheme for the first iterations) shows convergence to the exact solution;
- No convergence proof since the scheme is non monotone.

Value function $t = t_N = 0.1$

Value function $t = t_1$
Filtered schemes

Theorem (Godunov ('59))

Monotone linear schemes have order of consistency at most $\rho = 1$.

Idea of filtered schemes: Consider “almost-monotone” schemes obtained as a perturbation of high order schemes and for which convergence to the viscosity solution can be proved.

References:
- Froese-Oberman ('13): Monge-Ampère equation;
- Oberman-Salvador ('15): first order HJ equations;
- Bokanowski-Falcone-Sahu ('15): first order time dependent HJ eq.

Aim of this work:
- Generalise to second order time dependent HJB equations;
- Generalise to implicit finite difference schemes;
- Tests on some control problems arising in financial applications.
To define the filtered scheme, the following three ingredients are needed:

- A monotone scheme: $u^{k+1} = S_M(u^k)$;
- A higher order ($\rho > 1$) scheme: $u^{k+1} = S_H([\ldots, u^{k-1}, u^k])$;
- A filter function $F$.

We switch between the schemes using the following filter:

$$F(x) := \begin{cases} x & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$ 

The filtered scheme is defined by

$$S_F := S_M + \varepsilon \tau \left( \frac{S_H - S_M}{\varepsilon \tau} \right).$$
The monotone scheme

We assume that $\mathbb{I} = \{1, \ldots, J\}$ and consider a monotone, consistent and stable scheme $\mathcal{S}_M$ (⇒ convergent) of the following form:

$$
\mathcal{S}_M(t_k, x_i, u_{i}^{k}, u_{i}^{k-1})_i \equiv \frac{1}{\tau} \sup_{a \in \Lambda} \left\{ \mathbb{M}^{a,k} u^k_ - G^{a,k-1}(u^{k-1}) \right\}_i
$$

with $\mathbb{M}^{a,k} \in \mathbb{R}^{J \times J}$ and $G^{a,k-1}(\cdot) \in \mathbb{R}^J$, such that

$$
\mathcal{E}_M(\tau, h) \leq C_M(\tau + h).
$$

Under suitable assumptions on $\mathbb{M}$:

$$
\begin{cases}
    u_i^k = S_M(u_{i}^{k-1})_i & k = 1, \ldots, N, \quad i \in \mathbb{I} \\
    u_i^0 = \psi(x_i) & i \in \mathbb{I}
\end{cases}
$$

Examples: Implicit Euler, Semi-Lagrangian;
Consider a higher order scheme:

\[
\begin{aligned}
    u^k_i &= S_H([\ldots, u^{k-2}, u^{k-1}])_i & k = 1, \ldots, N, & i \in \mathbb{I} \\
    u_0^i &= \psi(x_i) & i \in \mathbb{I}
\end{aligned}
\]

such that

\[
\mathcal{E}_H(\tau, h) \leq C_H(\tau^p + h^q)
\]

for some \( p, q > 1 \).

Examples \((p = q = 2)\): CN, BDF2 :

\[
\frac{3u^k_i - 4u^{k-1}_i + u^{k-2}_i}{2\tau} + \sup_{a \in \Lambda} \left\{ - \frac{1}{2} \sigma^2(t_k, x_i, a)D^2u^k_i + b^+(t_k, x_i, a)D^{1,-}u^k_i \right. \\
+ \left. b^-(t_k, x_i, a)D^{1,+}u^k_i + f(t_k, x_i, a)u^k_i - \ell(t_k, x_i, a) \right\} = 0,
\]

where

\[
D^{1,-}u_i := \frac{3u_i - 4u_{i-1} + u_{i-2}}{2h} \quad \text{and} \quad D^{1,+}u_i := \frac{3u_i - 4u_{i+1} + u_{i+2}}{2h}.
\]
Main results

<table>
<thead>
<tr>
<th>Theorem</th>
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<tbody>
<tr>
<td>The solution of the filtered scheme converges locally uniformly to the unique viscosity solution of the HJB equation as $(\tau, h, \varepsilon) \to 0$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Let the solution $v$ be smooth in $\mathcal{O}$. If $S_F$ is stable, there exists $C_0 &gt; 0$ such that if $\varepsilon = C_0(\tau + h)$ and $\tau, h$ are small enough then $S_F(v) = S_H(v)$ and $\mathcal{E}_F(\tau, h) = \mathcal{E}_H(\tau, h)$ in $\mathcal{O}$.</td>
</tr>
</tbody>
</table>

**Proof.**

One has to prove that $\frac{|S_H([\ldots, v^{n-1}, v^n]) - S_M(v^n)|}{\varepsilon \tau} \leq 1$. For simplicity let us consider explicit schemes. From consistency of the schemes there is $C_M > 0$ such that:

$$
\frac{|S_M(v^n) - S_H([v])|}{\varepsilon \tau} \leq \frac{\mathcal{E}_M(\tau, h) + \mathcal{E}_H(\tau, h)}{\varepsilon} \leq \frac{C_M(\tau + h) + o(\tau + h)}{\varepsilon}.
$$

The result follows for $\tau, h$ small enough and $C_0 > C_M$ big enough.
Recall the HJB equation

\[
\begin{aligned}
\nu_t + \sup_{a \in A} \left\{ -\frac{1}{2} (\sigma ax)^2 \nu_{xx} - (c + x(r + a\sigma \xi)) \nu_x \right\} &= 0, \\
\nu(0, x) &= \left(x - \frac{\gamma}{2}\right)^2.
\end{aligned}
\]
Test 1: Mean-variance asset allocation problem

Monotone scheme: Implicit Euler scheme. One has:

\[ C_M \sim 35. \]

High order scheme: BDF2, Crank-Nicolson
Consider the following HJB equation in \([0, T] \times ([-\pi, \pi] \times [-\pi, \pi]):

\[
\begin{aligned}
\nu_t - \inf_{a \in \Lambda} \left( \frac{1}{2} Tr[\sigma \sigma^T (a) D_x^2 \nu] + \ell(t, x, a) \right) &= 0, \\
\nu(0, x) &= 2 \sin x_1 \sin x_2
\end{aligned}
\]

with

\[
\Lambda = \{ a \in \mathbb{R}^2 : a_1^2 + a_2^2 = 1 \}, \quad \sigma(a) = \sqrt{2} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix},
\]

\[
\ell(t, x, a) = (1 - t) \sin x_1 \sin x_2 + (2 - t)(a_1^2 \cos x_1 + a_2^2 \cos x_2).
\]

This problem has exact solution:

\[
\nu(t, x) = (2 - t) \sin x_1 \sin x_2.
\]
Test 2: a two dimensional example

Monotone scheme: Explicit semi-Lagrangian scheme. One has:

\[ C_M \sim \frac{20}{3}. \]

High order scheme: Implicit Euler timestepping and

\[
\begin{align*}
D_{xx}^2 v_{i,j} &:= \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h_x^2}, \\
D_{yy}^2 v_j &:= \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h_y^2} \\
D_{xy}^2 v_{i,j} &:= \frac{v_{i+1,j+1} - v_{i+1,j-1} + v_{i-1,j-1} - v_{i-1,j+1}}{4h_xh_y}.
\end{align*}
\]
Test 2: a two dimensional example

Value function at $T = 0.5$

Activity of the filter
Recall the HJB equation:

\[
\begin{cases}
    v_t + \sup_{\sigma \in \{\sigma_{\text{min}}, \sigma_{\text{max}}\}} \left( -\frac{\sigma^2}{2} x^2 v_{xx} \right) - r x v_x + rv &= 0, \\
    v(0, x) &= \psi(x)
\end{cases}
\]

where we use a “butterfly” payoff $\psi(x)$ of mixed convexity:
Pooley-Forsyth-Vetzal (‘01): in absence of the CFL, the CN scheme may converge to a wrong solution.
Bokanowski-AP-Reisinger (‘17): the BDF2 scheme shows good performances.

Value function $t = t_N = 0.1$:

Value function $t = t_1$:

- Monotone scheme
- Crank-Nicolson (no CFL)
- BDF2
Test 3: butterfly option with UV

Applying the filter to the Crank-Nicholson scheme:

- The filter becomes active to correct the scheme;
- the order of convergence is not more than one in the relevant region.

<table>
<thead>
<tr>
<th>Filtered CN</th>
<th>BDF</th>
<th>Rannacher</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.25</td>
<td>0.72</td>
<td>1.28</td>
</tr>
<tr>
<td>0.85</td>
<td>2.34</td>
<td>1.86</td>
</tr>
<tr>
<td>0.90</td>
<td>1.70</td>
<td>1.88</td>
</tr>
<tr>
<td>1.10</td>
<td>1.82</td>
<td>1.94</td>
</tr>
<tr>
<td>0.94</td>
<td>1.95</td>
<td>1.98</td>
</tr>
<tr>
<td>1.10</td>
<td>1.98</td>
<td>1.99</td>
</tr>
<tr>
<td>0.92</td>
<td>1.83</td>
<td>1.96</td>
</tr>
</tbody>
</table>

Applying the filter on Rannacher or BDF2:
The schemes converges, but the unboundedness of the derivatives prevents second order convergence.
Conclusions

- We have defined filtered schemes for second order time dependent HJB equations;
- We have showed convergence and high order behavior for smooth solutions;

...moreover...

Motivated by the results in our numerical tests, we have studied a second order (non monotone) BDF2 scheme (mainly in one dimension):

- Stability proof in $| \cdot |_{{H_1}}$ and $\| \cdot \|_2$ for the BDF2 scheme;
- Error estimates for piecewise smooth solutions.

Future directions of work:

- Stability proof for BDF2 scheme in $L^\infty$ norm;
- Multiple dimensions.
